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Reçu par la Rédaction le 20. 9. 1973

Stable sets, a characterization of β_2-models of full second order arithmetic and some related facts*

by

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Abstract. We study here stable sets i.e. transitive sets with Σ_1-reflection property. As a result we get the following characterization of β_2-models of A_2: M is a β_2-model for A_2 iff there is stable transitive model N of ZFC$^-$ such that $M = N \cap \wp(\omega)$. We get a generalization of both theorems of Kripke and Platek on stability of L_{δ_2} and Lévy on stability of HC.

Zbierski (in [16]) gives the following characterization of β-models for full second order arithmetic A_2 (i.e. arithmetic with the scheme of choice):

M is a β-model of A_2 iff $M = N \cap \wp(\omega)$ for some transitive model N of ZFC$^-$.

We give similar characterization of β_n-models of A_2. The characterization is especially nice in case of β_2-models. Namely we prove:

M is a β_2-model of A_2 iff $M = N \cap \wp(\omega)$ for some transitive model N of ZFC$^-$ *such that* $N \prec_1 V$.

The proof of these and related facts (for instance we prove that the sets $\mathrm{Th}(\wp(\omega))$ and $\mathrm{Th}(\mathrm{HC})$ are recursively isomorphic) takes first two paragraphs of the paper.

In the third paragraph we prove theorem of Kripke and Platek about stability of δ_2. We generalize this theorem getting result generalizing both aforenamed theorem and theorem of Lévy.

Paragraph four is devoted to the study of levels of constructible hierarchy from the point of definability. As shown in [10] pointwise definability of levels is related with gaps, one of important means while studying fine structure of constructible universe. We show that wide class of stable ordinals gives pointwise definable levels. We finally prove a result complementary to the one of Friedman, Jensen and Saks on characterization of countable admissible ordinals as ω_1^A for $A \subseteq \omega$.

* Part of the results was obtained in the summer of 1972 when the author worked at S.U.N.Y. at Buffalo. We express our gratitude for the Department of Mathematics of that University.

Most of the paper is devoted to study of the properties of Σ_1-formulas. We find this first non-constructive but yet fairly regular class of formulas interesting and think that studies of their properties are important.

We wish to express our gratitude to professor Mostowski and Paweł Zbierski whose efforts in explaining us the method of trees led us to these results. The comments by Marian Srebrny and Krzysztof Apt helped us to smooth some details.

0. Preliminaries. Throughout the paper we accept standard set-theoretic notation. $V = L$ denotes the Gödel's axiom of constructibility, and by analytical form of the axiom of constructibility we mean a Π^1_3 formula which can be found in Addison [1]. $V = HC$ denotes the statement "everything is countable".

An ordinal α is called $\Delta^{1,B}_k$ if there is a $\Delta^{1,B}_k$ set of natural numbers A, such that $\overline{\{\langle x, y\rangle: 2^x(2y+1) \in A\}} = \alpha$.

δ_n is the first ordinal which is not Δ^1_n.

ZFC^- is a theory formulated in the language of set theory containing all the axioms of ZF with the exception of the power set axiom but with the following scheme of choice — replacement added:

$$(x)_y(Ez)\Phi(x, z) \rightarrow (Ef)\big(\mathrm{Func}(f) \;\&\; Df = y \;\&\; (x)_y\Phi(x, f(x))\big) .$$

A_2 denotes full second order arithmetic with the scheme of choice, i.e. second order arithmetic with the scheme

$$(x)(EY)\Phi(x, Y) \rightarrow (EY)(x)\Phi(x, Y^{(x)}) ,$$

$$D^1_n = \{x \subseteq \omega: x \text{ is } \Delta^1_n\} .$$

If M is a transitive set then M is said to have the Σ_1-*reflection property* iff for every Σ_1-formula $\Phi(x, \vec{p})$, where $\vec{p}$ is a sequence of parameters from M,

$$(Ex)\Phi(x, \vec{p}) \rightarrow M \models (Ex)\Phi[\vec{p}] ,$$

$$HC = \{x: \overline{\overline{TC(x)}} \leqslant \omega\} .$$

An ordinal α is called *stable* iff L_α has the Σ_1-reflection property.

If M, N are transitive sets, then $M \prec_n N$ means that $M \subseteq N$ and for every Σ_n-formula Φ of the language of set theory and every sequence of parameters $\vec{p}$ from M, $M \models \Phi[\vec{p}] \leftrightarrow N \models \Phi[\vec{p}]$.

If X, Y are two subfamilies of $\wp(\omega)$, then $X \prec^1_n Y$ if for every Σ^1_n-formula Φ of the language of second order arithmetic and every sequence of parameters $\vec{p}$ from X, $X \models \Phi[\vec{p}] \leftrightarrow Y \models \Phi[\vec{p}]$.

If X is a model of full second order arithmetic and $X \prec^1_n \wp(\omega)$, then X is called a β_n-*model*.

§ 1. Some results to be used in further parts.

The results of this paper are strongly based on the methods of Zbierski [16] and Marek [9], and also Leeds and Putnam [7], and Marek and Srebrny [10].

A) Results of Zbierski [16]. A tree is a function $A \subseteq \omega \times \omega$ such that

(a) $(Y)(Y \subseteq DA \;\&\; Y \neq \emptyset \Rightarrow Y \nsubseteq A * Y)$,

(b) $(Ea)(x)_{DA}(En)(\underbrace{A \circ \ldots \circ A}_{n}(x) = a)$,

(c) A has no automorphisms.

Since "being a tree" is a Π^1_1-formula it is (by the results of Mostowski [12]) absolute with respect to β-models. One can define relations Eps and Eq between trees in such a way that Eps is a well-founded relation and Eq is a congruence relation with respect to Eps. If we consider the family of trees of a β-model M, divide it by Eq and collapse it (this is possible because the axiom of extensionality is true among trees), we get a transitive model N of $ZFC^- + V = HC$ such that $M = N \cap \wp(\omega)$. The relations Eq and Eps are both Σ^1_1.

The relation between a tree and the set it codes is definable as follows. Let X be a tree, $a \in DX$. Then $\|a\|_X = \{\|b\|_X: X(b) = a\}$ and $\|X\| = \|\mathrm{MAX}_X\|_X$, where MAX_X is the maximal element of X (whose existence is guaranted by (b) above). We describe the fact $\|X\| = x$ as $\mathrm{Code}(X, x)$. One can prove that this relation is absolute with respect to transitive models of ZFC^-.

Let us denote by $\overline{M}$ a model of $ZFC^- + V = HC$ arising from M by the procedure described above. As we noted, $M = \overline{M} \cap \wp(\omega)$. The analytical form of the axiom of constructibility is true in M iff $V = L$ is true in $\overline{M}$.

B) Results of Leeds and Putnam [7] **and Marek and Srebrny** [10]. α is called a *gap ordinal* iff $(L_{\alpha+1} - L_\alpha) \cap \wp(\omega) = \emptyset$.

α is called the *beginning of a gap* iff α is a gap ordinal but

$$(\beta)_\alpha\big(L_\alpha \cap \wp(\omega) \neq L_\beta \cap \wp(\omega)\big) .$$

Putnam and Leeds prove that if α is a gap ordinal then $L_\alpha \cap \wp(\omega)$ is a β-model of A_2.

Marek and Srebrny prove that α is the beginning of a gap iff L_α is a model of $ZFC^- + V = HC$.

They also prove that in this case $L_\alpha = \overline{L_\alpha \cap \wp(\omega)}$.

C) Result of Shoenfield [13]. If $A \subseteq \omega$, then $A \in D^1_2$ iff there is a Δ^1_2 ordinal α such that $A \in L_{\alpha+1} - L_\alpha$.

D) Result of Lévy [8]. HC has the property of Σ_1-reflection (which (informally) may be written as $HC \prec_1 V$).

§ 2. Translation procedure, β_2-models. It is clear that $\overline{\wp(\omega)} = \mathrm{HC}$, we now describe a uniform procedure that allows us to translate set-theoretic formulas into analytical ones.

We use the following lemma from Marek [9].

LEMMA 2.1. *Let X, Y be the trees. Then*

$$\|X\| \in \|X\| \leftrightarrow X \,\mathrm{Eps}\, Y,$$
$$\|X\| = \|Y\| \leftrightarrow X \,\mathrm{Eq}\, Y,$$

i.e. $\mathrm{Code}(X, x) \,\&\, \mathrm{Code}(Y, y) \to [(X \,\mathrm{Eps}\, Y) \leftrightarrow x \in y) \,\&\, (X \,\mathrm{Eq}\, Y \leftrightarrow x = y)]$.

Now let Φ be a Δ_0 (i.e. bounded) formula of the language of set theory with free variables $V_1 - V_k$. We will construct two formulas of the language of second order arithmetic, Φ_1^T and Φ_2^T, such that:

(a) Φ_1^T is Σ_2^1,

(b) Φ_2^T is Π_2^1,

(c) $A_2 \vdash \Phi_1^T \leftrightarrow \Phi_2^T$,

(d) if $x_1, \ldots, x_k$ are elements of HC, $x_1 = \|X_1\|, \ldots, x_k = \|X_k\|$ then

$$\mathrm{HC} \models \Phi[x_1, \ldots, x_k] \leftrightarrow \wp(\omega) \models \Phi_1^T[X_1, \ldots, X_k] .$$

The construction: For atomic formulas the construction is clear since both Eps and Eq are Σ_1^1 hence Σ_2^1 and Π_2^1.

For boolean connectives the construction is clear. For restricted quantifiers the construction of Φ_1^T and Φ_2^T proceeds as follows. Let $\Phi = (\mathrm{E}x)_y \Psi$ and Ψ_1^T and Ψ_2^T be given.

The formulas Φ_1^T and Φ_2^T are produced from them by eliminating Y_a from the formula:

$$(\mathrm{E}a)\big(Y(a) = \mathrm{MAX}_Y \,\&\, \Psi_i^T(Y_a, Y_1, \ldots)\big)$$

(where Y_a is the tree arising from Y by taking a as a maximal element and "cutting out" all elements bigger or incomparable with a in the smallest transitive relation containing Y).

The fact that the interpretation of Δ_0-formulas of set theory leads to provably — Δ_2^1-formulas of second order arithmetic matches two facts: provably — Δ_2^1-formulas are absolute with respect to the β-models of A_2 [1], and $\Delta_0^{\mathrm{ZFC}^-}$-formulas are absolute with respect to transitive models of ZFC^-.

Since Δ_0-formulas are interpretable as Σ_2^1 formulas therefore also Σ_1-formulas are interpretable this way.

Let Φ^T be appropriate interpretation. Since it is Σ_2^1 it is absolute with respect to β_2-models.

(1) Added in proof. Using the fact that both Eps and Eq are not only Σ_1^1 but also Π_1^1 we may find $\varphi^T \Delta_1^1$ (for $\varphi \in \Delta_0$). This however does not improve φ^T for $\varphi \in \Sigma_1$.

LEMMA T. *Let M be a β-model of A_2, Φ a formula of set theory, Φ^T the interpretation of Φ as described above. Assume $\overline{M} \models \mathrm{Code}[X_1, x_1], \ldots, \overline{M} \models \mathrm{Code}[X_k, x_k]$. Then*

$$\overline{M} \models \Phi^T[X_1, \ldots, X_k] \leftrightarrow \overline{M} \models \Phi[x_1, \ldots, x_k] .$$

Proof. Direct from the construction.

THEOREM 2.2. *Assume M is a β_2-model. Then $\overline{M}$ has the property of Σ_1-reflection.*

Proof. Assume $(\mathrm{E}x)\Phi(x, a_1, \ldots, a_n)$, where Φ is Σ_1 formula and $a_1, \ldots, a_n \in \overline{M}$. Then in particular $a_1, \ldots, a_n \in \mathrm{HC}$. By Levy's result (cf. 1.D), $\mathrm{HC} \models (\mathrm{E}x)\Phi(x, a_1, \ldots, a_n)$. Let $a_1 = \|A_1\|, \ldots, a_n = \|A_n\|$, $A_1, \ldots, A_n \in M$.

Then $\wp(\omega) \models (\mathrm{E}X)\Phi^T[X, A_1, \ldots, A_n]$, and since Φ^T is Σ_2^1 and M is a β_2-model, X can be found in M. So $x = \|X\| \in \overline{M}$, and by Lemma T, $\overline{M} \models (\mathrm{E}x)\,(x, a_1, \ldots, a_n)$.

The same reasoning to the following theorem.

THEOREM 2.3. *If a is the beginning of a gap and $L_a \cap \wp(\omega)$ is a β_2-model, then a is stable.*

LEMMA 2.4. *Let Φ be a Π_1^1-formula. Let Φ^+ be the usual interpretation of the formula of second order arithmetic in set theory. Then there is a Δ_1-formula Φ_1 such that $\mathrm{ZFC}^- \models \Phi^+ \leftrightarrow \Phi_1$.*

Proof. Mostowski [10] shows how to transform a formula Φ into another formula Ψ equivalent to it, but of the form "something is a well-ordering". But the last formula is $\Delta_1^{\mathrm{ZFC}^-}$.

As a corollary we get:

LEMMA F. *Let Φ be a Σ_2^1-formula. Then there is a Σ_1-formula Φ such that*

$$\mathrm{ZFC}^- \vdash \Phi^+ \leftrightarrow \Phi_1 .$$

THEOREM 2.5. *Assume M is a transitive model of ZFC^- with the Σ_1-reflection property. Then $M \cap \wp(\omega)$ is a β_2-model.*

Proof. Assume $\wp(\omega) \models \Phi[A_1, \ldots, A_n]$, where Φ is a Σ_2^1-formula. Then $\mathrm{HC} \models \Phi_1[A_1, \ldots, A_n]$, where Φ_1 is the translation of Φ from Lemma F. Thus $\Phi_1[A_1, \ldots, A_n]$, and so $M \models \Phi_1[A_1, \ldots, A_n]$. Therefore $M \models \Phi^+[A_1, \ldots, A_n]$, and so $M \cap \wp(\omega) \models \Phi[A_1, \ldots, A_n]$.

THEOREM 2.6. *If a is stable and the beginning of a gap, then $L_a \cap \wp(\omega)$ is a β_2-model.*

COROLLARY 2.7. *Let M be a β-model. Then M is a β_2-model iff $\overline{M}$ has the property of Σ_1-reflection.*

COROLLARY 2.8. *Let a be the beginning of a gap. Then a is stable iff $L_a \cap \wp(\omega)$ is a β_2-model.*

The assumption in Corollary 2.8 that α is the beginning of a gap is necessary since non-gap ordinals may be stable (but do not give a model of ZFC^- then).

For instance, δ_2 is stable (as we will show later) but is not a gap ordinal (since $L_{\delta_2} \cap \wp(\omega) = D^1_2$ is not a model of A_2 as shown by Mostowski [12]).

Corollary 2.9. *There is a stable gap ordinal below* δ_3.

Proof. Enderton and Friedman [3] prove that there is a β_2-model M with height below δ_3. Its constructible sets L^M also form a β_2-model. The height of L^M is less than or equal to the height of M, so is also less than δ_3, and by Corollary 2.8 it is stable gap ordinal.

We give generalization which are proved exactly along the lines of the proofs above.

Theorem 2.10. *Let n be a natural number* $\geqslant 1$. *Then M is a β_n-model of A_2 iff* $\overline{M} \prec_{n-1} \mathrm{HC}$ & $\overline{M} \models \mathrm{ZFC}^-$.

Theorem 2.11. $M \prec \wp(\omega)$ *iff* $\overline{M} \prec \mathrm{HC}$.

As the result of our construction we get the following theorems:

Theorem 2.12. *If $n \geqslant 1$ then the set of Σ_n-sentences true in $\overline{M}$ is recursively isomorphic to the set of Σ^1_{n+1}-sentences true in M.*

Theorem 2.13. *The set of sentences true in M is recursively isomorphic to the set of sentences true in $\overline{M}$.*

Proof. We had shown that each of them is 1-1 reducible to the another, and then we use Myhill recursive isomorphism theorem ($A \leqslant_1 B$ & $B \leqslant_1 A \rightarrow A \underset{\mathrm{rec}}{\simeq} B$).

Corollary 2.14. *If $n \geqslant 1$ then the set of Σ_n sentences true in* HC *is recursively isomorphic to the set of Σ^1_{n+1} sentences true in $\wp(\omega)$.*

Corollary 2.15. *The set of sentences true in* HC *is recursively isomorphic to the set of sentences true in $\wp(\omega)$.*

§ 3. A proof of the theorem of Kripke and Platek.

Theorem (Kripke-Platek) 3.1. *δ_2 is the least stable ordinal.*

Lemma L. $x \in L_{\delta_2}$ *iff* $x = \|A\|$ *for some* $A \in D^1_2$.

Proof. $\leftarrow$ Assume $x = \|A\|$ for some $A \in D^1_2$. By the Addison-Kondo basis theorem, there is a set $B \in D^1_2$ such that B is a code for a countable family $M \subseteq \wp(\omega)$, M is a β-model of second order arithmetic with the axiom of constructibility, and $A \in M$. Clearly all elements of M belong to D^1_2. The height of M, i.e. the first ordinal not represented in M, is $\Delta^{1,B}_2$ and so is a Δ^1_2-ordinal. Since M is a β-model, $\overline{M}$ is a transitive model of $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{L}$. So $\overline{M} = L_\alpha$, where α is the height of M. Clearly $\alpha \in \delta_2$.

We show now that δ_2 is a limit of gap ordinals. Assume it is false. Then, since δ_2 is a limit number, there is $\xi \in \delta_2$ such that there is no gap ordinal between ξ and δ_2. By the definition

$$(L_{\delta_2} - L_\xi) \cap \wp(\omega) \neq \emptyset .$$

Let $C \in (L_{\delta_2} - L_\xi) \cap \wp(\omega)$. Then $C \in D^1_2$ (by Shoenfield's result), and by the above reasoning we can find a Δ^1_2 set D such that D codes a β-model N of second order arithmetic and the axiom of constructibility and such that $C \in N$. $\overline{N} = L_\varrho$ for some $\varrho \in \delta_2$, and since $\overline{N} \cap \wp(\omega) = N$, $C \in L_\varrho$. But since L_ϱ is a model of $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$, ϱ is the beginning of a gap, which contradicts the choice of ξ.

Since δ_2 is a limit of gap ordinals, therefore if $A \in L_{\delta_2}$ there is the beginning of a gap $\xi \in \delta_2$ such that $A \in L_\xi$. But then by (1.B) L_ξ is a model of ZFC^-, so $L_\xi \models (\mathrm{E}y)\mathrm{Code}(y, X)[A]$, and thus $\|A\| \in L_\xi$. But $L_\xi \subseteq L_{\delta_2}$, which finishes half of the proof.

$\rightarrow$ Assume $x \in L_{\delta_2}$. Since δ_2 is a limit of gap ordinals, $x \in L_\varrho$ for some ϱ which is the beginning of a gap. But by (0.B) $L_\varrho \models \mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$.

Now $\mathrm{ZFC}^- \vdash (\mathrm{V} = \mathrm{HC} \leftrightarrow (x)(\mathrm{E}Y)\mathrm{Code}(x, Y))$, so there is a tree in L_ϱ coding (in L_ϱ) x. Since this last relation is absolute we conclude that there is a tree in $L_\varrho \cap \wp(\omega)$ coding x.

Since $L_\varrho \cap \wp(\omega) \subseteq L_{\delta_2} \cap \wp(\omega) = D^1_2$, we get the proof of $\rightarrow$.

Proof of the Kripke-Platek Theorem (3.1). Assume

$$(\mathrm{E}x)\Phi(x, a_1, \ldots, a_n), \quad \text{where} \quad a_1, \ldots, a_n \in L_{\delta_2} .$$

Then by Lévy's result (1.D) $\mathrm{HC} \models (\mathrm{E}x)\Phi[a_1, \ldots, a_n]$. So there is a tree X and trees $A_1, \ldots, A_n$ such that $\wp(\omega) \models \Phi^T[X, A_1, \ldots, A_n]$, Φ^T is Σ^1_2. By Lemma L we can choose $A_1, \ldots, A_n$ in D^1_2. By the Novikoff-Addison-Kondo basic theorem there is $X \in D^{1, A_1, \ldots, A_n}_2$ such that

$$\wp(\omega) \models \Phi^T[X, A_1, \ldots, A_n] .$$

Since $A_1, \ldots, A_n$ are elements of D^1_2, so is X. Applying our lemma once more (in the opposite direction) we find $x \in L_{\delta_2}$ such that $\Phi(x, a, \ldots, a_n)$. But this clearly is enough.

In order to prove that δ_2 is the least stable ordinal we show the following.

Lemma 3.2. *If σ is stable ordinal then all Σ^1_2 sets belong to* $L_{\sigma+1}$.

Proof. Let $A \subseteq \omega$ be Σ^1_2. Let Φ be a Σ^1_2 definition of A. Let Φ^+ be natural set-theoretical version of Φ (Σ_1 as shown above). We have:

$$n \in A \leftrightarrow \wp(\omega) \models \Phi[n] \leftrightarrow \mathrm{HC} \models \Phi^+[n] \leftrightarrow L_\sigma \models \Phi^+[n] .$$

Thus $A \in L_{\sigma+1}$. Now, since $L_{\delta_2} \cap \wp(\omega) = D^1_2$, the first complete Σ^1_2 set occurs in L_{δ_2+1}, so δ_2 is the least stable ordinal. It may be shown (as noted by Krzysztof Apt) that the definition of the complete set needs exactly two unbounded quantifiers.

The proof of the theorem of Kripke and Platek relativizes, in fact, using the same reasoning we get.

THEOREM 3.1. *If* $A \in \wp(\omega) \cap L$ *then* δ_2^A *is a stable ordinal.*

It is obvious that the enumeration of consecutive stable ordinals is continuous (by contrast with the consecutive enumeration of admissible ordinals).

THEOREM 3.3 (Srebrny). (a) *"Next" stable ordinal* $\alpha < \omega_1^L$ *is of the form* δ_2^A *for some* $A \in \wp(\omega) \cap L$, A *may be found* α*-finite.*

(b) *If* α *is limit in the consecutive enumeration of stable ordinals then* α *is not of the form* δ_2^A *for* $A \in \wp(\omega) \cap L$.

The proof of 3.3 may be found in [10] and [15].

By contrast H. Friedman [4] had shown that under suitable conditions (ω_1 inaccessible in L) every countable stable ordinal is of form δ_2^A for some $A \in \wp(\omega)$. Let us mention that he conjectured, that all stable ordinals are of the form δ_2^A for some $A \in \wp(\omega)$ is equivalent to $\wp(\omega) - L \neq \emptyset$.

Let us mention some facts concerning notion of stability.

Fact 3.4. The notion of stability is not absolute for transitive models for KP (or other "reasonable" set theory (Like ZF$^-$, ZF etc.) though the notion of admissibility is.

Fact 3.5. (a) If $\wp(\omega) \subseteq L$ then for every $n > 1$, $A \in \wp(\omega)$ δ_n^A is stable, non gap ordinal.

(b) Let $\delta_n^{L,A}$ be first ordinal not $\Delta_n^{1,A}$ in L. Then $\delta_n^{L,A}$ is stable, non gap ordinal.

(c) If $\wp(\omega) \subseteq L$ then $\delta = \bigcup_{n \in \omega} \delta_n$ is stable gap.

(d) $\delta^L = \bigcup_{n \in \omega} \delta_n^L$ is stable gap.

Fact 3.6. If $\wp(\omega) \subseteq L$ then Lemma L holds with δ_2 and D_2^1 changed for δ_n and D_n^1 for $n > 1$.

We had shown that $L_{\delta_2} = \overline{L_{\delta_2} \cap \wp(\omega)}$. This sort of property $\left(A = \overline{A \cap \wp(\omega)}\right)$ holds for wide class of transitive structures.

Property "To be a tree" is Π_1^1 in second order arithmetic and similarly Π_1^{KP}. Yet this property is not absolute with respect to transitive admissible sets. In particular $L_{\omega_1^{\mathrm{CK}}}$ has elements A with this property (ω_1^{CK} is recursive ω_1); using Gandy's recursive ordering without hyperarithmetic descending sequence (but not being wellordering) it is easy to construct set of natural numbers satisfying inside of $L_{\omega_1^{\mathrm{CK}}}$ formula "To be a tree" but not being a tree. Yet, using settheoretical — and not arithmetical definition of the tree we can define trees inside of admissible set. Let us change in the definition of the tree condition (a) (wellfoundness) for the following. There is a norm into an ordinal (i.e. a function $f\colon DX \to \alpha$ s.t. $X(y) = z \to f(y) \in f(z)$. The latter condition is Σ_1^{KP}.

LEMMA 3.7. *If* x *is transitive admissible set,* $On \cap x = \alpha$, $X \in x$ *is a tree then there is a norm for* X *in* x.

Proof. It is enough to prove that rank function for X, $rk_x(\cdot)$ is in X. Define

$$\varphi(\eta) = \left\{n \in DX \cup RX\colon\ (m)_{X^{-1} * \{n\}} (\mathrm{E}\beta)_\eta \big(m \in \varphi(\beta)\big)\right\}.$$

Since X is a tree (i.e. is wellfounded) therefore there must be β such that $\bigcup_{\xi<\beta} \varphi(\xi) = DX \cup RX$.

We show $\beta \in \alpha$. Assume $\alpha \subseteq \beta$. Then there must be $a \in DX \cup RX$ such that $rk_X(a) = \alpha$ $\left(\text{i.e. } a \in \varphi(\alpha) - \bigcup_{\beta \in \alpha} \varphi(\beta)\right)$. But then

$$(b)_{X^{-1} * \{a\}} (\mathrm{E}\beta)\big(b \in \varphi(\beta)\big).$$

By Σ-collection we have $\gamma < \alpha$ such that $(b)_{X^{-1}*\{a\}}\big(b \in \varphi(\gamma)\big)$. i.e. $a \in \varphi(\gamma+1)$. But α is limit therefore $\gamma+1 < \alpha$ which contradicts our assumption.

Notice that this proof resembles analogous proofs like: If $\langle x, < \rangle$ is a wellordering belonging to x then $\overline{\langle X, < \rangle} \in \alpha$ and similarity function is in x.

Using Lemma 3.7 we get:

LEMMA 3.8. *If* $x \models \mathrm{KP} + \mathrm{V} = \mathrm{HC}$, x *transitive,* $X \in x$ *is a tree, then* $\|X\| \in x$.

Proof. Using rank functions on X we define inductively $\|n\|_X = \{\|m\|_X\colon X(m) = n\}$.

THEOREM 3.9. *If* x *is countable transitive admissible set,*

$$x \models \mathrm{V} = \mathrm{HC}, \quad \textit{then} \quad x = \overline{x \cap \wp(\omega)}$$

Proof. Lemma 3.7 shows $\overline{x \cap \wp(\omega)} \subseteq x$. Now let $y \in x$. Then $\mathrm{TC}(y) \in x$, $x \models \overline{\overline{\mathrm{TC}(y)}} = \omega$. Thus we get a copy of $\mathrm{TC}(y)$ on ω. The transformation of this copy into a tree is obvious.

Theorem 3.9 generalizes for admissible sets in which strong forms of the axiom of choice holds; namely in which every set is equipollent with some ordinal, only the definition of the tree must be changed to allow $D(X) \subseteq On$.

We may use Theorem 3.9 as alternative of lemma in the proof of the theorem of Kripke and Platek. However this would need additional lemma showing absoluteness of the notion of tree with respect to L_{δ_2}.

Let us introduce the following abbreviations; if class x satisfies $\langle x, \in \rangle \prec_1 \langle \mathrm{V}, \in \rangle$ then x is called *stable*. If $\langle x, \in \rangle \prec_1 \langle \mathrm{L}, \in \rangle$ then x is called *strongly stable*. In this way Levy's theorem ([8], Thm 36) states that $H_\varkappa$ (i.e. set of all x such that $\overline{\overline{\mathrm{TC}(x)}} < \varkappa$) is stable. Lévy-Shoenfield theorem ([8], Thm 43) states that $L_{\omega_1^L}$ is both stable and strongly stable.

As pointed to us by G. Sacks stability of L implies V = L. Indeed assume that there is nonconstructible set z. We may assume $z \subseteq L_\alpha$ for some α. Then the statement $(\mathrm{E}x)$ $(x \subseteq L_\alpha \ \& \ \neg\, x \in L_{\alpha^+})$ is true Σ_1 statement which is false in L.

LEMMA 3.10. (a) *Assume $x \subseteq L_{\omega_1^L}$, x transitive. Then x is strongly stable iff x is stable.*

(b) *If x is stable set, $x \subseteq \mathrm{HC}$ then $\langle x, \epsilon\rangle \prec_1 \langle \mathrm{HC}, \epsilon\rangle$ and so $\langle x, \epsilon\rangle$ $\models \mathrm{V} = \mathrm{HC}$.*

Proof. (a) follows from Lévy-Shoenfield theorem.

(b) is obvious from general model-theoretic reasons.

LEMMA 3.11. (a) *If $A \subseteq \omega$ then $L_{\delta_2^A}[A]$ is countable transitive stable set.*

(b) *If x is countable, transitive, stable set and $A \subseteq \omega$, $A \in x$, then $L_{\delta_2^A}[A] \subseteq x$.*

Proof. (a) Using Shoenfield's lemma, relativized form, we get $\overline{\overline{D_2^{1,A}}} = L_{\delta_2^A}[A]$. By 3.10 (a) $L_{\omega_1^{L[A]}} \prec \mathrm{HC}$. Relativized version of Kripke-Platek theorem gives $L_{\delta_2^A}[A] \prec_1 L_{\omega_1^{L[A]}}[A]$.

(b) All $\Delta_2^{1,A}$ ordinals are Σ_1 definable with parameter A and so they are in x. Since $x \models \mathrm{KP}$ so $L_{\delta_2^A}[A] \subseteq x$.

THEOREM 3.12. *If x is countable stable transitive set then*

$$x = \bigcup_{A \in x \cap \wp(\omega)} L_{\delta_2^A}[A]\,.$$

Proof. Clearly, in view of 3.11 (b) $R \subseteq L$. Let $y \in x$. Then $\mathrm{TC}(y) \in x$ and is countable in x (using 3.10. (b)). From the enumeration of $\mathrm{TC}(y)$ we get $A \in x$ such that $\|A\| = y$. But $\|A\| \in L_{\delta_2^A}[A]$ which shows $L \subseteq R$.

THEOREM 3.13. (Basis property for hereditarily countable sets). *Let $\Phi(x, \vec{y})$ be Σ_1 formula, $\vec{y} \in \mathrm{HC}$ and let A be any tree such that $\vec{y} = \|A\|$. If $(\mathrm{E}x)\Phi(x, \vec{y})$ then $(\mathrm{E}x)_{L_{\delta_2^A}[A]}\Phi(x, \vec{y})$.*

Proof. Assume $(\mathrm{E}x)\Phi(x, \vec{y})$. Then $\mathrm{HC} \models (\mathrm{E}x)\Phi(x, \vec{y})$, by translation lemma $\wp(\omega) \models (\mathrm{E}X)\Phi^T(X, A)$. By basis theorem there is $X \in D_2^{1,A}$ satisfying Φ. Thus $\|X\| \in L_{\delta_2^A}[A]$.

Since δ_2^A is limit it is clear that $\|X\|$ from the proof of 3.13 belongs to some $L_\xi[A]$ for some $\xi \in \delta_2^A$. Uniform evaluation of such a ξ is possible. We give here only sketch of the proof, since the details are beyond of the limits of this paper. Let Φ be Σ_1 formula, $\vec{y} \in \mathrm{HC}$, $\vec{y} = \|A\|$.

Consider Φ^T. We may write Φ^T as $(\mathrm{E}X)\Psi(X, A)$ with Ψ being Π_1^1 formula. Let α be height of the sieve connected with Ψ and A (cf. [14], pp. 180 and 188). Let α^1 be first limit of admissibles bigger then α (this is to ensure that there are admissibles between α and α^1 and that α^1 has β-property i.e. preserves wellfoundness). Then a tree X being a witness for Ψ may be found in $L_{\alpha^1}[A]$. Using 3.8 we get $\|X\|$ in $L_{\alpha^1}[A]$. (This idea comes from conversation with D. Guaspari). Let us note that the Theorem 3.13 may be expressed as follows: Every stable countable transitive set is d termined by its continuum. Moreover the continuum of it is basis feor Σ_2^1 formulas with the parameters from it.

§ 4. Pointwise definability of L_α for stable α.

DEFINITION 4.1. (a) Let $a \subseteq x$. The set x is *pointwise definable* from a iff $\mathrm{Def}^a x = x$ i.e. every element of x is (implicitely) definable in $\langle x, \epsilon\rangle$ using parameters from a.

(b) Let $a \subseteq x$. The set x is *Σ_n^a-pointwise definable* iff every element of x is implicitely definable by Σ_n formula with parameters from a.

(c) In case $a = \emptyset$ we say pointwise definable and Σ_n pointwise definable.

LEMMA 4.2. *L_{δ_2} is Σ_1-pointwise definable, and consequently it is pointwise definable.*

Proof. Using Σ_1-uniformization of Jensen [6] we find that the set B of all Σ_1 implicitely definable elements forms 1-elementary subsystem of L_{δ_2}. Since $L_{\delta_2} \models \mathrm{V} = \mathrm{HC}$, the set B is transitive (see Marek and Srebrny [11]). Thus B is L_ξ for some $\xi \leqslant \delta_2$. But then ξ is stable. Since δ_2 is least stable we get $\xi = \delta_2$. Since however B is Σ_1-pointwise definable we get the desired result.

We informally use V in our further considerations.

LEMMA 4.3 (Barwise). *L_{δ_2} consists exactly of all Σ_1-implicitely definable elements of V.*

Proof. Clearly Σ_1 implicitely definable elements of V are constructibly hereditarily countable and so, by stability of δ_2 they are in L_{δ_2}. This shows inclusion from the right hand side to the left hand side. If φ is a Σ_1-definition in L_{δ_2} then it is also Σ_1-definition in V. For assume $\varphi(a), \varphi(b)$, $a \neq b$. Then $(\mathrm{E}x)(\mathrm{E}y)\big(x \neq y \ \& \ \varphi(x) \ \& \ \varphi(y)\big)$ is true and Σ_1. Thus it holds in L_{δ_2} contradicting the fact that φ is a definition. Now conclusion follows from 4.2.

Let σ_α be consecutive enumeration of stable ordinals. Using the same reasoning as in 4.3 we get.

THEOREM 4.4. (Barwise). (a) *$L_{\sigma_{\alpha+1}}$ is $\Sigma_1^{L_{\sigma_\alpha} \cup \{L_{\sigma_\alpha}\}}$-pointwise definable.*

(b) *$L_{\sigma_{\alpha+1}}$ consists exactly of sets $\Sigma_1^{L_{\sigma_\alpha} \cup \{L_{\sigma_\alpha}\}}$ definable in V.*

LEMMA 4.5. *Among constructible levels, L_{δ_2} is biggest Σ_1-pointwise definable.*

Proof. Assume L_α is Σ_1-pointwise definable, $\alpha \geqslant \delta_2$. Every Σ_1-definition in L_α is a definition in V and so we use reasoning of 4.3.

LEMMA 4.6 (*). *Let $\Phi(\cdot)$ be Σ_1 formula such that $\Phi(\delta_2)$ is true and $\Phi(x)\to\mathrm{Ord}(x)$. Then $\delta_2 = \bigcup\{\alpha\colon\ \alpha\in\delta_2 \,\&\, \Phi(\alpha)\}$.*

Proof. Let $\beta\in\delta_2$. Then $(\mathrm{E}x)(\beta\in x \,\&\, \Phi(x))$ is true. Hence it is true in L_{δ_2}. Thus there is α such that $\beta\in\alpha\in\delta_2$ and $\Phi(\alpha)$.

COROLLARY 4.7. *δ_2 is a supremum of ordinals α such that L_α is Σ_1-pointwise definable.*

Proof. Check that $(\mathrm{E}x)(x = L_\alpha \,\&\, x$ is Σ_1-pointwise definable) is Σ_1-property.

Let us consider now problem of pointwise definability of L_α for stable α.

THEOREM 4.8. *$L_{\sigma_{\alpha+1}}$ is pointwise definable for every $\alpha\in\omega_1^L$.*

Proof. By Srebrny's result 3.2 $\sigma_{\alpha+1}$ is of the form δ_2^A for some $\sigma_{\alpha+1}$-finite A. Therefore it is not a gap ordinal and so, by results of [10] $L_{\sigma_{\alpha+1}}$ is pointwise definable.

Yet much wider class of stable ordinals given pointwise definable constructible levels.

LEMMA 4.9. *If $L_\alpha \prec L_\beta$ and $\alpha\in\beta$ then L_α is a model of* ZFC$^-$.

Proof. It is clear that it is enough to prove replacement in L_α. But the image of $x\in L_\alpha$ under L_α definable function belongs to $L_{\alpha+1}\subseteq L_\beta$. So it must belong to L_α.

Note close analogy of 4.9 and the following theorem of Montague and Vaught: If $R_\alpha \prec R_\beta$ and $\alpha\in\beta$ then R_α is a model of ZF.

LEMMA 4.10. *If α is stable but less then first stable gap then L_α is pointwise definable.*

Proof. If L_α is not pointwise definable then $\mathrm{Def}\,L_\alpha \prec L_\alpha$ and $\mathrm{Def}\,L_\alpha \neq L_\alpha$. But α is stable and so $L_\alpha \models V = \mathrm{HC}$. By [11] $\mathrm{Def}\,L_\alpha$ is transitive and so there is $\xi\subseteq\alpha$ such that $\mathrm{Def}\,L_\alpha = L_\xi$. Since L_α is not pointwise definable but $\mathrm{Def}\,\overline{L}_\alpha$ is, we have $\xi\in\alpha$. But then both ξ and α must be gaps which contradicts assumption.

We could get the theorem simpler because α is not gap and so by [10] L_α is pointwise definable but we gave it because the same reasoning shows that also first stable gap gives pointwise definable level. We will get more general

LEMMA 4.11. *If α is stable ordinal, $\beta\in\alpha$ then $L_\alpha\models\beta$ is stable iff β is stable.*

Proof. Assume $(\mathrm{E}x)\varphi(x, y)$ where $y\in L_\beta$. By stability of L_α we have an example in L_α. Since β is stable in α therefore we get an example in L_β. Other direction follows from absoluteness of satisfaction.

(*) As noted by M. Srebrny it is enough to assume $\Phi(\alpha)$ for some $\alpha \geqslant \delta_2$.

Fact 4.12. Assumption that α is stable (in 4.11) can not be omitted.

COROLLARY 4.13. *If α is stable, $\beta\in\alpha$ then $L_\alpha\models\beta$ is stable gap $\leftrightarrow\beta$ is stable gap.*

LEMMA 4.14. *If α is stable gap then $\alpha = \sigma_\alpha$.*

Proof. By Lemma 4.11 *$\varphi(\xi) = \sigma_\xi$ is definable over L_α and absolute for L_α. Since α is a beginning of the gap therefore it is regular with respect to φ and so α must be σ_α.*

Let τ_α be an enumeration of stable gap ordinals.

THEOREM 4.15. *If $\alpha\in\tau_\alpha$ then L_{τ_α} is pointwise definable.*

Proof. If $\alpha\in\tau_\alpha$ then using 4.13 we find that α is definable in L_{τ_α}. If L_{τ_α} is not pointwise definable then $\mathrm{Def}\,L_{\tau_\alpha}$ is L_ξ for some $\xi\in\tau_\alpha$ (As in the proof of 4.10). Clearly $\alpha\in\xi$. Thus all τ_β for $\beta\in\alpha$ belong to ξ. But since $L_\xi \prec L_{\tau_\alpha}$ ξ is stable gap. Since $\xi > \tau_\beta$ for all $\beta\in\alpha$, ξ is τ_η for some $\eta \geqslant \alpha$. Contradiction.

Let γ_0 be least μ such that $L_\mu \prec L_{\omega_1^L}$. Clearly L_{γ_0} is pointwise definable and γ_0 is stable gap. We do not know if there are non pointwise definable stable gaps below γ_0 (²). Yet there are non pointwise definable stable ordinals below ω_1^L. For instance γ_1, least ordinal μ such that $\gamma_0\in\mu$ and $L_\mu \prec L_{\omega_1^L}$. Reasoning of 4.2, and 4.3 leads to the following.

THEOREM 4.16. *Let $n \geqslant 2$.* (a) $L_{\delta_n^L} \prec_{n-1} L_{\omega_1^L}$,

(b) *δ_n^L is least ordinal with this property,*

(c) *$L_{\delta_n^L}$ is Σ_{n-1}-pointwise definable,*

(d) *$L_{\delta_n^L}$ consists exactly of Σ_{n-1}-definable elements of $L_{\omega_1^L}$*

(δ_n^L is δ_n in sense of constructible universe).

(We doubt if the analogon of 4.5 holds for $n > 2$ (³).

The reasoning of the Lemma 3.6 allows us to get more information on admissible ordinals. Sacks, Friedman and Jensen [13] proved that all countable admissible ordinals are of the form ω_1^A for some $A\in\wp(\omega)$. One may ask when A can be found in L_α i.e. when $\alpha = \omega_1^A$ for some α-finite A.

Let α be admissible and α^+ be next admissible ordinal.

LEMMA 4.17. *The following conditions are equivalent.*

(a) *α^+ is of the form ω_1^A for some $A\in L_{\alpha^+}$,*

(b) $(L_{\alpha^+} - L_\alpha)\cap\wp(\omega)\neq\emptyset$,

(c) $L_{\alpha^+}\models V = \mathrm{HC}$.

(²) Added in proof. As noted by M. Srebrny there are non-pointwise definable stable gaps below γ_0. As we noted later they may be even found below δ_3^L.

(³) Added in proof. We recently proved that there are arbitrarily big admissibles α less then ω_1^L such that L_α is Σ_2-pointwise definable. The proof follows from the fact that "α is stable" is Π_1 and 4.13. Similarly there are arbitrarily big admissibles α less then ω_1^L such that L_α is Σ_3 but not Σ_2 pointwise definable.

Proof. (a) $\Rightarrow$ (c) Since α^+ is next admissible ordinal there can be no beginning of the gap between α and α^+. Thus either there is $\beta < \alpha^+$ such that in every step between β and α^+ there is a real constructed or in none. Second case is impossible because then $L_\alpha \cap \wp(\omega) = L_{\alpha^+} \cap \wp(\omega)$ and so if $\alpha^+ = \omega_1^A$ for some $A \in L_{\alpha^+} \cap \wp(\omega)$ then in particular A is in L_α but then ω_1^A is $\leqslant \alpha$ (here we use reasoning of 3.6). Thus there is always a real constructed and so α^+ is a limit of nongaps. Thus (using results of [10]) $L_{\alpha^+} \models V = HC$.

(c) $\Rightarrow$ (b) If $L_{\alpha^+} \models V = HC$ then $\wp(\omega) \cap L_\alpha$ is countable in L_{α^+}. Diagonal procedure implies existence of new real in L_{α^+}.

(b) $\Rightarrow$ (a) Since $(L_{\alpha^+} - L_\alpha) \cap \wp(\omega) \neq \emptyset$ there must be an arithmetical copy A of L_α in L_{α^+} (see [2]). For this particular A clearly $\omega_1^A > \alpha$. But $\omega_1^A \leqslant \alpha^+$ so $\omega_1^A = \alpha^+$.

THEOREM 4.18. *Let α be admissible. Then α is of the form ω_1^A for some α-finite A iff*

(a) $L_\alpha \models V = HC$,

(b) *α is not recursively inaccessible.*

Proof. $\Leftarrow$ If (b) holds then $\alpha = \beta^+$ for some $\beta \in \alpha$ (not necessarily admissible). Combining reasoning (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) of 4.17 we get appriopriate real.

$\Rightarrow$ If α is recursively inaccessible then in particular it is a limit of admissibles and so can not be ω_1^A for any $A \in L_\alpha$. Other part follows from reasoning of (a) $\Rightarrow$ (c) of 4.17.

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Reçu par la Rédaction le 15. 10. 1973