An uncountable collection of mutually exclusive planar atriodic tree-like continua with positive span

by

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Abstract. In this paper is constructed an uncountable collection of mutually exclusive continua in the plane each element of which is an atriodic tree-like continuum with positive span. Thus each element of the collection is an atriodic tree-like continuum which is not chainable.

1. Introduction. In this paper we construct in the plane an uncountable collection of mutually exclusive atriodic tree-like continua each having positive span. The construction is similar to that carried out in [1], and throughout this paper many references to that paper will be made.

2. Property L. In this section we obtain the technical lemmas necessary to the inverse limit constructions of this paper. In order to conserve space the proof of Lemma 1 contains frequent references to [1]. $T=\{(\varrho,\theta)|\ 0\leqslant\varrho\leqslant 1,\ \theta=0,\ \theta=\frac{1}{2}\pi \text{ or } \theta=\pi\}$ (in polar coordinates in the plane), while f denotes the mapping of T onto T in [1], and O denotes $(0,0)$, A denotes $(0,\frac{1}{2}\pi)$, B denotes $(0,\pi)$ and C denotes $(0,1)$.

DEFINITION. A continuum Z in $T\times T$ is said to have *property* L provided (a) Z is the union of twelve continua $\langle OB, OC\rangle$, $\langle OC, OB\rangle$, $\langle OA, OB\rangle$, $\langle OB, OA\rangle$, $\langle OA, OC\rangle$, $\langle OC, OA\rangle$, $\left\langle O\frac{A}{2}, OB\right\rangle$, $\left\langle OB, O\frac{A}{2}\right\rangle$, $\left\langle O\frac{A}{2}, OC\right\rangle$, $\left\langle OC, O\frac{A}{2}\right\rangle$, $\left\langle O\frac{A}{2}, \frac{3A}{4}A\right\rangle$, and $\left\langle\frac{3A}{4}A, O\frac{A}{2}\right\rangle$ where $\langle t, u\rangle$ denotes a continuum having first projection the arc t and second projection the arc u and $\langle t, u\rangle^{-1}=\langle u, t\rangle$, (b) there exist four points x_1, x_2, x_3, and x_4 such that x_1 is in $\left[\frac{3A}{4}A\right]$, x_2 is in $\left[\frac{2B}{3}B\right]$, x_3 is in $\left[\frac{2C}{3}C\right]$, and x_4 is in $\left[\frac{A}{4}\frac{A}{2}\right]$ and (x_1, O) is in $\langle OA, OB\rangle\cap\langle OA, OC\rangle\cap\left\langle\frac{3A}{4}, O\frac{A}{2}\right\rangle$, (x_2, O) is in $\langle OB, OA\rangle\cap\langle OB, OC\rangle\cap\left\langle OB, O\frac{A}{2}\right\rangle$, (x_3, O) is in $\langle OC, OA\rangle\cap$

$\cap \langle OC, OB\rangle \cap \left\langle OC, O\frac{A}{2}\right\rangle$, and (x_4, O) is in $\left\langle O\frac{A}{2}, OB\right\rangle \cap \left\langle O\frac{A}{2}, OC\right\rangle$, and (c) there exists six points z_1, z_2, z_3, z_4, z_5 and z_6 such that z_1 is in $\left[\frac{C}{3}\,\frac{C}{2}\right]$, z_2 is in $[OA]$, z_3 is in $\left[\frac{B}{3}\,\frac{B}{2}\right]$, z_4 is in $[OA]$, z_5 is in $\left[O\,\frac{B}{3}\right]$, and z_6 is in $\left[O\,\frac{C}{3}\right]$ and (B, z_1) is in $\langle OB, OC\rangle$, (B, z_2) is in $\langle OB, OA\rangle$, (C, z_3) is in $\langle OC, OB\rangle$, (C, z_4) is in $\langle OC, OA\rangle$, (C, z_5) is in $\langle OC, OB\rangle$, and (B, z_6) is in $\langle OB, OC\rangle$.

LEMMA 1. *If Z is a subcontinuum of $T\times T$ with property L, then there is a subcontinuum Z' of $T\times T$ with property L such that $f\times f(Z') = Z$.*

Proof. The continua α_i, $1\leqslant i\leqslant 12$, constructed in the proof of Theorem 2 of [1] serve as the twelve continua whose union is the continuum Z' and thus $f\times f(Z') = Z$. To see that Z' has property L we need to add a few observations to the proof of Theorem 2 [1]. The four points x_1', x_2', x_3', and x_4' are chosen here exactly as they were there, but here we require that x_3' be a point of $\left[\frac{2C}{3}\,C\right]$. Since $x_3' = \left(f\middle|\frac{C}{2}\,C\right)^{-1}(z_1)$ and z_1 is a point of $\left[\frac{C}{3}\,\frac{C}{2}\right]$, x_3' is actually a point of $\left[\frac{2C}{3}\,\frac{3C}{4}\right]$.

The six points $z_1', z_2', z_3', z_4', z_5'$, and z_6' are obtained as follows. (z_1') Let $z_1' = \left(f\,|\,O\,\frac{C}{2}\right)^{-1}(z_5)$. Since z_5 is in $\left[O\,\frac{B}{3}\right]$, z_1' is in $\left[\frac{C}{3}\,\frac{C}{2}\right]$. Since (C, z_5) is in $\langle OC, OB\rangle$, from the construction of $\alpha_1 = \langle OB, OC\rangle'$ we see that (B, z_1') is in the subset L_5^1 of α_1. (z_2') Since some point in $\langle OB, OA\rangle$ has first coordinate B there is a point z_2' in $[OA]$ such that (B, z_2') is in $\langle OB, OA\rangle'$. (z_3') Since there is a point in $\left\langle O\frac{A}{2}, OC\right\rangle$ with second coordinate C, suppose (Q_1, C) is such a point. Let $z_3' = \left(f\middle|\frac{B}{3}\,\frac{B}{2}\right)^{-1}(Q_1)$, a point of $\left[\frac{B}{3}\,\frac{B}{2}\right]$. Then (z_3', C) is a point of the subset L_2^1 of α_1 and (C, z_3') is in $\alpha_2 = \langle OC, OB\rangle'$. (z_4') Since some point in $\langle OC, OA\rangle'$ has first coordinate C there is a point z_4' in $[OA]$ such that (C, z_4') is in $\langle OC, OA\rangle'$. (z_5') Since $\langle OB, OC\rangle$ contains a point with second coordinate C, suppose (Q_2, C) is such a point. Let $z_5' = \left(f\,|\,O\,\frac{B}{3}\right)^{-1}(Q_2)$. Note that z_5' is in $\left[O\,\frac{B}{3}\right]$ while (z_5', C) is a point of the subset L_1^1 of α_1 so (C, z_5') is a point of $\alpha_2 = \langle OC, OB\rangle'$. (z_6') Let $z_6' = \left(f\,|\,O\,\frac{C}{2}\right)^{-1}(z_3)$. Since z_3 is in $\left[\frac{B}{3}\,\frac{B}{2}\right]$, z_6' is in $\left[\frac{C}{4}\,\frac{C}{3}\right]$. The point (B, z_6') is in the subset L_5^1 of $\alpha_1 = \langle OB, OC\rangle'$. This concludes the proof.

DEFINITION. If Z is a continuum with property L, a continuum Z' with property L such that $f\times f(Z') = Z$ will be called a *lift of Z with respect to $f\times f$.*

DEFINITION. Denote by r the mapping of T onto T defined by

$$r(x,\theta) = \begin{cases} (x, \frac{1}{2}\pi) & \text{if } \theta = \frac{1}{2}\pi\,, \\ (x, 0) & \text{if } \theta = \pi\,, \\ (x, \pi) & \text{if } \theta = 0\,. \end{cases}$$

Denote by g the mapping $r\circ f$.

The following lemma is easy to establish.

LEMMA 2. *If Z is a subcontinuum of $T\times T$ with property L, then $r\times r(Z)$ is a subcontinuum of $T\times T$ with property L.*

3. Span and inverse limits. In this section we show that any continuum which is the inverse limit of an inverse limit sequence $\{T_n, f_n\}$ where for each n $T_n = T$ and f_n is in $\{r, f\}$ has positive span.

LEMMA 3. *Suppose, for each n f_n is in $\{r, f\}$ and $f_1^n = f_1 f_2 \cdots f_{n-1}$ if $n > 1$. Then, if $n > 1$, $\sigma f_1^n \geqslant \frac{1}{2}$.*

Proof. Let

$$\begin{aligned} Z_1 = {} & \big(([OB]\times\{C\}) \cup (\{B\}\times[OC])\big) \cup \big(([OC]\times\{B\}) \cup (\{C\}\times[OB])\big) \cup \\ & \cup \big(([OA]\times\{C\}) \cup (\{A\}\times[OC])\big) \cup \big(([OC]\times\{A\}) \cup (\{C\}\times[OA])\big) \cup \\ & \cup \big(([OA]\times\{B\} \cup (\{A\}\times[OB])\big) \cup \big(([OB]\times\{A\}) \cup (\{B\}\times[OA])\big) \cup \\ & \cup \left(\left(\left[O\frac{A}{2}\right]\times\{C\}\right) \cup \left(\left\{\frac{A}{2}\right\}\times[OC]\right)\right) \cup \left(\left([OC]\times\left\{\frac{A}{2}\right\}\right) \cup \left(\{C\}\times\left[O\frac{A}{2}\right]\right)\right) \cup \\ & \cup \left(\left(\left[O\frac{A}{2}\right]\times\{B\}\right) \cup \left(\left\{\frac{A}{2}\right\}\times[OB]\right)\right) \cup \left(\left([OB]\times\left\{\frac{A}{2}\right\}\right) \cup \left(\{B\}\times\left[O\frac{A}{2}\right]\right)\right) \cup \\ & \cup \left(\left(\left[O\frac{A}{2}\right]\times\{A\}\right) \cup \left(\{O\}\times\left[\frac{3A}{4}A\right]\right)\right) \cup \left(\left(\left[\frac{3A}{4}A\right]\times\{O\}\right) \cup \left(\{A\}\times\left[O\frac{A}{2}\right]\right)\right). \end{aligned}$$

If (p, q) is in Z_1, $d(p, q)\geqslant\frac{1}{2}$. Further note that Z_1 has property L.

Suppose Z_n is a subcontinuum of $T\times T$ having property L such that if $n > 1$, $f_1^n\times f_1^n(Z_n) = Z_1$. If $f_n = r$, let $Z_{n+1} = r\times r(Z_n)$ and note that $f_n\times f_n(Z_{n+1}) = Z_n$. If $f_n = f$, let Z_{n+1} be the lift (use Lemma 1) of Z_n with respect to $f\times f$. In either case Z_{n+1} has property L. Since $f_1^{n+1}\times f_1^{n+1}(Z_{n+1}) = Z_1$, $\sigma f_1^{n+1}\geqslant\frac{1}{2}$.

THEOREM 1. *Suppose, for each n, $T_n = T$, f_n is in $\{r, f\}$ and M is the inverse limit of the inverse limit sequence $\{T_n, f_n\}$. Then $\sigma M > 0$.*

Proof. Apply Lemma 3 and then Theorem 4 of [1].

THEOREM 2. *Suppose, for each n, $T_n = T$, f_n is in $\{f, g\}$ and M is the inverse limit of the inverse limit sequence $\{T_n, f_n\}$. Then M is atriodic and $\sigma M > 0$.*

Proof. The argument that M is atriodic is similar to that given for Theorem 1 of [1]. That $\sigma M > 0$ is a direct consequence of Theorem 1 above.

4. Plane embedding. In this section we construct an uncountable collection of mutually exclusive tree-like continua in the plane. We then show each of them has positive span by showing each is homeomorphic to the inverse limit of an inverse limit sequence on simple triods with bonding maps from $\{f, g\}$.

DEFINITIONS. A tree-chain S which is the union of three chains $A(A_1, A_2, \ldots, A_i)$, $B(B_1, B_2, \ldots, B_j)$ and $C(C_1, C_2, \ldots, C_k)$ of connected links such that $A_1 = B_1 = C_1$ will be called a *simple tree-chain*. Note that we are using conventions of chain notation as in [2]. The link $A_1 = B_1 = C_1$ will be called the *junction link of* S. For notational convenience when $i = j = k$ we will denote S by $S(A, B, C, i)$. S will be called *taut* if each two non-intersecting links of S have mutually exclusive closures.

Suppose n is an even integer not less than 4. The statement that the simple tree-chain $E(E^1, E^2, E^3, 12n)$ follows the f-pattern with respect to the simple tree-chain $D(D^1, D^2, D^3, n)$ means E has the following three properties with respect to D: (1) E^1 is the union of four chains $(E^1, E^1_2, \ldots, E^1_{3n})$ running straight through D^2 from D^2_n to D^2_1, $(E^1_{3n+1}, \ldots, E^1_{6n})$ running straight through D^1 from D^1_1 to D^1_n, $(E^1_{6n+1}, \ldots, E^1_{9n})$ running straight through D^1 from D^1_n to D^1_1, and $(E^1_{9n+1}, \ldots, E^1_{12n})$ running straight through D^3 from D^3_1 to D^3_n where each of these four subchains of E^1 strongly refines the corresponding subchain of D with three links per link, (2) E^2 is the union of four chains $(E^2_1, \ldots, E^2_{4n})$ running straight through D^2 from D^2_n to D^2_1, $(E^2_{4n+1}, \ldots, E^2_{6n})$ running straight through $(D^1_1, \ldots, D^1_{n/2})$ from D^1_1 to $D^1_{n/2}$, $(E^2_{6n+1}, \ldots, E^2_{8n})$ running straight through $(D^1_1, \ldots, D^1_{n/2})$ from $D^1_{n/2}$ to D^1_1, and $(E^1_{8n+1}, \ldots, E^1_{12n})$ running straight through D^3 from D^3_1 to D^3_n where each of these four subchains of E^2 strongly refines the corresponding subchain of D with four links per link, and (3) E^3 is the union of two chains $(E^3_1, \ldots, E^3_{6n})$ running straight through D^2 from D^2_n to D^2_1 and $(E^3_{6n+1}, \ldots, E^3_{12n})$ running straight through D^3 from D^3_1 to D^3_n where each of these two subchains of E^3 strongly refines the corresponding subchain of D with six links per link.

By exchanging the roles of D^2 and D^3 in the definition of the f-pattern one obtains the meaning of the statement that E follows the g-pattern with respect to D.

THEOREM 3. *If $S_1, S_2, S_3, \ldots$ is a sequence of simple tree-chains such that* (1) *for each i S_i is taut and mesh less than* $1/i$ *and* (2) *if i is a positive integer, S_{i+1} follows the f-pattern with respect to S_i or S_{i+1} follows the g-pattern with respect to S_i, then $\bigcap_{i\geqslant 1} S^*_i$ is an atriodic tree-like continuum with positive span.*

Proof. Let $\{T_i, f^{i+1}_i\}$ be an inverse limit sequence such that for each i $T_i = T$ and $f^{i+1}_i = f$ if S_{i+1} follows the f-pattern with respect to S_i while $f^{i+1}_i = g$ if S_{i+1} follows the g-pattern with respect to S_i. Denote by H the inverse limit of $\{T_i, f^{i+1}_i\}$ and by K the tree-like continuum $\bigcap_{i\geqslant 1} S^*_i$. By Theorem 2 H is atriodic and has positive span. We complete the proof by showing H and K are homeomorphic.

Suppose $S_n = S_n(a_n, b_n, c_n, 12^{n-1}k)$. Since $T_n = [OA] \cup [OB] \cup [OC]$ we subdivide T_n by subdividing each of OA, OB, and OC into $12^{n-1}k$ subintervals of equal length. Denote by γ_n a transformation defined on the set of links of S_n as follows: If L is the junction link of S_n, $\gamma_n(L)$ $= \pi_n^{-1}\left(\left[O\,\frac{A}{12^{n-1}k}\right] \cup \left[O\,\frac{B}{12^{n-1}k}\right] \cup \left[O\,\frac{C}{12^{n-1}k}\right]\right)$. If L is not the junction link of S_n then for some $j > 1$ L is the jth link of a_n and $\gamma_n(L)$ is $\pi_n^{-1}\left(\left[\frac{(j-1)A}{12^{n-1}k}, \frac{jA}{12^{n-1}k}\right]\right)$ or L is the jth link of b_n and $\gamma_n(L)$ is $\pi_n^{-1}\left(\left[\frac{(j-1)B}{12^{n-1}k}, \frac{jB}{12^{n-1}k}\right]\right)$ or L is the jth link of c_n and $\gamma_n(L)$ is $\pi_n^{-1}\left(\frac{(j-1)C}{12^{n-1}k}, \frac{jC}{12^{n-1}k}\right)$. If x is a point of K, then x belongs to no more than two links of S_n. Denote by L_x a link of S_n containing x. If two links of S_n contain x denote the other by L'_x, while, for convenience, if not, denote by L'_x the link L_x. Then, let $R_n(x) = \gamma_n(L_x) \cup \gamma_n(L'_x)$. Denote by h the function throwing K onto H defined by $h(x) = \bigcap_{i\geqslant 1} R_i(x)$. It is not difficult to argue that h is a reversible, continuous transformation throwing K onto H.

THEOREM 4. *There exist uncountably many mutually exclusive atriodic tree-like continua in the plane each having positive span.*

Proof. If $S = S(A, B, C, i)$ is a taut simple tree-chain in the plane, there exist two taut simple tree-chains S_f and S_g each with mesh less than half the mesh of S, each strongly refining S so that S_f follows the f-pattern with respect to S, S_g follows the g-pattern with respect to S, and $\overline{(S^*_f)}$ does not intersect $\overline{(S^*_g)}$.

Thus, there exists an uncountable collection U of sequences such that if $S_1, S_2, S_3, \ldots$ is one of them then for each i (1) S_i has mesh less than $1/i$, (2) S^*_i is a subset of the plane, (3) S_{i+1} follows the f-pattern with respect to S_i or S_{i+1} follows the g-pattern with respect to S_i. Further, if $S_1, S_2, S_3, \ldots$ and $S'_1, S'_2, S'_3, \ldots$ are two sequences in the collection U, there is a positive integer n such that $\overline{(S^*_n)}$ does not intersect $\overline{(S'_n)}^*$.

By Theorem 3, each sequence in this uncountable collection U determines a planar atriodic tree-like continuum with positive span. Further, if H and K are two continua each determined by a sequence in U, then H does not intersect K.

5. Remark. In 1939 Waraszkiewicz [3] published a paper in which he claimed that the plane contains no atriodic tree-like continuum which is not chainable. However, each continuum in the collection of plane continua described in this paper has positive span and, thus, is not chainable.

References

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On smooth continua

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Abstract. A metric continuum X is said to be *smooth at a point* p if for each subcontinuum K of X which contains p and for each open set V which contains K there exists an open connected set U such that $K \subset U \subset V$. If the continuum X is hereditarily unicoherent at p, then the definition of smoothness at p mentioned above is equivalent to the definition of smoothness at p introduced by G. R. Gordh in [7]. Moreover, it is proved that if the continuum X is hereditarily unicoherent at some point (or if X is an irreducible continuum), then X is hereditarily unicoherent at each point at which it is smooth; thus X is smooth in the sense of Gordh at each such point. Therefore, we conclude that the notion of smoothness at a point (introduced in this paper) is independent of hereditary unicoherence at a point, and smoothness at a point may be defined for all continua.

The set of all points of X at which X is smooth is called the *initial set* and is denoted by $I(X)$. It is proved that if a mapping f on a continuum X is monotone (or open, or quasi-interior), then $f(I(X)) \subset I(f(X))$. This is a generalization of Theorem 4.1 in [7] (Theorem 4 and Corollary 6 in [13]). Furthermore, we give a new relation between the initial set of the preimage and the initial set of the image for confluent mappings. Namely, if a mapping f on a continuum X is conflunet, then $f(X) \setminus I(f(X)) \subset f(X \setminus I(X))$.

§ 1. Introduction. Investigating smooth continua, defined by G. R. Gordh in [7], we have observed that the notion of smoothness of a continuum at some point at which it is hereditarily unicoherent, can be easily extended to the notion of smoothness of a continuum at some point at which it need not to be hereditarily unicoherent, i.e., that the notion of the smoothness of a continuum is independent of the notion of its hereditary unicoherence. Moreover, the idea of the smoothness of a continuum at a point is, as will be seen, a generalization of the idea of the local connectedness of the continuum at that point in some sense.

In this paper we study some properties of smooth continua, in particular we give some characterizations of them, and, incidentally, we investigate the invariability of smoothness under some classes of continuous mappings, and the co-existence of smoothness at some point and hereditary unicoherence at another point in an arbitrary continuum.

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