Nonsymmetric weakly complete G-spaces

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Abstract. The theory of G-spaces which include complete Finsler, and hence Riemannian, spaces was developed by H. Busemann. Busemann and, later, Zautinsky investigated nonsymmetric distance in finitely compact G-spaces. The present paper develops the basic theory of spaces with only a weaker compactness condition (which implies weaker geodesic completeness). A class of examples of these spaces is also constructed. The last section extends the topological property of noncontractibility of small spheres to weakly complete G-spaces of finite topological dimension. Many of the methods apply to the study of locally compact spaces also.

1. Introduction. A geometric axiomatic theory of complete Finsler spaces with a symmetric distance was developed by H. Busemann in *The Geometry of Geodesics* and was brought up to date by him in *Recent Synthetic Differential Geometry*, see [3] and [5]. When the distance xy is not symmetric, completeness can be given a strong or weak form: both the balls $px \leqslant \varrho$ and $xp \leqslant \varrho$ are compact or only the balls $px \leqslant \varrho$, say, are compact. The theory for the case of strong completeness was studied by H. Busemann [2] and developed principally by Zautinsky [10]. Here most arguments carry over without serious difficulties from the symmetric spaces. Section 6 of the present paper where we prove the noncontractibility of small spheres $px = \varrho$ in finite dimensional spaces provides an example: here considerable complications stem from the lack of symmetry rather than completeness.

Entirely new phenomena appear in weakly complete spaces when questions in the large are considered. Precise formulation would make this introduction too long so we mention merely the following: A complete geodesic has the form $x(t)$, $a < t < \infty$, if t is arclength, where a may be finite or $-\infty$. If $x_\nu(t)$ $(\nu = 0, 1, 2, \ldots;\ a_\nu < t < \infty)$ are geodesics and $x_\nu(t) \to x_0(t)$ for $t \geqslant 0$, it does not follow that $a_\nu \to a_0$, only $\limsup a_\nu \leqslant a_0$. We discuss in detail, therefore, prolongation of segments and the convergence of extremals which are the topics of sections 3 and 4. In section 5 we construct a class of examples of weakly complete spaces and illustrate

(1) Work supported by the National Science Foundation of the USA and the National Research Council of Canada.

the above phenomenon by means of these examples. In the last section we discuss the topology of finite dimensional weakly complete G-spaces.

2. The axioms and examples. A set R with distance xy defined on $R \times R$ is a general metric space (or a metric space with a not necessarily symmetric distance) if $xy \geqslant 0$, $xy = 0$ if and only if $x = y$, $xy + yz \geqslant xz$ and $xx_\nu \to 0$ if and only if $x_\nu x \to 0$ for any sequence $\{x_\nu\}$ in R. The last condition allows us to use the topology induced by the symmetric metric $\delta(x, y) = \max(xy, yx)$. When $xy + yz = xz$, we say that y *lies between* x *and* z and denote this by writing (xyz). The metric xy is said to be *intrinsic* if the distance from p to q is equal to the infimum of the lengths of the curves joining p to q, see [5, page 3]; for spheres and balls we use the following notation:

$$S^+(p, \varrho) = \{x|\ px < \varrho\},\ S^-(p, \varrho) = \{x|\ xp < \varrho\},$$
$$K^+(p, \varrho) = \{x|\ px = \varrho\},\ K^-(p, \varrho) = \{x|\ xp = \varrho\},$$
$$S^n(p, \varrho) = S^+(p, \varrho) \cap S^-(p, \varrho) \quad \text{and}$$
$$S^u(p, \varrho) = S^+(p, \varrho) \cup S^-(p, \varrho).$$

For a set A, $\bar{A}$ denotes its topological closure. In an intrinsic metric space we always have $\bar{S}^+ = S^+ \cup K^+$ and $\bar{S}^- = S^- \cup K^-$, see [5, page 3].

A general metric space R with an intrinsic metric is said to be a *nonsymmetric weakly complete G-space* if all the closed balls $px \leqslant \varrho$ are compact and if the following two conditions hold:

(i) Local prolongability: Every point p has a neighbourhood $U(p)$ such that for $x, y \in U(p)$ there exist points u, v such that (uxy) and (xyv).

(ii) Uniqueness of prolongation: (xyu) and (xyv) with $yu = yv$ implies $u = v$ and (wxy) and (zxy) with $wx = zx$ and $x \neq y$ implies $w = z$.

Featherstone [7] calls these spaces *spaces with nonsymmetric distance* and compactness because only the positive balls are assumed to be compact. This is also called *semi-finite compactness*. Since this weaker condition of compactness implies a weaker geodesic completeness we call these spaces *weakly complete spaces*.

Because of the compactness of all closed positive balls, for any two points p, q a curve $T(p, q)$ joining p to q and of length pq exists, see [5, page 3]. We say that $T(p, q)$ is a *segment joining p to q* and represent it always as $x(t)$ with t as the arc length so that $x(t)x(s) = s - t$, $s > t$. When $U(p) = R$ for all $p \in R$ the space is said to be a *straight space*.

We note that we could also define locally compact G-spaces to be locally compact general metric spaces with an intrinsic metric satisfying conditions (i) and (ii) above. In fact the results of sections 3, 4 and 6 below remain valid in such space with little change. However we stick to the case with all $px \leqslant \varrho$ compact because interesting examples have so far been found only in this case.

Since our definition of nonsymmetric weakly complete G-spaces does not exclude the case of symmetric distance or finite compactness (i.e. both $px \leqslant \varrho$ and $xp \leqslant \varrho$ are compact), our spaces include all the G-spaces and hence complete Finsler spaces with symmetric distance, Riemannian spaces, see [1] and [3], Minkowskian spaces with symmetric and nonsymmetric distance [2] and [10], projective metric spaces [6] and the quasihyperbolic spaces [4] and [10].

The Funk space (*Geometrie der spezifischen Massbestimmung* discovered by Funk [8]) is an example where only the balls $px \leqslant \varrho$ are compact. It is a metrization of the interior of a closed strictly convex curve C in the euclidean plane with the metric $xy = \log[e(x, u)/e(y, u)]$ where $e(p, q)$ denotes the euclidean distance between p and q and u is the point in which the oriented ray joining x to y meets the curve C. For an account of some of the interesting properties of the Funk space, see Zautinsky [10, Appendix I]. In section 5 we construct a class of other examples of weakly complete spaces.

3. The radius of prolongation $\gamma(p)$. Because of the axiom of local prolongability, we can assign to every point p a number $\varrho(p) > 0$ defined as follows:

$$\varrho(p) = \operatorname{Sup}\{\varepsilon|\ x, y \in S^n(p, \varepsilon) \text{ implies that there exist } u, v \text{ with } (uxy) \text{ and } (xyv)\}.$$

From a general principle given in [3, page 33], it follows that $\varrho(p)$ is Lipschitzian in the sense that it satisfies $|\varrho(p) - \varrho(q)| \leqslant \max(pq, qp)$ for all p and q, see also [10, page 11]. In finitely compact spaces, $\varrho(p)$ measures the amount of prolongation from the left as well as to the right. In a weakly complete space $\varrho(p)$ can no longer measure the amount of prolongation of a segment from the left, we need a function $\gamma(p)$ different from $\varrho(p)$ and also different from a function defined by Zautinsky [10, p. 13]. However, unlike $\varrho(p)$, our function $\gamma(p)$ will not turn out to be Lipschitzian in general, we can only prove that it is continuous.

To define $\gamma(p)$ we put

$$\gamma^-(p) = \operatorname{Sup}\{\varepsilon|\ S^-(p, \varepsilon) \subset S^n(p, \varrho(p))\},$$
$$\gamma^+(p) = \operatorname{Sup}\{\varepsilon|\ S^+(p, \varepsilon) \subset S^n(p, \varrho(p))\},$$

and

$$\gamma(p) = \min\{\gamma^-(p), \gamma^+(p)\}.$$

We prove that $\gamma^-(p)$ is continuous. Since the proof that $\gamma^+(p)$ is continuous is similar, this proof will imply the continuity of $\gamma(p)$.

3.1. Consider first the case when $\gamma^-(p)=\infty$ for some point p. Then for any x, $px, xp<\varrho(p)$. This means that for any two x and y there exist points u and v with (xyu) and (vxy). Thus $\varrho(q)=\infty$ for all q. Then $\gamma^-(q)=\infty$ also. Thus we have proved that $\gamma^-(p)=\infty$ for one point implies $\gamma^-(q)=\infty$ for all points q.

3.2. In view of the above, to prove $\gamma^-(p)$ is continuous we may assume that $\gamma^-(p)<\infty$ for all points p. First we show that for all $\varepsilon>0$, there exists $\delta(p)>0$ such that pq, $qp<\delta(p)$ implies $\gamma^-(q)<\gamma^-(p)+\varepsilon$.

Since $\gamma^-(p)<\infty$ we can find a point x such that $\gamma^-(p)+\varepsilon>\gamma^-(p)+\varepsilon/2>xp>\gamma^-(p)$ and such that $\max(xp,px)>\varrho(p)$. Therefore we can find an $\eta>0$ with $\max(xp,px)>\eta+\varrho(p)$. Take $0<\delta<\frac14\eta, \frac14\varepsilon$. Since $|\varrho(p)-\varrho(q)|\leqslant\max(pq,qp)$ we have pq, $qp<\delta$ implies $\max(xq,qx)\geqslant\max(xp-qp,px-pq)>\varrho(p)+\frac34\eta>\varrho(q)-\frac14\eta+\frac34\eta=\varrho(q)+\frac12\eta$. Also we have $xq\leqslant xp+pq<\gamma^-(p)+\frac12\varepsilon+\frac14\varepsilon<\gamma^-(p)+\varepsilon$. Thus pq, $qp<\delta$ implies $xq<\gamma^-(p)+\varepsilon$. But $\max(xq,qx)>\varrho(q)$, so that pq, $qp<\delta$ implies $\gamma^-(q)<\gamma^-(p)+\varepsilon$. This is half of the proof of the continuity of $\gamma^-(p)$.

3.3. We must now show that for all $\varepsilon>0$ there exists $\delta=\delta(p)>0$ such that pq, $qp<\delta$, implies $\gamma^-(q)>\gamma^-(p)-\varepsilon$. We use the following notations:

$$\begin{aligned}
\alpha &= \gamma^-(p)\,,\\
\beta'_k &= (1-2/k)\,\varrho(p)\,,\\
\beta_k &= (1-1/k)\,\varrho(p)\,,\\
\alpha_k &= \operatorname{Sup}\{\varepsilon|\ S^-(p,\varepsilon)\subset S^n(p,\beta'_k)\}\,.
\end{aligned}$$

Then clearly $\alpha_k\to\alpha$ as $k\to\infty$.

Now given $\varepsilon>0$, choose k so large that $\alpha-\varepsilon<\alpha_k-1/k$ and take $0<\delta<\varrho(p)/k$. Then pq, $qp<\delta$ implies $\varrho(q)>\beta_k$ because $|\varrho(p)-\varrho(q)|\leqslant\max(pq,qp)$. Also $xq<\alpha_k-1/k$ implies $xp\leqslant xq+qp<\alpha_k-1/k+1/k=\alpha_k$. Therefore xp, $px<\beta'_k$ so that

$$qx\leqslant qp+px<\varrho(p)/k+\beta'_k=\beta_k<\varrho(q)$$

and

$$xq\leqslant xp+pq<\beta'_k+\varrho(p)/k=\beta_k<\varrho(q)\,.$$

Thus $\gamma^-(q)\geqslant\alpha_k-1/k$. Hence pq, $qp<\delta$ implies $\gamma^-(q)\geqslant\alpha_k-1/k>\alpha-\varepsilon$, i.e. $\gamma^-(q)>\gamma^-(p)-\varepsilon$. This proves the other part of the continuity of $\gamma^-(p)$.

That $\gamma(p)$ is continuous now follows as observed above because the continuity of $\gamma^+(p)$ can be proved similarly.

With the help of the function $\gamma(p)$ we can now state the following results, we omit the proofs which are obtained by methods similar to those of [3, sections 6, 7] and [5, sections 1, 2].

3.4. *If $0<gp<\gamma(p)$ and $a<\gamma(p)$ then there exists an r with $pr=a$ and (gpr). If $0<pq<\gamma(p)$ and $a<\gamma(p)$ then there exists an s with $sp=a$ and (spq).*

3.5. *If $x,y\in S^u(p,\gamma(p))$, then $T(x,y)$ and $T(y,x)$ are unique.*

3.6. *If (pxy), (xyq), $x\neq y$, $0<py<\gamma(y)$, $0<yq<\gamma(y)$ then (pyq) and (pxq).*

We note also the following facts which will be used in section 6.

3.7. *Given any compact set C and $\varepsilon>0$ there exists $\delta>0$ such that $x,y\in C$ and $xy<\delta$ implies $yx<\varepsilon$.*

3.8. *If C is a compact set then $\inf\{\gamma(x)|\ x\in C\}>0$.*

4. Extremals and their convergence.

4.1. Definitions. A partial geodesic is a map $x(t)$ of a connected set M (containing more than one point) on the real line into the space R such that for every point t in the interior of M there is a $\delta(t)>0$ such that $x(s)x(t)=s-t$ for s in M and $t+\delta(t)>s>t$. A partial geodesic $x(t)$: $(\alpha,\beta)\to R$ is said to be a *maximal partial geodesic* or an *extremal* if no partial geodesic $y(t)$: $(\gamma,\delta)\to R$ exists such that (α,β) is properly contained in (γ,δ) and $y(t)$ agrees with $x(t)$ on (α,β). An extremal $x(t)$ is a *straight line* if $x(t)x(s)=s-t$ for $s>t$. Hence if $\varrho(p)=\infty$, all extremals are straight lines and we say that the space is *straight*.

4.2. *Every partial geodesic can be extended to an extremal and this extension is unique.*

Proof. The existence of the extension follows from the Zorn's lemma and the uniqueness can be proved as in 8.4 of Chapter 1 of [3].

4.3. *Let $x(t)$ and $x_\nu(t)$ represent extremals which are all defined in a neighbourhood of a point t_0. If $t_\mu\to t_0$ and $x_\nu(t_\mu)\to x(t_\mu)$ for every ν then $x_\nu(t_0)\to x(t_0)$ and we have $x_\nu(t_\nu)\to x(t_0)$.*

The proof of this is obtained along the same lines as in 8.11, Chapter 1 of [3].

4.4. *Let $x(t)$ and $x_\nu(t)$ be extremals defined in (α_0,β_0) and (α_ν,β_ν) respectively. Suppose that $(\alpha,\beta)\subset(\alpha_0,\beta_0)$ and $(\alpha,\beta)\subset(\alpha_\nu,\beta_\nu)$ for each ν. Further suppose that $x_\nu\to x$ on some $(\alpha',\beta')\subset(\alpha,\beta)$. Then $x_\nu\to x$ on (α,β), uniformly on compact subsets of (α,β).*

Proof. Suppose, without loss of generality, that $o\in(\alpha',\beta')$, and consider the set $V=\{t|\ x_n\to x \text{ on } (0,t)\}$. We show that $\operatorname{Sup}V=\tau\geqslant\beta$. Put $x_n(\tau)=p_n$, $x(\tau)=p$, $\gamma(p)=\gamma$, if $\tau<\beta$.

Since $\gamma(x(t))\to\gamma(x(\tau))$ as $t\to\tau$ we can choose $\tau_1<\tau$ such that $\gamma(x(\tau_1))>\frac34\gamma$ and $\tau-\tau_1<\frac14\gamma$. Put $x(\tau_1)=q$ and $x_n(\tau_1)=q_n$. Since $x_n\to x$ on $(0,\tau)$ and $\tau_1<\tau$, $q_n\to q$. Therefore $\gamma(q_n)\to\gamma(q)$, so that there exists an N such that $n>N$ implies $\gamma(q_n)>\frac34\gamma(q)>\frac{9}{16}\gamma>\frac12\gamma$.

Using a basic result (Sec. 3.4), we can find points q', q'_n such that (qpq'), $(q_np_nq'_n)$ with $pq' = \frac{1}{2}\gamma$, $p_nq'_n = \frac{1}{2}\gamma$. Let τ_2 be the parametric value corresponding to q', q'_n i.e., $\tau_2 = \tau + \frac{1}{2}\gamma > \tau$. We prove $q'_n \to q'$. This is so because $qq'_n \leqslant qq_n + q_nq'_n = qq_m + \frac{1}{2}\gamma + (\tau - \tau_1)$, and since $qq_n \to 0$, this shows that the values qq'_n are bounded. By compactness of $\bar{S}^+(p, \varrho)$ the sequence $\{q'_n\}$ has at least one accumulation point while every accumulation point must coincide with q' because of the uniqueness of prolongation on $x(t)$. But $q'_n \to q'$ implies that $x_n \to x$ on $(0, \tau_2)$ and since $\tau_2 > \tau$ this is a contradiction to assuming that $\operatorname{Sup} V = \tau < \beta$. This proves that $x_n \to x$ on (α, β) because we can apply a similar argument to the interval $(\alpha, 0)$.

To show uniform convergence on compact subsets of (α, β) we use 4.3. For if convergence were not uniform on a compact subset W of (α, β) we could find points τ_ν in W such that $x_\nu(\tau_\nu)x(\tau_\nu) \geqslant \eta > 0$. If $\tau_\nu \to \tau$ we get a contradiction to 4.3 because $x(\tau)x(\tau_\nu) \leqslant |\tau - \tau_\nu|$. This completes the proof of 4.4.

4.5. *Under the same assumptions as in 4.4 if additionally all $\beta_\nu \geqslant 0$ then $\beta \leqslant \liminf \beta_\nu$. A similar statement holds about the left hand end points.*

Proof. If possible let a subsequence $\{\beta_n\}$ exist such that $\beta_n \to \tau < \beta$. Let $x(\tau) = p$, and choose $\tau_2 < \tau_1 < \tau$ with $r = x(\tau_2)$, $q = x(\tau_1)$ such that $\gamma(q) > \frac{3}{4}\gamma(p) = \frac{3}{4}\gamma$ and such that $\tau - \tau_1 < \frac{1}{2}\gamma$. Then since $x_n \to x$ for $t < \tau$ by 4.4, we have $q_n \to q$ where $q_n = x_n(\tau_1)$, so that there exists an N such that $n > N$ implies $\gamma(q_n) > \frac{3}{4}\gamma(q) > \frac{9}{16}\gamma(p) > \frac{1}{2}\gamma$. Then for each $n > N$ there exists p_n on x_n such that $(r_nq_np_n)$ with $r_n = x_n(\tau_2)$ and $q_np_n = \frac{1}{2}\gamma$. Thus x_n extends at least up to $\tau_1 + \frac{1}{2}\gamma > \tau + \frac{1}{4}\gamma$ implying $\beta_n > \tau + \frac{1}{4}\gamma$ for each $n > N$. This however is a contradiction to $\beta_n \to \tau < \tau + \frac{1}{4}\gamma$.

4.6. *If $x_\nu(t)$ represents an extremal, $\nu = 1, 2, \ldots$, and if for a subsequence $\{x_\mu(t_0)\}$, $x_\mu(t_0) \to q$ as $\mu \to \infty$, then $\{x_\mu\}$ contains a subsequence which converges to an extremal x. If x is defined on (α, β) then the convergence is uniform on compact subsets of (α, β).*

Proof. By local compactness there exists a $t_1 > t_0$ and a subsequence $\{\nu\}$ of $\{\mu\}$ such that $x_\nu(t_1) \to q_1$ say. Let $x(t)$ be that extremal which joins q to q_1. Then $x_\nu(t) \to x(t)$ on (t_0, t_1). The rest follows as in 4.4 and 4.5 above.

4.7. By the Hopf Rinow theorem [5, page 4], when $\bar{S}^u(p, \varrho)$ is compact for all $\varrho > 0$ we have $\alpha = -\infty$ and $\beta = \infty$, for each extremal and the space is geodesically complete. When only the spheres $px \leqslant \varrho$ are compact we still have $\beta = \infty$ so that only the part about the left hand end points in 4.5 remains significant. However we proved 4.5 for both the end points α and β so that our results remain valid in the locally compact case also. The example of an open disc punctured at a point in the euclidean plane shows that the statement $\beta \leqslant \liminf \beta_\nu$ cannot be improved to $\beta = \lim \beta_\nu$, in the locally compact case. The corresponding inequality $\limsup \alpha_\nu \leqslant \alpha$ regarding the left hand end points can also not be improved in the weakly complete case, this is shown by examples in section 5.

In view of the above, we formulate the following property of geodesic continuity for a space R. We say that the space is *geodesically continuous* if $\lim \alpha_\nu = \alpha$ and $\lim \beta_\nu = \beta$ for all sequences $\{x_\nu(t)\}$ of extremals such that x_ν is defined on (α_ν, β_ν) and $x_\nu(t) \to x(t)$ on an interval $(-\varepsilon, \varepsilon)$ where $x(t)$ is also an extremal defined on (α, β). In section 5 we give an example of a space which is not geodesically continuous.

5. A class of examples. In this section we construct a class of examples of nonsymmetric weakly complete straight G-spaces. We prove that given a family of curves satisfying certain conditions in the euclidean plane we can metrize the plane as a weakly complete space with the given curves as extremals. This metrization is obtained by suitably modifying a method of H. Busemann, see [3, section 11]. We state the theorem:

THEOREM. *In the plane P with an associated euclidean metric $e(x, y)$, let a system Σ of curves be given with the following two properties:*

I. *Each curve in Σ is representable in the form $p(t)$, $-\infty < t < \infty$, such that $p(t_1) \neq p(t_2)$ for $t_1 \neq t_2$ and $e(p(0), p(t)) \to \infty$ as $|t| \to \infty$.*

II. *There is exactly one curve in Σ throught two given distinct points of P. Then P may be metrized as a nonsymmetric straight space such that the curves in Σ are the extremals and such that all $\bar{S}^+(p, \varrho)$ are compact while not all $\bar{S}^-(p, \varrho)$ are compact.*

Proof. We only outline the modifications necessary. Since the family of curves Σ is the same as in [3, Theorem 11.2], all the topological properties of Σ carry over. We therefore obtain a countable collection of simple families $\varphi_1, \varphi_2, \ldots$ such that the union of the curves in these families is dense in Σ. Let each family be parametrized by means of a monotone continuous parameter and denote by t^i_x the parametric value of the curve through x in the ith family. We may assume that $|t^i_x| \to \infty$ when $e(x, z)$ and i simultaneously tend to infinity where z is a fixed point and also that $|t^i_x| < m(x) < \infty$ for all i. (The specific parametrization used in [3], for example, satisfies these two conditions).

Let f be a strictly monotonic, bounded real valued continuous function defined on the entire real line and put

$$\eta_i(x, y) = |f(t^i_x) - f(t^i_y)|, \qquad \delta(x, y) = \sum \frac{1}{2^i}\eta_i(x, y).$$

It can be shown, as in [3], that δ is a metric in terms of which $\delta(x, y) + \delta(y, z) = \delta(x, z)$ if and only if x, y, z lie on a member of Σ in that order. However $\delta(x, y) < 2m$ if $|f(u)| < m$ for all u, so that δ is neither finitely compact nor semifinitely compact. We therefore add another compatible metric to $\delta(x, y)$.

For this metric, choose two positive real valued continuous functions $h(t)$ and $g(t)$ defined on the real line such that $h(t)$ strictly increases and tends to ∞ as $t \to \infty$ and $g(t)$ strictly decreases and tends to ∞ as $t \to -\infty$. Define, for every integer i,

$$\delta_i(x, y) = h(t_y^i) - h(t_x^i) \quad \text{if} \quad t_y^i > t_x^i$$

and

$$\delta_i(x, y) = g(t_y^i) - g(t_x^i) \quad \text{if} \quad t_y^i < t_x^i .$$

We show that δ_i is a general (nonsymmetric) indefinite metric by showing that δ_i satisfies the triangle inequality.

Let x_1, x_2, x_3 be any three points and $\lambda_1, \lambda_2, \lambda_3$ be the numbers t_x^i, t_y^i, t_z^i respectively. Put $h_{ij} = h(\lambda_i) - h(\lambda_j)$ and $g_{ij} = g(\lambda_i) - g(\lambda_j)$. There are in all six different cases for the relative magnitudes of $\lambda_1, \lambda_2, \lambda_3$. We can prove that $\delta_i(x_1, x_2) + \delta_i(x_2, x_3) \geqslant \delta_i(x_1, x_3)$ in each of these six cases. We omit the details of this verification.

We make use of these metrics δ_i to obtain nonsymmetric compactness in the following manner. Let z be a fixed point in the space and let N_ν denote the disc $\{x|\ e(zx) < \nu\}$. For every positive integer ν we can construct, as in [3], using the topological properties of Σ, a Σ-convex polygon $Q_\nu = \bigcup_{\lambda=1}^{\beta(\nu)} T(x_\lambda, x_{\lambda+1})$ with $x_{\beta(\nu)+1} = x_1$, containing N_ν in its interior. Let $j(\nu, 1), j(\nu, 2), \ldots, j(\nu, \beta(\nu))$ be the indices of simple families containing g_λ where g_λ is the extremal containing $T(x_\lambda, x_{\lambda+1})$. Let $\{i\}$ be a single subscript indexing of the collection of indices of families used to construct $\bigcup Q_\nu$. Put

$$b(\nu) = \operatorname{Min}\{\delta_i(z, x)|\ x \in Q_\nu,\quad i = j(\nu, 1), j(\nu, 2), \ldots, j(\nu, \beta(\nu))\}$$

and

$$b^*(\nu) = \operatorname{Max}\{\delta_i(z, x)|\ x \in N_\nu,\quad i = j(\nu, 1), j(\nu, 2), \ldots, j(\nu, \beta(\nu))\} .$$

Then, in view of the condition $|t_x^i| \to \infty$ as $e(z, x) \to \infty$ and $i \to \infty$, $b^*(\nu) \to \infty$ as $\nu \to \infty$, so that $b(\nu) \to \infty$ also because $b(\nu) \geqslant b^*(\nu)$. Let $\varphi(\nu)$ be any function defined on $(0, \infty)$ which increases to ∞ as $\nu \to \infty$. We define

$$\psi_\nu(\tau) = \begin{cases} \dfrac{\tau}{j(\nu, \beta(\nu))2^\nu}, & \text{if } 0 \leqslant \tau \leqslant \tfrac{1}{2}b(\nu), \\ \varphi(b(\nu)) + (\tau - b(\nu))\left[\dfrac{2\varphi(b(\nu))}{b(\nu)} - \dfrac{1}{j(\nu, \beta(\nu))2^\nu}\right] & \text{otherwise.} \end{cases}$$

$$\varepsilon_\nu(x, y) = \sum_{\lambda=j(\nu,1)}^{j(\nu,\beta(\nu))} \psi_\nu(\delta_\lambda(x, y)) \quad \text{and} \quad xy = \delta(x, y) + \sum \varepsilon_\nu(x, y) .$$

Since ψ_ν are convex functions, xy is a general metric. We show that xy has the required properties.

First, given two points x, y there exists $m(x, y)$ such that $\delta_i(x, y) \leqslant m(x, y)$ for all i. Since $b(\nu) \to \infty$, there exists ν_1 such that $m(x, y) < \frac{1}{2}b(\nu)$ for all $\nu > \nu_1$ and ν_2 such that $x, y \in Q_\nu$ for all $\nu > \nu_2$. Then $\varepsilon_\nu(x, y) \leqslant m(x, y)/2^\nu$ for all but a finite number of ν and therefore xy is a finite number.

Since ψ_ν is a monotone function of τ the same argument as in [3] shows that with respect to the distance xy, the members of Σ are the extremals. We show that with this distance xy all $\bar{S}^+(p, \varrho)$ are compact while not all $\bar{S}^-(p, \varrho)$ are compact.

Let $e(z, x_n) \to \infty$ as $n \to \infty$ and let $G > 0$ be given. Then since $\varphi(b(\nu)) \to \infty$ as $\nu \to \infty$, there exists ν' such that $\varphi(b(\nu)) > G$ if $\nu' < \nu$. Let $\bar{\nu} = \nu' + 1$. Since $e(z, x_n) \to \infty$ there exists an n_0 such that $x_n \notin Q_{\bar{\nu}}$ if $n > n_0$. Hence $\delta_\lambda(z, x_n) \geqslant b(\bar{\nu})$ if $n > n_0$, for $\lambda = j(\bar{\nu}, 1), j(\bar{\nu}, 2), \ldots, j(\bar{\nu}, \beta(\bar{\nu}))$. Therefore $\varepsilon_{\bar{\nu}}(z, x_n) > \varphi(b(\bar{\nu})) > G$, for all $n > n_0$. Thus $zx_n > G$ for all $n > n_0$. Hence $e(z, x_n) \to \infty$ implies $zx_n \to \infty$. Therefore all $\bar{S}^+(p, \varrho)$ are compact.

On the other hand, if $|t_z^i| < m(z) < \infty$ then for any x and any i, $\delta_i(x, z) < \max[h(m(z)), g(m(z))]$. Thus there exists a $\bar{\nu} = \bar{\nu}(z)$ such that $\delta_i(x, z) < \frac{1}{2}b(\nu)$ for all $\nu > \bar{\nu}$. Consequently $xz < k(z) < \infty$ for all x. Hence $\bar{S}^-(p, \varrho)$ are not all compact.

This completes the proof of the theorem.

We note that the above method also gives us examples which do not have the property of geodesic continuity. For example consider the ordinary plane with ordinary straight lines forming the family Σ and consider a line L passing through a fixed point z. Let the family φ_1 consist of lines parallel to L. Let $\{L_\nu\}$ be a sequence of lines through z converging to L. Since none of L_ν is in φ_1, while L_ν intersects each line of the family φ_1, (consisting of lines parallel to L), we see that the contribution to the metric length of each L_ν from η_1 is positive and bounded below away from zero while η_1 makes zero contribution to the length of L. Thus the corresponding metric is not geodesically continuous.

In the above metrizations the curves are extremals traversed in both the orientations. A space with this property is said to be a space with reversible extremals. Given a family of curves Σ in the euclidean plane such that there is exactly one oriented curve through every ordered pair of points can we metrize the plane as a weakly complete G-space such that the two curves through a pair of points are extremals one in each direction? This problem is so far unsolved.

6. The topology of finite dimensional spaces. In this section we show that the finite dimensional weakly complete spaces share with the finite dimensional G-spaces the property of noncontractible small spheres. Our arguments based on an idea in [5, page 16] become quite complicated but they remain valid for the locally compact case also with very little change. The spaces in this section are assumed to be of constant topological dimension. We first prove the following:

6.1. THEOREM. *Let $\varrho > 0$ and $S^+(p, \varrho) \subset S^n(p, \varrho(p))$. Let q be a point of $S^+(p, \varrho)$. Then $K^+(p, \varrho)$ is a strong deformation retract of $\bar{S}^+(p, \varrho) - \{q\}$.*

Proof. The proof is quite long. We first note that if $q = p$ then the required deformation retract is obtained by means of segments $T(p, x)$ with $x \in K^+(p, \varrho)$. To consider the case when $q \neq p$ we use the following notation:

$\varepsilon_1 = \inf\{\gamma(x) \mid x \in \bar{S}^+(p, \varrho)\} > 0$, because of 3.8 above since $\bar{S}^+(p, \varrho)$ is compact.

$\varepsilon_2 = \frac{1}{8}\varepsilon_1$ and points $p_0 = p, p_1, p_2, \ldots, p_n = q$ on $T(p, q)$ are chosen in such a way that $p_i p_{i+1} + p_{i+1} p_i < \varepsilon_2$. This can be done by using 3.7 above.

$\varepsilon_3 = \max(p_i p_{i+1}, p_{i+1} p_i)$, $i = 0$ to $n-1$.

$\varepsilon_4 = \min(\frac{1}{8}\varepsilon_1, \varrho - pq)$.

$E_k = \{x \mid p_{k-1}x + xp_k < p_{k-1}p_k + \varepsilon_4\}$,

$F_k = \{x \mid p_{k-1}x + xp_k = p_{k-1}p_k + \varepsilon_4\}$,

$G_k = \bar{S}^+(p, \varrho) - E_k$.

Before we take up the proof of the theorem we need the following facts:

(i) $E_i \subset S^+(p, \varrho)$.

Proof. $x \in E_i$ implies $px \leqslant pp_i + p_i x \leqslant pp_i + p_i x + xp_{i+1} \leqslant pp_i + p_i p_{i+1} + \varepsilon_4 < pq + \varepsilon_4 \leqslant \varrho$. Hence $x \in S^+(p, \varrho)$.

(ii) $S^n(p_i, \frac{1}{2}\varepsilon_4) \subset E_i$.

Proof. If $x \in S^n(p_i, \frac{1}{2}\varepsilon_4)$ then $p_{i-1}x + xp_i \leqslant p_{i-1}p_i + p_i x + xp_i \leqslant p_{i-1}p_i + \varepsilon_4$.

(iii) $S^n(p_i, \frac{1}{2}\varepsilon_4) \subset E_{i+1}$.

Proof. If $x \in S(p_i, \frac{1}{2}\varepsilon_4)$ then $p_i x + xp_{i+1} \leqslant p_i x + xp_i + p_i p_{i+1} < \varepsilon_4 + p_i p_{i+1}$.

(iv) $S^+(p_{i+1}, \frac{3}{4}\varepsilon_1) \supset E_{i+1}$.

Proof. If $p_{i+1}x \geqslant \frac{3}{4}\varepsilon_1$ then $p_i x \geqslant p_{i+1}x - p_{i+1}p_i \geqslant \frac{1}{4}3\varepsilon_1 - \varepsilon_2 \geqslant \frac{3}{4}\varepsilon_1 - \frac{1}{8}\varepsilon_1 = \frac{5}{8}\varepsilon_1 > \frac{1}{8}\varepsilon_1 + \varepsilon_2 > p_i p_{i+1} + \varepsilon_4$.

(v) $S^-(p_{i+1}, \frac{3}{4}\varepsilon_1) \supset E_{i+1}$.

Proof. Let $xp_{i+1} \geqslant \frac{3}{4}\varepsilon_1$ then $p_i x + xp_{i+1} \geqslant \frac{3}{4}\varepsilon_1 > \varepsilon_1 + \frac{1}{8}\varepsilon_2 > p_i p_1 + \varepsilon_4$.

(vi) $\bar{E}_{i+1} = E_{i+1} \cup F_{i+1}$.

Proof. If $r \in K^-(p_{i+1}, \frac{3}{4}\varepsilon_1)$, $rp_{i+1} = \frac{3}{4}\varepsilon_1 < \gamma(p_{i+1})$ then the segment $T(r, p_{i+1})$ is unique. Therefore because of (v) above it suffices to prove that $T(r, p_{i+1})$ meets F_{i+1} in a unique point.

Now if $u, v \in T(r, p_{i+1})$ with (uvp_{i+1}) then $p_i v + vp_{i+1} \leqslant p_i u + uv + vp_{i+1} = p_i u + up_{i+1}$. Thus as x traverses $T(r, p_{i+1}) p_i x + xp_{i+1}$ decreases from $p_i r + rp_{i+1}$ which is $\geqslant p_i p_{i+1} + \varepsilon_4$ to $p_i p_{i+1}$ which is $< p_i p_{i+1} + \varepsilon_4$. Hence there exists $r_F \in T(r, p_{i+1})$ such that $p_i r_F + r_F p_{i+1} = p_i p_{i+1} + \varepsilon_4$.

The point r_F is unique because if $\bar{r}_F$ is another point and if $(r_F \bar{r}_F p_{i+1})$ then $p_i \bar{r}_F + \bar{r}_F p_{i+1} \leqslant p_i r_F + r_F \bar{r}_F + \bar{r}_F p_{i+1} = p_i r_F + r_F p_{i+1}$ shows that the equality holds only when $(p_i r_F \bar{r}_F)$. But then $(p_i r_F \bar{r}_F)$ and $(r_F \bar{r}_F p_{i+1})$ imply $(p_i \bar{r}_F p_{i+1})$ and $(p_i r_F p_{i+1})$, from 3.6, because $p_i \bar{r}_F + \bar{r}_F p_{i+1} = p_i p_{i+1} + \varepsilon_4 \leqslant \varepsilon_3 + \varepsilon_4 = \frac{1}{8}\varepsilon_1 + \frac{1}{8}\varepsilon_1 = \frac{1}{4}\varepsilon_1 < \gamma(\bar{r}_F)$. Since $\bar{r}_F \notin T(p_i p_{i+1})$ the above argument shows the uniqueness of r_F. This proves (vi).

(vii) F_{i+1} is a strong deformation retract of $\bar{E}_{i+1} - \{p_{i+1}\}$.

Proof. A retract f: $\bar{E}_{i+1} - \{p_{i+1}\}$ is obtained, in view of (vi) above, by sliding down points of $T(r, p_{i+1}) \cap \bar{E}_{i+1} - \{p_{i+1}\}$ to r_F. A similar retract f_θ can be obtained for every θ, $0 \leqslant \theta \leqslant 1$, from $\{x \mid p_i x + xp_{i+1} \leqslant p_i p_{i+1} + \theta\varepsilon_4\} - \{p_{i+1}\}$ to $\{x \mid p_i x + xp_{i+1} = p_i p_{i+1} + \theta\varepsilon_4\}$. Keeping all other points of $\bar{E}_{i+1} - \{p_{i+1}\}$ fixed this can be extended to $\bar{E}_{i+1} - \{p_{i+1}\}$. Since $f_0 =$ identity on $\bar{E}_{i+1} - \{p_{i+1}\}$ and $f_1 =$ the retract f as above, we have proved that F_{i+1} is a strong deformation retract of $\bar{E}_{i+1} - \{p_{i+1}\}$.

We now return to the proof of the theorem. As noted in the beginning $\bar{S}^+(p, \varrho) - \{p\}$ is strongly deformable over itself into $K^+(p, \varrho)$. Denote, without using a double subscript, this deformation homotopy by f_0. Similarly, denote by f_{i+1} the deformation homotopy obtained in (vii) above of $\bar{E}_{i+1} - \{p_{i+1}\}$ into F_{i+1}. Then f_{i+1} can be extended to $\bar{S}^+(p, \varrho - - \{p_{i+1}\}$ by keeping points of $G_{i+1} = \bar{S}^+(p, \varrho) - E_{i+1}$ pointwise fixed throughout the deformation. Without changing names let f_{i+1} itself denote this deformation homotopy of $\bar{S}^+(p, \varrho) - \{p_{i+1}\}$ into G_{i+1}.

Now define $g_1 =$ restriction of f_0 to G_1 and put $h_1 = g_1 \circ f_1$ so that h_1 gives a deformation homotopy of $\bar{S}^+(p, \varrho) - \{p_1\}$ into $K^+(p, \varrho)$. Next put $g_2 =$ restriction of h_1 to G_2 and put $h_2 = g_2 \circ f_2$. Thus h_2 gives a deformation homotopy of $\bar{S}^+(p, \varrho) - \{p_2\}$ into $K^+(p, \varrho)$. Continuing in this manner, having defined h_{i-1}, put $g_i =$ restriction of h_{i-1} to G_i and $h_i = g_i \circ f_i$. Thus h_i gives a deformation homotopy of $\bar{S}^+(p, \varrho) - \{p_i\}$ into $K^+(p, \varrho)$. Since $p_n = q$, h_n gives the required deformation homotopy of $\bar{S}^+(p, \varrho) - \{q\}$ into $K^+(p, \varrho)$.

Thus $K^+(p, \varrho)$ is a strong deformation retract of $\bar{S}^+(p, \varrho) - \{q\}$. This completes the proof of the theorem.

6.2. A finite dimensional generalized G-space is an r-space. We say that an open set V is *canonical* if $\bar{V}$ is compact and if for $p \in V$, the boundary

$\dot{V}$ of V is a strong deformation retract of $\bar{V}-\{p\}$. Although our spaces have nonsymmetric distance we define, following [5] and [9], an r-space as a locally compact general metric space of constant topological dimension where each point possesses arbitrarily small canonical neighborhoods. Since the above theorem says that arbitrarily small positive balls are canonical neighborhoods, we see that our spaces are also r-spaces.

6.3. Therefore all the results contained in the theory of r-spaces (see [5], § 3), carry over. For example, the spheres $K^+(p, \varrho)$, $0<\varrho<\gamma(p)$, are not contractible and the property of domain invariance holds in our spaces. Many important results on conjugate points proved by Busemann [see [5], pp. 14–20) depend only on these two properties and hence they c rry over to our spaces. Since his methods generalize, we omit the details.

This paper constituted a part of my Ph. D. thesis written at the University of Southern California. I am grateful to Professor H. Busemann for his guidance, encouragement and suggestions throughout the preparation of the thesis. The work was partially supported by the National Science Foundation of the United States. I thank also Professor H. S. M. Coxeter and the National Research Council of Canada for support at the University of Toronto.

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Reçu par la Rédaction le 24. 6. 1971

Weakly smooth dendroids

by

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Abstract. Let X be a dendroid. For each point in X let η_p denote the function from X into 2^X given by $\eta_p(x)=[p,x]$, where 2^X is the space of all nonempty closed subsets of X with the Vietoris topology and $[p,x]$ is the unique irreducible continuum from p to x. Observe that X is smooth if for some p, η_p is a homeomorphism of X onto its image $\mathfrak{D}(X,p)$. The converse is also true. The space $\mathfrak{D}(X,p)$ is studied for non-smooth dendroids. Define X to be *weakly smooth* if there exists a point p such that $\mathfrak{D}(X,p)$ is a compact subset of 2^X. Order-theoretic characterizations of weakly smooth dendroids are obtained.

1. Introduction. Throughout this paper *continuum* will mean a compact connected metric space containing more than one point. A continuum is *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. The *weak cut point order* on a hereditarily unicoherent continuum X with respect to p, $\leqslant_p$, is defined by $x\leqslant_p y$ if and only if $x\in[p,y]$, where $[p,y]$ denotes the intersection of all subcontinua of X containing p and y. *A dendroid* is an arcwise connected hereditarily unicoherent continuum. If X is a dendroid, then $\leqslant_p$ is a partial order and $[p,y]$ is an arc for all $y\in X$. For any point p in a dendroid X denote by $\mathfrak{D}(X,p)$ the set of all arcs in X of the form $[p,x]$. We view $\mathfrak{D}(X,p)$ as a subspace of 2^X, where 2^X denotes the space of nonempty closed subsets of X with the Vietoris topology [6].

Charatonik and Eberhart [1] investigate smooth dendroids (Definition 1). Here the more general notion of weakly smooth dendroids is introduced: A dendroid X is said to be *weakly smooth* if $\mathfrak{D}(X,p)$ is a compact subset of 2^X for some $p\in X$.

The work is divided into three sections. The first section deals with the structure of $\mathfrak{D}(X,p)$ and with two partial order characterizations of weakly smooth dendroids similar to those of smooth dendroids (Theorem 2). In the second section these results are applied to obtain necessary and sufficient conditions for a dendroid to be smooth and for a dendroid to be a dendrite ($\equiv$ locally connected dendroid). We discuss necessary and sufficient conditions for hereditarily unicoherent continua to be arcwise connected in the final section.