Periodic actions on (I-D) normed linear spaces

by

Raymond Y. T. Wong (*) (Santa Barbara, Cal.)

Abstract. We study various properties of periodic actions on infinite dimensional normed linear spaces. Typically we investigate the category Φ whose objects are spaces each equipped with a certain periodic map a, and whose morphisms are maps $m\colon X \to X_1$ for which m commutes with the a's. Homotopy classification of type (Z_q, n) spaces and results in infinite dimensional topology are employed. We then prove several results in the category Φ such as conjugation, embedding, extension of homeomorphisms, etc.

1. Introduction. The purpose of this paper is to employ a homotopy classification theorem (Proposition 1) for ANR and to apply it to periodic actions for infinite-dimensional spaces. Let Z_q denote the integers modulo q, $q \geqslant 0$. A connected, locally path connected metric space X is said to be an *Eilenberg-Maclane space of type* (Z_q, n), or simply, of *type* (Z_q, n), provided the fundamental group $\pi_n(X)$ is isomorphic to Z_q and $\pi_i(X) = \{0\}$ for all $i \neq n$. Let E denote a fixed (but arbitrary) infinite-dimensional (I-D) space which is homeomorphic ($\cong$) to F^ω or F^ω_f for some normed linear space (NLS) F, where F^ω denotes the countable infinite product of F by itself and $F^\omega_f \subset F^\omega$ denotes the subset consisting of all points having at most finitely many non-zero coordinates. The following proposition classifies, up to homotopy type, all metric absolute neighborhood retracts (ANR) of type (Z_q, n).

PROPOSITION 1. *Let Y, Y' be metrizable connected* ANR *of type (Z_q, n) and let $e \in \pi_1(Y)$, $e' \in \pi_1(Y')$ be generators. Then there is a homotopy equivalence $h\colon Y \to Y'$ such that $h_{\#}(e) = e'$.*

Proof. The case for $q = n = 1$ is contained in the corollary following Theorem 15 of Palais ([14]). In fact, Proposition 1 is a special case of a more generally known theorem (see [10, p. 127] and [12, p. 4]). Namely, let G be any abelian group. Then the homotopy type of type (G, n) spaces is determined by G and n. Moreover, if X, X' are respectively type (G, n) and type (G', n) spaces, then any isomorphism $G \overset{g}{\approx} G$ may be realized by a homotopy equivalence $X \overset{f}{\sim} X'$ for which $f_{\#} = g$.

(*) Partially supported by NSF Grant GP 20632. This research was prepared while the author was visiting the Mathematisch Centrum, Amsterdam.

It is now well known that E-manifolds can be classified by their homotopy types ([7], [8]) and the same is true in the C^∞-category for separable C^∞-Hilbert manifolds ([6], [11]). We summarize these results in the following proposition.

PROPOSITION 2. (A) *Each homotopy equivalence between E-manifolds is homotopic to a homeomorphism.*

(B) *Each homotopy equivalence between separable C^∞-Hilbert manifolds is homotopic to a C^∞-diffeomorphism.*

Since all E-manifolds are ANR, applying Proposition 1 and 2 we obtain the following proposition which classifies all metrizable connected E-manifolds (or C^∞-Hilbert manifolds) of type (Z_q, n).

PROPOSITION 3. (*Classification*) *Let M and M_1 be metrizable E-manifolds of type (Z_q, n) and let $e \in \pi_1(M)$, $e_1 \in \pi_1(M_1)$ be generators. Then there is a homeomorphism $h\colon M \to M_1$ such that $h_\#(e) = e_1$.*

Let l_2 denote the separable Hilbert space of all square summable complex sequences and let S denote its unit sphere. For any $q > 1$, define a fixed point free periodic homeomorphism $\alpha\colon S \to S$ of period q by

$$\alpha(z_0, z_1, \ldots) = (e^{2\pi i/q}z_0, e^{2\pi i/q}z_1, \ldots) .$$

Then α induces (by restrictions) periodic homeomorphisms $\alpha_n\colon S^{2n-1} \to S^{2n-1}$ of period q, where S^{2n-1} is the unit sphere of the $2n$-dimensional complex space C^n. The inductive limit of $\{S^{2n-1}/\alpha_n\}_{n\geqslant 1}$ (S^{2n-1}/α_n the orbit spaces), denoted by $\varinjlim S^{2n-1}/\alpha_n$, is a CW-complex of type $(Z_q, 1)$. Hence, by means of Proposition 3, we obtain

PROPOSITION 4. *Let M be a metrizable connected E-manifold of type $(Z_q, 1)$, then M has the same homotopy as $\varinjlim S^{2n-1}/\alpha_n$.*

Let M be as above with $q > 1$ a prime number. The universal covering space $\tilde{M}$ of M is a homotopically trivial E-manifold such that the projection $p\colon \tilde{M} \to M$ is a q-fold covering map. By Proposition 2(A), $\tilde{M} \cong E$. Let $\beta\colon \tilde{M} \to \tilde{M}$ be any fixed point free period q homeomorphism (β always exists, see [16]). Then the orbit space $\tilde{M}/\beta$ is an E-manifold of type $(Z_q, 1)$. By Proposition 3 there is a homeomorphism $h\colon \tilde{M}/\beta \to M$ which then induces a fibre homeomorphism $h_*\colon \tilde{M} \to \tilde{M}$. Let $\beta_* = h_* \circ \beta \circ h_*^{-1}$. We obtain the following

PROPOSITION 5. (*Representation*). *Let M be a metrizable connected E-manifold of type $(Z_q, 1)$, $q > 1$, a prime number. Then there is a q-fold covering projection $p\colon E \to M$ and a fixed point free periodic homeomorphism $\beta_*\colon E \to E$ of period q such that β_* induces a homeomorphism $\beta_0\colon E/\beta_0 \to M$ for which $\beta_0 \circ p_0 = p \circ \beta_*$.*

Non-trivial examples of involutions on s, H or Hilbert cube $Q = [-1, 1]^\omega$ are provided by the well-known facts $s \cong s \times Q$, $H \cong H \times Q$ and the following theorem of West [17, Corollary 5.1].

The product of a countable infinite collection of (non-degenerate) compact, contractible polyhedra is homeomorphic with the Hilbert cube, Q.

Let H be a Hilbert space and S the unit sphere of H. Let $m_1\colon H\setminus\{0\} \to S$ be any homeomorphism such that $m_1(S)$ is a equator of S (in an obvious sense). Let $\gamma_* = m_1 \gamma m_1^{-1}$, where $\gamma(x) = -x$ is the reflection on H. By Theorem 1 of section 2 there is a homeomorphism $m\colon (S, \gamma_*) \to (S, \gamma)$. Then let $mm_1(S) = S_0$, $A = mm_1(A_0)$ and $B = mm_1(B_0)$, where A_0, B_0 are respectively the bounded and unbounded components of $H\setminus S$. We then obtain the following non-trivial example in S. (Note that such an example trivially exists in H.)

EXAMPLE 1. *There is a bi-collared sub-Hilbert sphere $S_0 \subset S$ such that $S\setminus S_0 = A \cup B$, $A \cong H \cong B$ and S_0, A, B are all symmetric subsets of S.*

2. Application to periodic homeomorphisms and other results. Throughout this section let $q > 1$ denote a prime number. For any space X, let $G(X)$ denote the set of homeomorphisms of X.

THEOREM 1. (*Conjugation*). *Let β, $\beta_1\colon E \to E$ be fixed point free periodic homeomorphisms of period q. Then there is a homeomorphism $h_0\colon E \to E$ such that $h_0 \circ \beta = \beta_1 \circ h_0$.*

Moreover, if $E = l_2$ and β, β_1 are C^∞-smooth, we may choose h_0 to be a C^∞-diffeomorphism.

Proof. The C^0 case. Let $b \in E$ and suppose λ, $\lambda_1\colon ([0, 1], 0) \to (E, b)$ are maps (preserving base points) such that $\lambda(1) = \beta(b)$ and $\lambda_1(1) = \beta_1(b)$. Let $p\colon E \to E/\beta$, $p_1\colon E \to E/\beta_1$ denote the projections. Then $e = [p \circ \lambda] \in \pi_1(E/\beta)$ and $e_1 = [p_1 \circ \lambda_1] \in \pi_1(E/\beta_1)$ are generators. It follows from Proposition 3 that there is a homeomorphism $h\colon (E/\beta, p(b)) \to (E/\beta_1, p_1(b))$ such that $h_\#(e) = e_1$. The function h then induces a (fibre) homeomorphism $h_0\colon (E, b) \to (E, b)$ such that $p_1 \circ h_0 = h \circ p$ and $h_0 \circ \beta(b) = \beta_1 \circ h_0(b)$. For each $x \in E$, since $\{h_0(x), h_0 \circ \beta(x)\} \subset p_1^{-1}(h \circ p(x))$, there is an $1 \leqslant i \leqslant q$ for which $h_0 \circ \beta(x) = \beta_1^i \circ h_0(x)$. Let $A_i = \{x \in E\colon\ h_0 \circ \beta(x) = \beta_1^i \circ h_0(x)\}$. We easily verify that each A_i is closed and $\{A_i\}$ are pairwise disjoint. Since E is connected and $A_1 \neq \emptyset$, hence $A_1 = E$. The C^∞ case follows exactly the same considerations using Proposition 1 and Proposition 2 (B).

As a matter of convenience we introduce the category whose objects are pairs (X, β), (X_1, β_1), ... where X, X_1 are spaces equipped with periodic homeomorphisms β, β_1, ... of period q, and whose morphisms are maps $m\colon (X, \beta) \to (X_1, \beta_1)$ of pairs such that m is a map of X into X_1 which commutes with β, β_1· that is, $\beta_1 \circ m = m \circ \beta$. We can speak of m as an imbedding, homeomorphism, etc.

For any map $h\colon X \to X$, denote by $fp(h)$ the set of fixed points of h. The *reflection map* $x \to -x$ of any topological vector space will always be denoted by γ.

Homeomorphism extension. Let X be a space homeomorphic to $X \times F$, F a TVS. We say a set $Y \subset X$ is *F-deficient* if there is a homeomorphism $h\colon X \to X \times F$ such that $h(Y) \subset X \times \{0\}$. (See [1] and [5] for the equivalence of F-deficiency with the concept of Z-sets of Anderson.)

THEOREM 2. *Let A be a closed H-deficient subset of a complex Hilbert space H. Then each period q homeomorphism β on A extends to a period q homeomorphism $\tilde{\beta}$ on H such that $fp(\tilde{\beta}) = fp(\beta)$.*

Proof. First we remark that for any metric locally convex TVS $F \cong F \times F$, by a technique of Klee ([9]), any homeomorphism between two closed F-deficient subsets of F extends to one on F. Denote by Δ_q the diagonal $\{(x, x, \ldots)\colon x \in H\}$ of $H^q_{\Delta} = H \times H \times \ldots \times H$ (q times).

Let $\varphi\colon H \to H \times H$ be a homeomorphism such that $\varphi(A) \subset H \times \{0\}$. For any $a \in A$, denote $\varphi(a) = (a_0, 0)$ and $\varphi\big(\beta^n(a)\big) = (a_n, 0)$, $n = 1, \ldots, q-1$. Define $m_1\colon A \to H^q \times H^q$ by $m_1(a) = (a_0, \ldots, a_q) \times (0, 0, \ldots, 0)$ and $\gamma_\Delta\colon H^q \to H^q$ by $\gamma_\Delta(z_0, z_1, \ldots, z_{q-1}) = (z_{q-1}, z_0, z_1, \ldots, z_{q-2})$. Let $p_1\colon H^q \times H^q \to H^q$ be the projection onto the first factor. Denote $p_1 \circ m_1\big(fp(\beta)\big)$ by K_1. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "m_1"] \arrow[d, "\beta"'] & (H^q \setminus K) \times H^q \arrow[d, "\gamma_\Delta \times \gamma"] \\ A \arrow[r, "m_1"'] & (H^q \setminus K) \times H^q \end{tikzcd}$$

where $K = \Delta_q \setminus K_1$ and γ the reflection on H^q.

It is elementary to note that Δ_q is a H-deficient subset of H^q. Thus K is a locally closed (that is, a difference of two closed sets, Δ_q and K_1) H-deficient subset of $H^q \cong H$. Hence by Cutler ([5]), $H^q \setminus K \cong H$. So let $m_2\colon (H^q \setminus K) \times H^q \to H$ be a homeomorphism. Using the remark above we can extend $m_2 \circ m_1\colon A \to H$ to a homeomorphism λ on H. Let $\beta_1 = m_2 \circ (\gamma_\Delta \times \gamma) \circ m_2^{-1}\colon H \to H$. It is clear that β_1 is a period q homeomorphism such that $fp(\beta_1) = m_2(K_1)$. Then $\tilde{\beta} = \gamma^{-1} \circ \beta_1 \circ \gamma$ is the required extension of β.

Closed imbeddings.

THEOREM 3. *Let X be a space which can be imbedded as a closed subset of a Hilbert space H. Then for any two fixed point free period q homeomorphisms β, β_1 on X, H respectively, there is a closed imbedding $m\colon (X, \beta) \to (H, \beta_1)$.*

Proof. Let $m_1\colon X \to H \times H$ be a closed imbedding such that $m(X) \subset \{0\} \times H$. By Theorem 2 the induced map $m_1 \circ \beta \circ m_1^{-1}\colon m_1(X) \to m(X)$ extends to a fixed point free period q homeomorphism $\tilde{\beta}$ on $H \times H$. By Theorem 1 there is a homeomorphism $m_2\colon (H \times H, \tilde{\beta}) \to (H, \beta)$. Then let $m = m_2 \circ m_1$.

Let $X = M$ be a metrizable connected H-manifold. By Henderson-West [7, Theorem 6] M can be imbedded as a closed sub-manifold of H. Hence the proof above also yields

COROLLARY 1. *Let β, β_1 be fixed point free period q homeomorphisms on M, H respectively. Then there is a closed imbedding $m\colon (M, \beta) \to (H, \beta_1)$ such that $m(M)$ is a sub-manifold of H.*

Negligible subsets.

THEOREM 4. *Let $K_1, K_2, \ldots$ be closed H-deficient subsets of H. Suppose $\beta, \beta_1\colon H \to H$ are fixed point free periodic homeomorphisms of period q such that $\beta(K) = K$, where $K = \bigcup_{i \geqslant 1} K_i$, then there is a homeomorphism $m\colon (H \setminus K, \beta) \to (H, \beta_1)$.*

Proof. By Cutler ([5, Theorem 1]), there is a homeomorphism $m_1\colon H \setminus K \to H$. Let $\alpha_1 = m_1 \circ \beta|_{H \setminus K} \circ m_1^{-1}$. By Theorem 1 let $m_2\colon (H, \alpha_1) \to (H, \beta_1)$ be a homeomorphism. Then $m = m_2 \circ m_1$ is as required.

Homeomorphism spaces are contractible. For any space X, let $G_0(X)$ denote the subspace of $G(X)$ consisting of all periodic homeomorphisms and $G_n(X) = \{\beta \in G_0(X)\colon \text{period}\ (\beta) = n,\ n \geqslant 1\}$.

THEOREM 5. *For $k \geqslant 0$, each $G_k(E)$ is contractible and there is a contraction $\{\varphi_t\}\colon G(E) \to G(E)$ such that $\{\varphi_t|_{G_k(E)}\}$ induces a contraction for $G_k(E)$.*

Proof. Renz in [15] shows that $G(E)$ is contractible. If the construction of the contraction of [15] is replaced by

$$\varphi(h, t) = \varphi^{(n)}(\cdot, t)^{-1} \circ h^{(n)} \circ \varphi^{(n)}(\cdot, t)\,,$$

we then get a contraction denoted by $\{\varphi_t\}$ with the desired properties of the theorem.

Periodic stability of homeomorphisms. A subset K of a space X is said to be *deformable* if for each open set U in X, there is a $g \in G(X)$ such that $g(K) \subset U$. An open set $U \subset X$ is said to contain a *dilation system* if there is a sequence of pairwise disjoint open sets $B_0, B_1, \ldots$ in U converging to a point $p \in U$ and a homeomorphism r supported in U such that $r(B_{i+1}) = B_i$, $i \geqslant 0$. We sometimes call $(B_i, r)_{i \geqslant 0}$ a dilation system in U. For any $g \in G(X)$, let $\operatorname{supp}(g)$ denote the support of g.

THEOREM 6. *Suppose X is a metric space in which every open set contains a dilation system. Let $N \subset G(X)$ be the normal subgroup consisting of all finite compositions of $g \in G(X)$ such that $\operatorname{supp}(g)$ is deformable. Then N is simple.*

Proof. The following proof is derived from a technique of Fisher (On the group of all homeomorphisms of a manifold, Trans. A.M.S. 97 (1960), pp. 193–212). Suppose $h(\neq \mathrm{id}) \in G(X)$. Then for some open $U \subset X$, $h^{-1}(U) \cap U = \emptyset$. Let $(B_i, r)_{i\geqslant 0}$ be a dilation system in U. Denote $B = \bigcup_{i\geqslant 1} B_i$. Suppose $g \in G(X)$ such that $\mathrm{supp}(g) \subset B_1$, then define $\varphi\colon X \to X$ (supported in B) by $\varphi|_{B_i} = r^{1-i} \circ g \circ r^{i-1}|_{B_i}$ for $i \geqslant 1$ ($r^0 = \mathrm{id}$) and $\varphi(x) = x$ otherwise. Nore that $\varphi|_{B_1} = g|_{B_1}$. Consider

$$w = (r^{-1} \circ \varphi^{-1} \circ h^{-1} \circ \varphi \circ r)(r^{-1} \circ h \circ r) \circ h^{-1} \circ (\varphi^{-1} \circ h \circ \varphi) .$$

The same proof as [Fisher, p. 197] shows that $w = g$. If $g \in G(X)$ such that $\mathrm{supp}(g)$ is deformable, then by definition there is a $f \in G(X)$ such that $f(\mathrm{supp}(g)) \subset B_1$. Thus $f \circ g \circ f^{-1}$ is supported in B_1 and $g = f^{-1} \circ (f \circ g \circ f^{-1}) \circ f$. It follows that each $g \in N$ is a finite composition of conjugations of $\{h, h^{-1}\}$. Now suppose N_0 is any normal subgroup of N containing an h other than the identity. By what we have just shown, each $g \in N$ is a member of N_0. Thus $N_0 = N$ and Theorem 6 is proved.

It is known that for any normed linear space homeomorphic to E^ω, $G(X)$ is stable ([12], [18]), in the sense that every $f \in G(X)$ can be written as a finite composition $f_n \dots f_2 f_1$ of homeomorphisms of X such that each f_i is the identity on some non-void open subset of X. By well-known properties of X, it is routine to verify that:

(1) If $f \in G(X)$ is the identity on some non-void open subset of X, then $\mathrm{supp}(f)$ is deformable.

(2) Each open $U \subset X$ contains a dilation system. Hence we have

COROLLARY 2. *Let X be a space homeomorphic to Q, s or any normed linear space $E \cong E^\omega$. Then $G(X)$ is simple.*

For each fixed $k \geqslant 0$, the collection of all finite compositions of members in $G_k(X)$ clearly forms a non-trivial normal subgroup of $G(X)$. Hence $G_k(X)$ is entirely $G(X)$, which proves

THEOREM 7. *Let X be as above. Then for any $h \in G(X)$ and any $k \geqslant 0$, there are $h_1, \dots, h_n \in G_k(X)$ such that $h = h_n \circ \dots \circ h_2 \circ h_1$.*

References

[1] R. D. Anderson, *Hilbert space is homeomorphic to infinite product of lines*, Bull. Amer. Math. Soc. 72 (1966), pp. 515–519.

[2] C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sci. 14 (1966), pp. 27–31.

[3] — and A. Pełczyński, *Some remarks on homeomorphisms of F-spaces*, Bull. Acad. Pol. Sci. 10 (1962), pp. 265–270.

[4] T. A. Chapman, *Deficiency in infinite-dimensional manifolds*, General Topology and Appl. 1 (1971), pp. 263–272.

[5] W. Cutler, *Negligible subsets of non-separable Hilbert manifolds*, Proc. Amer. Math. Soc. 23 (1969), pp. 668–675.

[6] J. Eells Jr. and K. D. Elworthy, *On the differential topology of Hilbertian manifolds*, Proc. of the Summer institute of Global Analysis, Berkeley 1968.

[7] D. Henderson and J. E. West, *Triangulated infinite-dimensional manifolds*, Bull. Amer. Math. Soc. 76 (1970), pp. 655–660.

[8] D. W. Henderson and R. Schori, *Topological classification of infinite dimensional manifolds by homotopy type*, Bull. Amer. Math. Soc. 76 (1970), pp. 121–124.

[9] V. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. 74 (1953), pp. 10–43.

[10] A. Lundell and S. Weingram, *The Topology of CW Complexes*, 1969.

[11] N. H. Kuiper and D. Burghelea, *Hilbert manifolds*, Ann. of Math. 90 (1969), pp. 379–417.

[12] R. A. McCoy, *Annulus conjecture and stability of homeomorphisms in infinite-dimensional normed linear spaces*, Proc. Amer. Math. Soc. 24 (1970), pp. 272–277.

[13] R. Mosher and M. Tangora, *Cohomology Operations and Application in Homotopy Theory*, 1968.

[14] R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966), pp. 1–16.

[15] P. L. Renz, *The contractibility of the homeomorphism group of some product spaces by Wong's method*, Mathematica Scandinavia 28 (1971), pp. 182–188.

[16] J. E. West, *Fixed point sets of transformation group on infinite product spaces*, Proc. Amer. Math. Soc. 21 (1969), pp. 575–582.

[17] — *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. 150 (1970), pp. 1–25.

[18] R. Y. T. Wong, *On homeomorphisms of certain infinite dimensional spaces*, Trans. Amer. Math. Soc. 128 (1967), pp. 148–154.

UNIVERSITY OF CALIFORNIA
Santa Barbara, California

Reçu par la Rédaction le 21. 3. 1972