A Lindelöf space X such that X^2 is normal but not paracompact

by

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Abstract. The purpose of the paper is to prove that under the assumption of Martin's Axiom and $2^{\aleph_0} = 2^{\aleph_1}$ there exists a hereditarily Lindelöf first countable space X such, that X^2 is perfectly normal but not collectionwise normal, and hence not paracompact. Moreover, it is shown that the existence of such a space is equivalent to the existence of a normal, separable, non-metrizable Moore space, and consequently is independent of the ZFC axioms of set theory. The above theorems partially solve the problem (raised by M. Maurice, E. Michael and H. Tamano), whether or not the product of two paracompact spaces, which is normal must be necessarily paracompact.

M. Maurice (cf. [4], Problem 7.2), and in a special case H. Tamano [6], raised the problem whether or not the product of two paracompact spaces which is normal must necessarily be paracompact (1). In this paper we solve this problem in the negative under the assumption of Martin's Axiom and of the equality $2^{\aleph_0} = 2^{\aleph_1}$. Moreover, we show that a special case of this problem is equivalent to the problem of the existence of a separable, normal, non-metrizable Moore space and hence is independent of the usual axioms of set theory.

Let us recall (cf. [5]) that the assumption of Martin's Axiom plus $2^{\aleph_0} = 2^{\aleph_1}$ is independent of the ZFC axioms of set theory.

THEOREM (cf. [5]). *The following hypotheses are equivalent and independent of the* ZFC *axioms of set theory*:

H. 1. *There exists a separable, normal, non-metrizable Moore space*;

H. 2. *There exists a separable, first-countable, normal space containing an uncountable closed and discrete subset*;

H. 3. *There exists an uncountable subset T of the real line every subset of which is a relative F_σ.*

Moreover, Martin's Axiom plus $2^{\aleph_0} = 2^{\aleph_1}$ implies any of the above hypotheses.

The purpose of this paper is to prove the following theorems:

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(1) All undefined terms and symbols are as in [1]. The symbol $|A|$ denotes the cardinality of the set A.

THEOREM 1 (under the assumption of H. 3). *There exists a hereditarily Lindelöf space X such that X^2 is perfectly normal but not paracompact.*

Moreover, the space X is hereditarily separable and first-countable and the space X^2 is not collectionwise normal.

THEOREM 2. *The following hypothesis is equivalent to* H. 1, H. 2 *and* H. 3 *and hence is independent of the axioms of set theory*:

H. 4 *There exists a separable, first-countable, paracompact space X such that X^2 is normal but not collectionwise normal.*

LEMMA 1 (under the assumption of H. 3). *There exists an uncountable subset S of the interval $(0,1)$ with the following properties:*

(i) *every subset of S is a relative F_σ;*

(ii) *if $x \in S$, then $1-x \in S$.*

Proof of Lemma 1. We may assume that the set T in H. 3 is contained in the interval $(0, \frac{1}{2})$. It suffices to put

$$S = T \cup \{1-x\colon\ x \in T\}.$$

The following lemmas are well known.

LEMMA 2. *If A, B are subsets of the real line R and $f\colon A \twoheadrightarrow B$ is a monotone function of A onto B, then there exists a countable subset B_0 of B such that*

$$f|f^{-1}(B\setminus B_0)\colon\ f^{-1}(B\setminus B_0) \twoheadrightarrow B\setminus B_0$$

is a homeomorphism.

LEMMA 3. *If K, L are disjoint closed subsets of a topological space Z and $\{U_n\}_{n=1}^{\infty}$, $\{V_n\}_{n=1}^{\infty}$ are two sequences of open sets satisfying*

(i) $\bigcup_{n=1}^{\infty} U_n \supset K$, $\bigcup_{n=1}^{\infty} V_n \supset L$,

(ii) *for every* $n = 1, 2, \ldots$ $\overline{U}_n \cap L = \emptyset = \overline{V}_n \cap K$,

then there exist two open sets U, V such that $K \subset U$, $L \subset V$ and $U \cap V = \emptyset$.

Proof of Theorem 1. Consider the set $X = Y = S$, where S is as in Lemma 1, with the topology of the subspace of the Sorgenfrey line (see [1], Example 1.2.1). The space X is hereditarily Lindelöf, hereditarily separable, first-countable and by the theorem of R. W. Heath and E. A. Michael [2] the space $Z = X^2 = X \times Y$ is perfect. The set

$$D = \{(x, y) \in Z\colon\ y = 1-x\} = \{(x, 1-x)\colon\ x \in X\}$$

is uncountable, closed and discrete in Z. As Z is separable, it follows that Z is not collectionwise normal. To prove the theorem it remains to show that $Z = X^2$ is normal.

For every $z = (x, y) \in Z$ the sets

$$U\Big(z, \frac{1}{n}\Big) = \Big(\Big[x, x+\frac{1}{n}\Big) \times \Big[y, y+\frac{1}{n}\Big)\Big) \cap Z, \qquad n = 1, 2, \ldots$$

are open-and-closed in Z and form a neighbourhood base at the point z.

Let K and L be two disjoint and closed sets in Z. For every $z \in K \cup L$ there exists a natural number $n(z)$ such that

$$U\Big(z, \frac{1}{n(z)}\Big) \cap L = \emptyset \quad \text{or} \quad U\Big(z, \frac{1}{n(z)}\Big) \cap K = \emptyset.$$

Let us denote by E the set Z with the topology of the subspace of the Euclidean plane and put

$$U^0 = \mathrm{Int}_E(Z\setminus L), \qquad V^0 = \mathrm{Int}_E(Z\setminus K).$$

We can find sets U_n^0, V_n^0 open in E, and hence in Z, such that:

$$\text{(1)} \qquad \begin{aligned} U^0 &= \bigcup_{n=1}^{\infty} U_n^0, \quad \overline{U_n^0}^Z \subset \overline{U_n^0}^E \subset U^0 \subset Z\setminus L, \\ V^0 &= \bigcup_{n=1}^{\infty} V_n^0, \quad \overline{V_n^0}^Z \subset \overline{V_n^0}^E \subset V^0 \subset Z\setminus K. \end{aligned}$$

(2) Put $K_1 = K\setminus U^0$ and $L_1 = L\setminus V^0$. We have $\mathrm{Int}_E U\Big(z, \frac{1}{n(z)}\Big) \cap K_1 = \emptyset$ for every $z \in K$, $\mathrm{Int}_E U\Big(z, \frac{1}{n(z)}\Big) \cap L_1 = \emptyset$ for every $z \in L$.

For every rational number q in the interval $[0, 2]$ define:

$$\begin{aligned} T(q) &= \{z = (x, y) \in Z\colon\ x+y < q\}, \\ K(q) &= \Big\{z = (x, y) \in K_1\colon\ x+y < q < x+y+\frac{1}{n(z)}\Big\}, \\ L(q) &= \Big\{z = (x, y) \in L_1\colon\ x+y < q < x+y+\frac{1}{n(z)}\Big\}. \end{aligned}$$

The sets $T(q)$ are open-and-closed in Z, $K(q) \cup L(q) \subset T(q)$ and $K_1 = \bigcup_q K(q)$, $L_1 = \bigcup_q L(q)$. By Lemma 3 and (1) it suffices to prove that

(3) for every q there exists a countable family of sets open in Z which covers $K(q)$ (resp. $L(q)$) and is such that the closures of its elements in Z are disjoint with L (resp. K).

By the symmetry of assumptions it is enough to prove (3) for the set $K(q)$.

(4) If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ belong to $K(q)$ and $x_1 < x_2$, then $y_1 \geqslant y_2$.

Indeed, supposing that $y_2 > y_1$, by the definition of $K(q)$ we would have $(x_2, y_2) \in \operatorname{Int}_E U\left(z_1, \frac{1}{n(z_1)}\right)$, which is impossible by (2). Let p_x and p_y denote the projections of $K(q)$ onto the axes X and Y and let $A' = p_x(K(q))$. By (4) if $x_1, x_2 \in A'$, $x_1 < x_2$ and $y_1 \in p_y(p_x^{-1}(x_1))$, $y_2 \in p_y(p_x^{-1}(x_2))$, then $y_1 \geqslant y_2$. It follows that the set $A_0 = \{x \in A' \colon |p_y(p_x^{-1}(x))| > 1\}$ is countable and for every $x \in A = A' \setminus A_0$ there exists exactly one $f(x) \in p_y(p_x^{-1}(x))$. The function $f\colon A \twoheadrightarrow B = f(A)$ is by (4) non-increasing. By Lemma 2 we can find a countable set $B_0 \subset B$ such that $g = f|f^{-1}(B \setminus B_0)\colon f^{-1}(B\setminus B_0) \twoheadrightarrow B \setminus B_0$ is a homeomorphism, where we consider the subsets $C = f^{-1}(B \setminus B_0)$ and $D = B \setminus B_0$ of the sets X and Y with the topology of the subspace of the real line R. It can easily be seen that

$$K(q) = p_x^{-1}(A_0) \cup p_y^{-1}(B_0) \cup W_g,$$

where W_g denotes the graph of the homeomorphism g. As X and Y are hereditarily Lindelöf and $|A_0 \cup B_0| \leqslant \aleph_0$ the set $M = p_x^{-1}(A_0) \cup p_y^{-1}(B_0)$ is Lindelöf and hence the covering $\left\{U\left(z, \frac{1}{n(z)}\right)\right\}_{z \in M}$ of M, open-and-closed in Z, has a countable subcovering.

Thus to prove (3) it suffices to show that the set W_g has a countable open covering in Z consisting of elements whose closures in Z are disjoint with L.

By (i) of Lemma 1 we can find a family $\{F_n\}_{n=1}^{\infty}$ of subsets of C, closed with respect to the topology of X induced by R and such that $C = \bigcup_{n=1}^{\infty} F_n$. Similarly, for every $n = 1, 2, \ldots$ we can choose a family $\{H_{n,m}\}_{m=1}^{\infty}$ of sets closed with respect to the topology of Y induced by R and such that $\bigcup_{m=1}^{\infty} H_{n,m} = g(F_n)$. As g is continuous, the sets $F_{n,m} = F_n \cap g^{-1}(H_{n,m})$ are also closed in the real-line topology of X. Denoting by $W_{n,m}$ the graph of the homeomorphism $g_{n,m} = g|F_{n,m}\colon F_{n,m} \twoheadrightarrow H_{n,m}$, we obviously have $W_g = \bigcup_{n,m=1}^{\infty} W_{n,m}$.

Let us define $U_{n,m} = \left(\bigcup_{z \in W_{n,m}} U\left(z, \frac{1}{n(z)}\right)\right) \cap T(q)$. The family $\{U_{n,m}\}_{n,m=1}^{\infty}$ forms a covering of W_g open in Z and countable. To prove our theorem it suffices to show that the sets $U_{n,m}$ are open-and-closed in Z, i.e. that no point $z = (x, y) \notin U_{n,m}$ belongs to $\overline{U_{n,m}}$. Obviously we can assume that $z \in T(q)$. There are three cases to investigate:

1) $x \in F_{n,m}$, 2) $y \in H_{n,m}$, 3) $x \notin F_{n,m}$ and $y \notin H_{n,m}$.

In the first case we have $g_{n,m}(x) > y$. Indeed, otherwise we would have $z = U\left(z_1, \frac{1}{n(z_1)}\right)$, where $z_1 = (x, g_{n,m}(x))$ belongs to $W_{n,m}$, which is impossible. By the continuity of $g_{n,m}$ there exist $a, b \in R$ such that $g_{n,m}(x) > b > y$, $a > x$ and if $x_0 \in F_{n,m}$, $x \leqslant x_0 < a$, then $g_{n,m}(x_0) > b$. As $g_{n,m}$ is non-increasing, for every $x_0 \in F_{n,m} \cap (-\infty, a)$ we have $g_{n,m}(x_0) > b$. It is easy to check that the neighbourhood $([x, a) \times [y, b)) \cap Z$ of z in Z is disjoint with $U_{n,m}$.

In the second case the proof is analoguous and uses the continuity of $g_{n,m}^{-1}$.

Finally, in the case (3), there exist $a, b, c, d \in R$ such that $c < x < a$, $d < y < b$ and $(c, a) \cap F_{n,m} = \emptyset$, $(d, b) \cap H_{n,m} = \emptyset$. The neighbourhood $([x, a) \times [y, b)) \cap Z$ of z in Z is then disjoint with $U_{n,m}$. This completes the proof of Theorem 1.

Remark. The theorem of D. J. Lutzer [3] implies that the space X^2 is subparacompact (²). By a modification of the proof of Theorem 1 one can show that $\operatorname{Ind} X^2 = \dim X^2 = 0$.

Proof of Theorem 2. Theorem 1 shows that H. 3 implies H. 4. On the other hand, if H. 4 holds, the space $Z = X^2$ must contain an uncountable closed and discrete subset, for otherwise it would be collectionwise normal. As Z is separable and first-countable, H. 2 holds, which completes the proof.

PROBLEMS. Is the existence of a paracompact separable, first-countable space X such that X^2 is normal but not paracompact independent of the axioms of set theory? Is the existence of a paracompact first-countable space X such that X^2 is normal but not collectionwise normal equivalent to the existence of the normal non-metrizable Moore space?

References

[1] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[2] R. W. Heath and E. A. Michael, *A property of the Sorgenfrey line*, Compositio Math. 23 (1971), pp. 185–188.

(²) A topological space Z is called *subparacompact* if every open covering of Z admits a σ-discrete closed refinement. Every collectionwise normal subparacompact space is paracompact.

[3] D. J. Lutzer, *Another property of the Sorgenfrey line*, to appear.
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