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An example of a Φ-algebra whose uniform closure is a ring of continuous functions

by

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1. Introduction. In this note, X will denote an arbitrary completely regular Hausdorff space and $C(X)$ will denote the ring of all real-valued continuous functions on X. A ring which is of the form $C(X)$, for some space X as above, will be referred to as a ring of continuous functions.

In the notation of [3], an archimedean lattice-ordered f-algebra with unity is called a Φ-algebra. In this note, an example is given of a Φ-algebra which is not isomorphic to a ring of continuous functions but whose uniform closure is. Furthermore, this Φ-algebra has the property that its structure space is homeomorphic to the structure space of a ring of continuous functions. This example is used to show that it is possible for non-isomorphic Φ-algebras to have isomorphic maximal ideals, whereas this cannot happen if the Φ-algebras are both rings of continuous functions. (See [4], 6.6.)

2. Preliminaries and notations. For a subset I of $C(X)$, we denote by mI the set of all functions f in $C(X)$ such that $f \in fI$, and we denote by I^m the closure of I in the m-topology. (See [2] for definition.) We denote by I^u the closure of I in the uniform topology. It is proved in [2] that if I is any ring ideal of $C(X)$, then $(mI)^m = I^m$ and $mI = m(I^m)$. If P is a prime ideal in $C(X)$ and M is a maximal ideal containing P, then $P \supseteq mM$.

For a function $f \in C(X)$, we denote $\{x \in X | f(x) = 0\}$ by $Z(f)$.

A maximal ideal M in $C(X)$ is said to be *real in* $C(X)$ if $C(X)/M$ is isomorphic to $\boldsymbol{R}$, the ring of real numbers. If M is not real, we say that M is *hyper-real*.

Discussions of structure spaces can be found in many places such as [1] page 105 and Section 2.3 in [4]. The structure space of $C(X)$ is homeomorphic to βX, the Stone-Čech compactification of X.

For each $f \in C(X)$, there exists a continuous function f^* from βX into the one-point compactification of the reals so that f^* agrees with f on X. (See [1], 7.5.)

3. The example. Let M_0 be any maximal ideal in $C(X)$ and let

$$S = \{f+c \mid f \in M_0 \text{ and } c \in \boldsymbol{R}\}.$$

In the notation of [4], $S = (M_0)$. It is clear that S is a subring of $C(X)$. Of course, $S = C(X)$ if and only if M_0 is real in $C(X)$, so we shall assume for the remainder of this note that M_0 is hyper-real.

3.1. Lemma. *Let I be an ideal in S and let $\bar{I}$ denote $I^m \cap S$. Then $\bar{I}$ is an ideal of S, and $\bar{I}$ is proper if I is.*

Proof. The proof that $\bar{I}$ is an ideal of S is routine. Assume $1 \in \bar{I}$. Then $1 \in I^m$, which implies that there exists $f \in I$ so that $|f-1| < \frac{1}{2}$. Now f is of the form m_0+c for some $m_0 \in M_0$ and $c \in \boldsymbol{R}$, so we must have $c > \frac{1}{2}$. Let $g_0 = \dfrac{1}{m_0+c} - \dfrac{1}{c} \in C(X)$. Since $Z(m_0) \subseteq Z(g_0)$, $g_0 \in M_0$ (see [1] 2.7), whence $g_0 + \dfrac{1}{c} = \dfrac{1}{f} \in S$. But this implies that $1 \in I$.

3.2. Remark. In the proof of 3.1 we have shown that if $s \in S$ with $s(x) \neq 0$ for all $x \in X$ then $1/s \in S$.

3.3. Lemma. *Let K be a maximal ideal in S. Then there exists a maximal ideal M in $C(X)$ such that $K \supseteq M \cap M_0$.*

Proof. Since K is maximal in S, it is prime in S (see [4], 3.3). Let G denote $S \setminus K$, a multiplicative semigroup in $C(X)$. Since $G \cap \{0\} = \emptyset$, there exists a prime ideal P in $C(X)$ such that $G \cap P = \emptyset$ (see [3], p. 6) and hence $K \supseteq P \cap S$. Let M be the maximal ideal in $C(X)$ containing P. We then have

$$K \supseteq mM \cap S \supseteq mM \cap mM_0 = m(M \cap M_0),$$

whence $K^m \supseteq [m(M \cap M_0)]^m = M \cap M_0$. This implies that $S \cap K^m = \bar{K} \supseteq M \cap M_0$, and $\bar{K} = K$ by the maximality of K.

3.4. Theorem. *Let M be a maximal ideal in $C(X)$. Then $M \cap S$ is a maximal ideal of S. Conversely, if K is a maximal ideal of S, then there exists a unique maximal ideal M in $C(X)$ such that $K = S \cap M$.*

Proof. Let M be a maximal ideal in $C(X)$. Then clearly $M \cap S$ is an ideal of S. Assume $M \cap S \nsubseteq I$ where I is an ideal of S. Let $s \in I \setminus M$. Then there exists $m \in M$ and $f \in C(X)$ such that $fs+m = 1$. It is clear that if $M = M_0$ the result is true, so assume $M \neq M_0$. Then there exist $g \in M$ and $g_0 \in M_0$ so that $g+g_0 = 1$. Since $g = -g_0+1 \in M \cap S$, we have that $g \in I$. But $g_0 = g_0 fs + g_0 m$ is also a member of I, whence $1 \in I$.

Conversely, suppose K is a maximal ideal of S. If $K = M_0$, the result follows trivially, so assume $K \neq M_0$. By 3.3, there exists a maximal ideal M in $C(X)$ so that $K \supseteq M \cap M_0$. To complete the proof, it suffices to show that $K \supseteq M \cap S$. (The uniqueness is easily established.) To this end, let $s \in M \cap S$ and assume $s \notin K$. Then for any $m_0 \in M_0$, $sm_0 \in M \cap M_0 \subseteq K$, whence $m_0 \in K$ by the primeness of K. But this says that $M_0 \subseteq K$, contrary to our assumption.

3.5. Remark. If we let $\mathcal{M}(S)$ denote the structure space of S, then the natural mapping $M \to M \cap S$ can be easily shown to be a homeomorphism of the structure space of $C(X)$ onto $\mathcal{M}(S)$.

3.6. Remark. If $M \neq M_0$ then the natural mapping $s+(M \cap S) \to s+M$ can be easily shown to be an isomorphism of $S/M \cap S$ onto $C(X)/M$.

3.7. Remark. In view of 3.6 it is clear that the real maximal ideals of S are M_0 itself and all ideals of the form $M \cap S$ where M is real in $C(X)$. If we let vS denote the set of all real maximal ideals in S and endow vS with the hull-kernel topology, then vS is homeomorphic to the subspace $vX \cup \{M_0\}$ of βX. We then have that $C(vS)$ is isomorphic to $(M_0^u) = \{f+c \mid f \in M_0^u \text{ and } c \in R\}$ using 5.6 in [4]. (In the notation of [4], $X(M_0)$ is homeomorphic to $vX \cup \{M_0\}$ and $C(X(M_0))$ is isomorphic to (M_0^u).)

3.8. Theorem. $S^u = (M_0^u)$.

Proof. Let $s \in (M_0^u)$, say $s = g_0 + c$ where $g_0 \in M_0^u$ and $c \in R$, and let $\varepsilon > 0$ be given. Then there exists $m_0 \in M_0$ such that $|g_0 - m_0| < \varepsilon$, which implies that $|s-(m_0+c)| < \varepsilon$. It follows that $s \in S^u$.

Conversely, suppose $s \in S^u$. Then there exist $m_0 \in M_0$ and $c \in R$ such that $|s-(m_0+c)| < 1$. In particular, s is bounded on $Z(m_0)$, whence $|s+M_0|$ is not infinitely large (see [1], p. 70). Thus there is a real number k such that $s^*(M_0) = k$, whence $(s-k)^*(M_0) = 0$ and $s-k \in M_0^u$. (See Section 2.4 in [4].)

3.9. Theorem. *S is a Φ-algebra.*

Proof. It suffices to show that S is a sublattice of $C(X)$. To this end, let

$$s = (f_1+c_1) \vee (f_2+c_2),$$

where $f_i \in M_0$ and $c_i \in R$. Then $Z(f_1) \cap Z(f_2) \subseteq Z(s - c_1 \vee c_2)$, whence $s - c_1 \vee c_2 \in M_0$ and $s \in S$.

Thus S is a Φ-algebra whose structure space is homeomorphic to the structure space of $C(X)$. Since $(M_0^u) \neq (M_0)$, S is not uniformly closed so it cannot be isomorphic to a ring of continuous functions. However, S^u is isomorphic to a ring of continuous functions. We observe that M_0 is a real ideal in S but a hyper-real ideal in $C(X)$. Of course, the identity isomorphism $M_0 \to M_0$ can not be extended to an isomorphism between S and $C(X)$ (compare with 6.6 in [4]).

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Partial order and collapsibility of 2-complexes (1)

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A partial order $\leqslant$ on a topological space X is said to be *closed* if $\leqslant$ is a closed subset of $X\times X$ (where $X\times X$ has the product topology). An element $\theta\in X$ is called the *zero* of X if and only if $\theta\leqslant x$ for each $x\in X$. For each $x\in X$ the *principal ideal* determined by x is denoted by $L(x)$ and

$$L(x)=\{y\in X|\ y\leqslant x\}.$$

If $A\subset X$ we let

$$L(A)=\bigcup\{L(y)|\ y\in A\}.$$

A compact, Hausdorff space is said to be *acyclic* if and only if it has the Spanier cohomology groups of a space with exactly one point. In [2] it was proved that the Spanier cohomology groups coincide with the Čech cohomology groups on compact Hausdorff spaces. We shall need the following theorem of A. D. Wallace [6]:

THEOREM (Wallace). *Let X be a compact space with a closed partial order. If X has a zero and if all of the principal ideals of X are acyclic then X is acyclic.*

A metric for a metric space X is said to be *strongly convex* if and only if for each pair of distinct points x and y of X there is a unique line segment in X with endpoints x and y.

Warren White proved in [7] that a 2-complex K is collapsible if and only if $|K|$ admits a strongly convex metric ϱ. The following theorem weakens rather dramatically the condition that $|K|$ admit a strongly convex metric.

THEOREM. *Let K be a finite 2-complex. Then K is collapsible if and only if $|K|$ admits a closed partial order with zero and with acyclic principal ideals.*

Proof. ($\Rightarrow$) If K is collapsible then by White's theorem [7] $|K|$ admits a strongly convex metric ϱ. Let $\theta\in|K|$. Define $x\leqslant y$ in $|K|$ if and only

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