predicate symbols be interpreted on $|\mathcal{A}_0'|$ so as to make $\mathcal{A}_0$ admissible and f_0 an isomorphism of $\mathcal{A}_0|L(A)$ and $\mathcal{A}_0'|L(A)$. Let R_0' be the unique equivalence relation on $(|\mathcal{A}_0'|-p(\mathcal{A}_0'))\times\omega$ such that R_0 is isomorphic to R_0' via f_0. Then f_0 induces an $L(A)$-isomorphism between $(\mathcal{A}_0, R_0)$ and $(\mathcal{A}_0', R_0')$. By the induction hypothesis $(\mathcal{A}_0', R_0')\Vdash A[i_1',\ldots,i_k']$ since $(\mathcal{A}_0, R_0)\Vdash A[i_1,\ldots,i_k]$. Thus some extension of $(\mathcal{A}', R')$ forces $A[i_1',\ldots,i_k']$. Since f^{-1} induces an $L(A)$-isomorphism between $(\mathcal{A}', R')$ and $(\mathcal{A}, R)$, if some extension of $(\mathcal{A}', R')$ forces $A[i_1',\ldots,i_k']$ then some extension of $(\mathcal{A}, R)$ forces $A[i_1,\ldots,i_k]$. This completes the proof of the lemma.

Using the lemma we can now prove that $\mathcal{B}\lesssim\mathcal{B}'$. We first observe that if $A\in S_0(L)$ then $\mathcal{B}'(A)=T$ if and only if $(\mathcal{A}_n, R_n)\Vdash A$ for some $n\in\omega$. Let $A\in S_1(L)$ and suppose that $(\mathcal{A}, R)\Vdash A[\omega]$. We define an extension $(\mathcal{A}', R')$ of $(\mathcal{A}, R)$ as follows. Let $|\mathcal{A}'|=|\mathcal{A}|\cup\{a\}$ where $a=(n, m)$ is chosen in $\omega\times\omega-|\mathcal{A}|$ so that $q_m\notin L(A)$. Let the predicate symbols be interpreted on $|\mathcal{A}'|$ so that $\mathcal{A}'$ is admissible and so that $f\colon |\mathcal{A}'|\to|\mathcal{A}'|$ is an automorphism of $\mathcal{A}'|L(A)$, where $f(\omega)=a$, $f(a)=\omega$, and f is the identity on $|\mathcal{A}'|-\{a,\omega\}$. Let R' be the least equivalence relation on $(|\mathcal{A}'|-p(\mathcal{A}'))\times\omega$ which extends R and which is such that (a, m) and (ω, m) are R'-equivalent for each m such that either the R-equivalence class of (ω, m) has power >1 or $q^m\in L(A)$. Now f induces an $L(A)$-automorphism of $(\mathcal{A}', R')$. It follows from the lemma that $(\mathcal{A}', R')\Vdash A[i]$ since $(\mathcal{A}', R')\Vdash A[\omega]$. Thus if $A[\omega]$ is true in $\mathcal{B}'$ so is $A[i]$ for some $a\neq\omega$. This demonstrates that $\mathcal{B}\lesssim\mathcal{B}'$.

References

[1] J. T. Baldwin, *Countable theories categorical in uncountable power*, Ph. D. thesis, Simon Fraser University 1971.

[2] — and A. H. Lachlan, *On strongly minimal sets*, J. Symb. Logic 36 (1971), pp. 79–96.

[3] A. H. Lachlan, *The transcendental rank of a theory*, Pacific J. Math. 37 (1971), pp. 119–122.

[4] M. D. Morley, *Categoricity in power*, Trans. Amer. Math. Soc. 114 (1965), pp. 514–538.

[5] — and R. L. Vaught, *Homogeneous universal models*, Math. Scandinavica 2 (1962), pp. 37–57.

[6] S. Shelah, *Stable theories*, Israel J. Math. 7 (1969), pp. 187–202.

[7] — *Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory*, Ann. Math. Logic 3 (1971), pp. 271–362.

[8] J. R. Shoenfield, *Mathematical Logic*, Reading 1967.

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An isomorphism theorem of the Hurewicz-type in Borsuk's theory of shape

by

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Introduction. In Hurewicz's well-known paper [6] is a homomorphism φ defined from the nth homotopy group $\pi_n(X)$ into the nth singular homology group $H_n(X)$ with integral coefficients, for any compact, pathwise-connected space X, and it is proved there (for $n\geqslant 2$) that if the space X is $(n-1)$-connected (that is, if $\pi_1(X)\approx\pi_2(X)\approx\ldots\ldots\approx\pi_{n-1}(X)\approx 0$), then the homomorphism φ is an isomorphism.

In this note an analogous homomorphism with similar properties will be constructed on the ground of Borsuk's theory of shape (introduced in [1]).

The singular homology groups of a pointed compactum (X, x_0) will be replaced by the Vietoris–Čech homology groups of (X, x_0), denoted by $\check{H}_q(X, x_0)$, and the homotopy groups $\pi_q(X, x_0)$ will be replaced by the so called fundamental groups $\underline{\pi}_q(X, x_0)$, defined by K. Borsuk (see also [1]). In the general case, Hurewicz's assumption of the $(n-1)$-connectednes of X will be replaced by approximative q-connectedness (for $q=0,1,\ldots\ldots,n-1$) of (X, x_0) (see for instance [2], p. 266, or Definition 3.1 in this paper). But if the pointed compactum (X, x_0) is connected and movable (see [2], § 4) then the assumption of the approximative q-connectedness (for $q=0,1,\ldots,n-1$) is equivalent to $\underline{\pi}_1(X, x_0)\approx\underline{\pi}_2(X, x_0)\approx\ldots\ldots\approx\underline{\pi}_{n-1}(X, x_0)$.

§ 1 of this paper contains a modified definition of the homology groups $\check{H}_q(X, x_0)$ and a proof of the equivalence of this definition to the original Vietoris definition. § 2 contains a construction of a homomorphism $\underline{\varphi}\colon \underline{\pi}_n(X, x_0)\to\check{H}_n(X, x_0)$, called the limit Hurewicz homomorphism. In § 3 the following theorem is proved:

If the pointed compactum (X, x_0) is approximatively q-connected for $q=0,1,\ldots,n-1$ $(n\geqslant 2)$, then the limit Hurewicz homomorphism $\underline{\varphi}$ is an isomorphism.

§ 1. The groups $\Gamma_q(X, x_0)$. Let Q denote the Hilbert-cube, X — a non-empty closed subset of Q and x_0 — a point lying in X. For any positive real number ε, the open neighbourhood of X consisting of all points $x \in Q$ with $\varrho(x, X) < \varepsilon$ will be denoted by U^ε. The term "mapping" will always denote continuous mapping.

In this paper we will base ourselves on the definition of the singular homology groups and all concepts related to this definition, described in [5], chapter VII.

1.1. Definition. A singular q-simplex $T: \Delta_q \to Q$ such that the set $T(\Delta_q)$ is contained in a set $A \subset Q$ is said to be *lying in* A. If $A = U^\varepsilon$, then T is called an (ε, q)-*simplex*. If $A = \{x_0\}$, then T is said to be *lying at* x_0. Now, let $a = \sum_{i=1}^{m} a_i T_i$ be a singular q-chain of Q (a_i-integers, T_i-singular q-simplexes). The chain a is said to be *lying in* A (resp. *at* x_0) if each T_i is lying in A (resp. at x_0); it is said to be an (ε, q)-*chain* if each T_i is an (ε, q)-simplex, and it is said to be *smaller* than $\delta > 0$ whenever for each T_i, the image $T_i(\Delta_q)$ is of diameter less than δ.

1.2. Definition. Let $\lambda = \{\lambda_k\}$ be a sequence of singular chains. The sequence λ is called an *infinite singular* q-*chain of* X if there exists a sequence $\{\varepsilon_k\}$ of positive real numbers converging to zero such that λ_k is an (ε_k, q)-chain. The infinite singular q-chain $\lambda = \{\lambda_k\}$ is said to be *lying at* x_0 if each λ_k is lying at x_0.

The addition of two infinite singular q-chains $\lambda = \{\lambda_k\}$ and $\mu = \{\mu_k\}$ is defined by the formula $\lambda + \mu = \{\lambda_k + \mu_k\}$; the set of all infinite singular q-chains with this operation is an Abelian group.

If $\lambda = \{\lambda_k\}$ is an infinite singular q-chain of X, then the sequence $\{\partial\lambda_k\}$ is an infinite singular $(q-1)$-chain of X; this chain will be denoted by $\partial\lambda$. If $\partial\lambda$ is lying at x_0, then λ will be called an *infinite singular* q-*cycle of the pair* (X, x_0).

An infinite singular q-cycle a of (X, x_0) is said to be *homologous to zero in* (X, x_0) (written $a \sim 0$) if there exists an infinite singular $(q+1)$-chain λ such that $a - \partial\lambda$ is lying at x_0.

An infinite singular q-cycle $a = \{a_k\}$ of (X, x_0) is called a *fundamental* q-*cycle of* (X, x_0) whenever the infinite singular cycle $\beta = \{a_k - a_{k+1}\}$ is homologous to zero in (X, x_0). Fundamental cycles of (X, x_0) will be denoted by underlined Greek letters, e.g. $\underline{a} = \{a_k\}$. It is easy to see, that the set of all fundamental q-cycles of (X, x_0) is a subgroup of the group of all infinite singular q-chains; this subgroup will be denoted by $\underline{Z}_q(X, x_0)$. The subgroup of $\underline{Z}_q(X, x_0)$ consisting of all infinite singular q-cycles of (X, x_0) which are homologous to zero in (X, x_0) will be denoted by $\underline{B}_q(X, x_0)$. Let $\Gamma_q(X, x_0)$ denote the factor group $\underline{Z}_q(X, x_0)/\underline{B}_q(X, x_0)$.

The qth Vietoris homology group $\check{H}_q(X, x_0)$ of the pointed compactum (X, x_0) with integers as the coefficients, is usually defined as follows (compare [3], chap. II, § 3):

Let ε be a positive number. A q-dimensional ε-simplex lying in X is an ordered set of $q+1$ points of X (called the *vertices of the* ε-*simplex*) with diameter less than ε. An ε-simplex is said to be *lying at* x_0, if any vertex of it is equal to x_0. A formal finite linear combination $a = \sum_{i=1}^{m} b_i \sigma_i$ of q-dimensional ε-simplexes σ_i with integral coefficients b_i is called a q-*dimensional* ε-*chain in* X. If any ε-simplex of this combination is lying at x_0, then the ε-chain a is said to be *lying at* x_0.

For any two q-dimensional ε-chains $a = b_1\sigma_1 + \ldots + b_m\sigma_m$ and $a' = b_1'\sigma_1' + \ldots + b_k'\sigma_k'$ define the sum $a+\beta$ as the q-dimensional ε-chain $b_1\sigma_1 + \ldots + b_m\sigma_m + b_1'\sigma_1' + \ldots + b_k'\sigma_k'$. This addition is a group operation on the set of all q-dimensional ε-chains. Equivalently, the group of all q-dimensional ε-chains can be defined as the free Abelian group generated by the set of all q-dimensional ε-simplexes.

Let $\sigma = (v_0, v_1, \ldots, v_q)$ be a q-dimensional ε-simplex. The $(q-1)$-dimensional ε-chain $\partial\sigma = \sum_{i=0}^{q} (-1)^i (v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_q)$ is called the *boundary of* σ. Then one extends linearly the boundary-operation ∂ on the whole group of q-dimensional ε-chains, i.e. $\partial\left(\sum_{i=1}^{m} b_i\sigma_i\right) = \sum_{i=1}^{m} b_i \partial\sigma_i$.

A sequence $\gamma = \{\gamma_k\}$ of q-dimensional ε_k-chains, where ε_k converges to zero, is called an *infinite* q-*dimensional chain* (in the Vietoris sense). If each γ_k is lying at x_0, then the infinite q-dimensional chain γ is said to be *lying at* x_0. For any two infinite q-dimensional chains $\gamma = \{\gamma_k\}$ and $\gamma' = \{\gamma_k'\}$ define the sum $\gamma + \gamma'$ as the infinite q-dimensional chain $\{\gamma_k + \gamma_k'\}$. The set of all infinite q-dimensional chains together with this addition is an Abelian group.

For an infinite q-dimensional chain $\gamma = \{\gamma_k\}$, the infinite $(q-1)$-dimensional chain $\{\partial\gamma_k\}$ is called the *boundary of* γ and denoted by $\partial\gamma$. An infinite q-dimensional chain γ is said to be an *infinite* q-*dimensional cycle in* (X, x_0), if $\partial\gamma$ is lying at x_0. An infinite q-dimensional cycle γ is said to be *homologous to zero in* (X, x_0), if there is an infinite $(q+1)$-dimensional chain $\varkappa$ such that the infinite q-dimensional chain $\partial\varkappa - \gamma$ is lying at x_0. An infinite q-dimensional cycle $\gamma = \gamma_k$ in (X, x_0) is called a q-*dimensional true cycle in* (X, x_0), if the infinite cycle $\{\gamma_k - \gamma_{k+1}\}$ is homologous to zero in (X, x_0).

Let $Z_q(X, x_0)$ denotes the group of all q-dimensional true cycles in (X, x_0) and $B_q(X, x_0)$ the group of all q-dimensional infinite cycles in (X, x_0), which are homologous to zero in (X, x_0). Certainly, $B_q(X, x_0)$ is a subgroup of $Z_q(X, x_0)$. The quotient group $Z_q(X, x_0)/B_q(X, x_0)$ is denoted by $\check{H}_q(X, x_0)$ and called the qth *homology group* (in the Vietoris sense) of the pointed compactum (X, x_0), with integers, as the coefficients.

1.3. THEOREM. *The groups $\Gamma_q(X, x_0)$ and $\check{H}_q(X, x_0)$ are isomorphic.*

Proof. Let $p\colon Q \to X$ be a (not necessary continuous) function such that $\varrho(x, p(x)) = \varrho(x, X)$ for each $x \in Q$.

Let $T\colon \Delta_r \to Q$ be an $(\frac{1}{3}\varepsilon, r)$-simplex of diameter less than $\frac{1}{3}\varepsilon$ and let $d^0, d^1, \ldots, d^r$ denote the vertices of Δ_r. Then the system

$$\big(p(T(d^0)), p(T(d^1)), \ldots, p(T(d^r))\big)$$

is an r-dimensional ε-simplex (in the Vietoris sense) lying in X, let it be denoted by $P(T)$. Now, if $\lambda = \sum_{i=1}^{m} a_i T_i$ is an $(\frac{1}{3}\varepsilon, r)$-chain smaller than $\frac{1}{3}\varepsilon$, then $\sum_{i=1}^{m} a_i P(T_i)$ is an r-dimensional ε-chain (in the Vietoris sense) lying in X; let it be denoted by $P(\lambda)$. Observe, that $P(\lambda_1 + \lambda_2) = P(\lambda_1) + P(\lambda_2)$ and $P(\partial\lambda) = \partial P(\lambda)$. Observe also, that for any $(\frac{1}{3}\varepsilon, r)$-cycle λ there exists an $(\frac{1}{3}\varepsilon, r)$-cycle λ which is smaller than $\frac{1}{3}\varepsilon$ such that λ and λ' are homologous in $(U^{\varepsilon/3}, x_0)$ (for instance, λ' can be obtained as the result of an iterated barycentric subdivision of λ; compare Theorem 8.2 in [4], chap. VII, p. 197).

Let $\underline{\alpha}$ be a fundamental q-cycle in (X, x_0). There exists an element $\underline{\alpha}' = \{\alpha_k\}$ of the homology class $[\underline{\alpha}]$ such that α_k is an $(\frac{1}{3}\varepsilon_k, q)$-cycle smaller than $\frac{1}{3}\varepsilon_k$ (where $0 < \varepsilon_k \to 0$ for $k = 1, 2, \ldots$). For this element $\underline{\alpha}'$, define $\underline{P}(\underline{\alpha}') = \{P(\alpha_k)\}$. The sequence $\{P(\alpha_k)\}$ is an infinite cycle (in the Vietoris sense) in (X, x_0). In order to prove, that it is a true cycle, observe, that there exists an infinite singular $(q+1)$-chain $\mu = \{\mu_k\}$ such that $(\alpha_k - \alpha_{k+1}) - \partial\mu_k$ is lying at x_0, since $\underline{\alpha}' = \{\alpha_k\}$ is a fundamental q-cycle in (X, x_0). Moreover, the sequence μ can be assumed to be such that μ_k is a $(\frac{1}{3}\delta_k, q+1)$-chain smaller than $\frac{1}{3}\delta_k$, where $0 < \delta_k \to 0$. Therefore $P(\alpha_k) - P(\alpha_{k+1}) - \partial P(\mu_k)$ is lying at x_0, where $P(\mu_k)$ is a δ_k-chain (in the Vietoris sense), and the sequence $\{P(\alpha_k)\}$ is a (Vietorian) true cycle. An analogous argumentation shows, that the homology class $[\underline{P}(\underline{\alpha}')]$ does not depend of the choice of $\underline{\alpha}'$ in the homology class $[\underline{\alpha}]$. Thus, the formula $\omega([\underline{\alpha}]) = [\underline{P}(\underline{\alpha}')]$ defines a homomorphism $\omega\colon \Gamma_q(X, x_0) \to \check{H}_q(X, x_0)$ $(q = 0, 1, \ldots)$.

Now, let $\sigma = (v_0, v_1, \ldots, v_r)$ be an r-dimensional (Vietorian) ε-simplex in X. The linear mapping $T\colon \Delta_r \to Q$ with $T(d^i) = v_i$ for $0 \leqslant i \leqslant q$ is a singular (ε, r)-simplex such that $P(T) = \sigma$. Denote this singular simplex by $F(\sigma)$. If $\gamma = \sum_{i=1}^{m} b_i \sigma_i$ is an r-dimensional (Vietorian) ε-chain, then $F(\gamma) = \sum_{i=1}^{m} b_i F(\sigma_i)$ is a singular (ε, r)-chain such that $P(F(\gamma)) = \gamma$. Thus, if $\varkappa = \{\varkappa_k\}$ is an infinite (Vietorian) r-dimensional chain, then $\underline{F}(\varkappa) = \{F(\varkappa_k)\}$ is an infinite singular r-chain such that $\underline{P}(\underline{F}(\varkappa)) = \varkappa$. Observe that $\underline{F}(\partial\varkappa) = \partial\underline{F}(\varkappa)$, and $\underline{F}(\varkappa' + \varkappa'') = \underline{F}(\varkappa') + \underline{F}(\varkappa'')$. This shows that if $\underline{\gamma}$ is a true (Vietorian) q-dimensional cycle, then $\underline{F}(\underline{\gamma})$ is a fundamental q-cycle such that $\underline{P}(\underline{F}(\underline{\gamma})) = \underline{\gamma}$. This proves that ω is an epimorphism.

To prove that ω is a monomorphism, suppose that $\alpha = \sum_{j=1}^{m} a_j T_j$ is a singular $(\frac{1}{3}\varepsilon, q)$-cycle smaller than $\frac{1}{3}\varepsilon$. Then $FP(\alpha)$ is a singular (ε, q)-cycle; moreover, the cycles α and $FP(\alpha)$ are homologous in (U^ε, x_0). In fact: $FP(\alpha) = \sum_{j=1}^{m} a_j T'_j$ where $T'_j\colon \Delta_q \to Q$ is the linear map with $T'_j(d^i) = pT_j(d^i)$ for $0 \leqslant i \leqslant q$ and each $1 \leqslant j \leqslant m$, and the cycle $\sum_{j=1}^{m} a_j(T_j - T'_j)$ is equal to the boundary of the $(q+1)$-dimensional prism-chain (see [5], chap. VII, § 6–7) $DT = \sum_{j=1}^{m} a_j DT_j$ where the mapping $DT_j\colon I \times \Delta_q \to Q$ for each j is defined by the formula

$$DT_j(t, v) = tT_j(v) + (1-t)T'_j(v) \quad \text{for} \quad t \in I,\ v \in \Delta_q$$

(the sign $+$ denotes here addition in the linear Hilbert space $H \supset Q$). Therefore, if $\underline{\alpha}$ is a fundamental q-cycle, then $\underline{\alpha}$ and $\underline{FP}(\underline{\alpha})$ are homologous in (X, x_0) and, in particular, if $\underline{P}(\underline{\alpha})$ is homologous to zero in (X, x_0), then $\underline{\alpha}$ is also homologous to zero in (X, x_0), i.e. ω is a monomorphism, which completes the proof.

§ 2. The limit Hurewicz homomorphism. The fundamental groups $\underline{\pi}_n(X, x_0)$ are defined by K. Borsuk ([1], pp. 246–252) as follows:

Let X denote, as before, a closed subset of the Hilbert-cube Q and let $x_0 \in X$. The q-sphere will be denoted by S^q; Let $s_0 \in S^q$ be a base-point of S^q. A sequence of pointed mappings $\xi_k\colon (S^q, s_0) \to (Q, x_0)$ $(k = 1, 2, \ldots)$ will be called an *approximative map of* (S^q, s_0) *towards* (X, x_0) whenever for any neighbourhood U of X the pointed homotopy $\xi_k \simeq \xi_{k+1}$ in (U, x_0) holds for almost all k. This approximative map is denoted by $\{\xi_k, (S^q, s_0) \to (X, x_0)\}$ or, more briefly, by $\underline{\xi}$.

Two approximative maps

$$\underline{\xi} = \{\xi_k, (S^q, s_0) \to (X, x_0)\} \quad \text{and} \quad \underline{\xi}' = \{\xi'_k, (S^q, s_0) \to (X, x_0)\}$$

are said to be *homotopic* (written $\underline{\xi} \simeq \underline{\xi}'$) whenever for any neighbourhood U of X the pointed homotopy $\xi_k \simeq \xi'_k$ in (U, x_0) holds for almost all k, i.e., whenever the "mixed" sequence $\{\xi_1, \xi'_1, \xi_2, \xi'_2, \ldots\}$ is an approximative map of (S^q, s_0) towards (X, x_0). The equivalence class of the approximative map $\underline{\xi}$ under the relation $\simeq$ is called the *homotopy class of* $\underline{\xi}$ or the *approximative class from* (S^q, s_0) *towards* (X, x_0), represented by the approximative map $\underline{\xi}$.

Now, let $[\underline{\xi}]$ and $[\underline{\eta}]$ be approximative classes, where $\underline{\xi} = \{\xi_k, (S^q, s_0) \to (X, x_0)\}$ and $\underline{\eta} = \{\eta_k, (S^q, s_0) \to (X, x_0)\}$. Define $[\underline{\xi}] + [\underline{\eta}] = \{\xi_k + \eta_k, (S^q, s_0)$

$\to(X, x_0)\}$, where $\xi_k + \eta_k$ is the homotopic sum (see [1], p. 250, the join of maps) of the mappings ξ_k and η_k. The set of all approximative classes from (S^q, s_0) towards (X, x_0) with the above addition is a group, called the *q-th fundamental group of the pointed compact space* (X, x_0) and denoted by $\underline{\pi}_q(X, x_0)$.

To define the limit Hurewicz homomorphism $\underline{\varphi}$: $\underline{\pi}_q(X, x_0) \to \check{H}_q(X, x_0)$ (for $q \geqslant 1$), let us take an approximative class $[\underline{\xi}] \in \underline{\pi}_q(X, x_0)$ represented by an approximative map $\underline{\xi} = \{\xi_k, (S^q, s_0) \to (X, x_0)\}$. It follows by the definition of the approximative maps that there exists a sequence ε_k of positive numbers which converges to zero and is such that $\xi_k \simeq \xi_{k+1}$ in (U^{ε_k}, x_0), where U^{ε_k} denotes, as before, the ε_k-neighbourhood of X in Q. In particular, $\xi_k(S^q) \subset U^{\varepsilon_k}$ and therefore ξ_k can be considered as a mapping of the pair (S^q, s_0) into the pair (U^{ε_k}, x_0). Let ξ_{k*}: $H_q(S^q, s_0) \to H_q(U^{\varepsilon_k}, x_0)$ denote the homomorphism of the singular homology groups, induced by ξ_k and let e be a fixed generator of the group $H_q(S^q, s_0)$. The element $\xi_{k*}(e) \in H_q(U^{\varepsilon_k}, x_0)$ is the homology class of a singular q-dimensional cycle a_k in (U^{ε_k}, x_0) (i.e. of a singular (ε_k, q)-cycle a_k). The cycles a_k and a_{k+1} are homologous in (U^{ε_k}, x_0) since the homotopy $\xi_k \simeq \xi_{k+1}$ holds in (U^{ε_k}, x_0). Thus, the sequence $\{a_k\}$ is a fundamental q-cycle of (X, x_0).

It is easy to see that the element $[\{a_k\}]$ of the group $\Gamma_q(X, x_0)$ does not depend either on the choice of the element $\underline{\xi}$ of the class $[\underline{\xi}] \in \underline{\pi}_n(X, x_0)$ or on the singular (ε_k, q)-cycles a_k representing the elements $\xi_{k*}(e)$, $k = 1, 2, \ldots$ Thus, the formula $\psi[\underline{\xi}] = [\{a_k\}]$ defines a function ψ: $\underline{\pi}_q(X, x_0) \to \Gamma_q(X, x_0)$. The function ψ is a homomorphism (observe the analogy between the definition of ψ and the classical definition of the Hurewicz homomorphism φ). The composed homomorphism $\underline{\varphi} = \omega \circ \psi$: $\underline{\pi}_q(X, x_0) \to \check{H}_q(X, x_0)$ will be called the *limit Hurewicz homomorphism* (the isomorphism ω: $\Gamma_q(X, x_0) \to \check{H}_q(X, x_0)$ is defined in § 1).

§ 3. The main theorem.

3.1. Definition. The pair (X, x_0) is called *approximatively q-connected* $(q = 0, 1, 2, \ldots)$, whenever for any neighbourhood U of X there is a neighbourhood V of X such that each mapping f: $(S^q, s_0) \to (V, x_0)$ is inessential in (U, x_0) (see [2], p. 266).

3.2. Theorem. *If the pointed compactum* (X, x_0) *is approximatively q-connected for all* $q = 0, 1, \ldots, n-1$ $(n \geqslant 2)$, *then the limit Hurewicz homomorphism* $\underline{\varphi}$: $\underline{\pi}_n(X, x_0) \to \check{H}(X, x_0)$ *is an isomorphism.*

The proof of this theorem will be preceded by three lemmas and some definitions.

A singular simplex T in (Q, x_0) will be called *reduced* whenever each s-dimensional face of T (for any $s \leqslant n-1$) lies at x_0. A singular chain (resp. cycle) which is a linear combination of reduced singular simplexes will be called a *reduced singular chain* (resp. *cycle*).

Let us remember that the ith $(q-1)$-dimensional face $T^{(i)}$ of the singular q-dimensional simplex T: $\Delta_q \to Q$ $(0 \leqslant i \leqslant q)$ is in [5] (chap. VII) defined as the composition $T^{(i)} = T \circ e_q^i$, where e_q^i is the suitable simplicial inclusion of Δ_{q-1} into Δ_q. Let us write $\Delta_q^{(i)} = e_q^i(\Delta_{q-1})$ and $\dot{\Delta}_q = \bigcup_{i=0}^{q} \Delta_q^{(i)}$.

For any subset U of Q, the group of all q-dimensional singular chains lying in U will be denoted by $C_q(U)$.

Observe that for any homotopy h: $I \times A \to B$ (where I is the unit interval $[0, 1]$ and A, B are topological spaces) there is a (unique) corresponding family of mappings h_t: $A \to B$ $(0 \leqslant t \leqslant 1)$ defined by the formula $h_t(x) = h(t, x)$, $x \in A$, $t \in I$. Thus, the homotopy h will be sometimes considered as the family $\{h_t\}$.

3.3. Lemma. *If the pair* (X, x_0) *is approximatively s-connected for all* $s = 0, 1, \ldots, n-1$ $(n \geqslant 2)$, *then for any neighbourhood* $U \supset X$ *there exists a neighbourhood* $U_0 \supset X$ *such that to each singular q-simplex* T: $\Delta_q \to U_0$ $(q = 0, 1, 2, \ldots)$ *can be assigned a homotopy* h^T: $I \times \Delta_q \to U$ *such that the following conditions are satisfied:*

(i) $h_0^T = T$;

(ii) *If* $T^{(i)}$ *is the i-th face of* T $(0 \leqslant i \leqslant q)$, *then* $h_t^{T^{(i)}}$ *is the i-th face of the simplex* h_t^T *for each* $t \in I$, *i.e.* $h_t^{T^{(i)}} = h_t^T \circ e_q^i$;

(iii) h_1^T *is a reduced singular simplex;*

(iv) *If* T *is a reduced singular simplex, then* $h_t^T = T$ *for each* $t \in I$.

Proof. Observe first that approximative s-connectedness for all $s = 0, 1, \ldots, n-1$ implies that

(*) there is a system of neighbourhoods $U_0, U_1, \ldots, U_n$ with $X \subset U_0 \subset U_1 \subset \ldots \subset U_n = U$ such that each mapping f: $(S^q, s_0) \to (U_q, x_0)$ is inessential in (U_{q+1}, x_0) for $q = 0, 1, \ldots, n-1$.

The homotopy h^T will be defined inductively with respect to the dimension of the singular simplex T.

1° Let T be a 0-dimensional singular simplex. If T maps Δ_0 into x_0, then let h^T be the constant map into x_0. Otherwise, let h^T be a homotopy in U_1 (U_1 as defined in (*)) such that $h_0^T = T$ and h_t^T is the constant map into x_0 for any $t \in [1/n, 1]$.

2° Suppose that $1 \leqslant q \leqslant n-1$ and the homotopies h^T are defined for all singular simplexes T of dimension $\leqslant q-1$, so that the conditions (i)–(iv) are satisfied and, moreover, it T is of dimension $q-1$, then for any $t \in [q/n, 1]$ h_t^T is the constant map into x_0.

Let T: $\Delta_q \to U_0$ be a q-dimensional singular simplex. If T is reduced, then let $h_t^T = T$, for each $t \in I$. Suppose now that T is not reduced. Let $T^{(i)}$: $\Delta_{q-1} \to U_0$ and $T^{(j)}$: $\Delta_{q-1} \to U_0$ denote respectively the ith and the jth face of T. Observe that if $T'\colon \Delta_{q'} \to U_0$ is a common face of $T^{(i)}$ and $T^{(j)}$, i.e. if there are simplicial inclusions $e'\colon \Delta_{q'} \to \Delta_{q-1}$ and $f'\colon \Delta_{q'} \to \Delta_{q-1}$ such that $T' = T^{(i)} \circ e' = T^{(j)} \circ f'$, then by (ii) the equality

$$h^{T^{(i)}}|_{I\times e'(\Delta_{q'})} = h^{T^{(j)}}|_{I\times f'(\Delta_{q'})} \text{ holds.}$$

Thus, a continuous function

$$g\colon \left[0, \frac{q}{n}\right] \times \dot{\Delta}_q \cup \{0\} \times \Delta_q \to U_q$$

is well defined by the following formula:

$$g(t, v) = \begin{cases} h_t^{T^{(i)}}(w) & \text{if } v = e_q^i(w) \text{ for some i and } w \in \Delta_{q-1}, \\ T(v) & \text{if } t = 0. \end{cases}$$

It is easy to see that there is a retraction

$$r\colon \left[0, \frac{q}{n}\right] \times \Delta_q \to \left[0, \frac{q}{n}\right] \times \dot{\Delta}_q \cup \{0\} \times \Delta_q;$$

let us write

$$h' = r \circ g\colon \left[0, \frac{q}{n}\right] \times \Delta_q \to U_q .$$

It follows by the inductive assumption that

$$h'\left(\left\{\frac{q}{n}\right\} \times \dot{\Delta}_q\right) = \{x_0\},$$

and by (*) there exists a homotopy

$$h''\colon \left[\frac{q}{n}, \frac{q+1}{n}\right] \times \Delta_q \to U_{q+1}$$

such that

$$h''|_{\{q/n\}\times\Delta_q} = h'|_{\{q/n\}\times\Delta_q} \quad \text{and} \quad h''\left(\left[\frac{q}{n}, \frac{q+1}{n}\right] \times \dot{\Delta}_q \cup \left\{\frac{q+1}{n}\right\} \times \Delta_q\right) = \{x_0\}.$$

The homotopy $h^T\colon I \times \Delta_q \to U_{q+1}$, defined by the following formula

$$h^T(t, v) = \begin{cases} h'(t, v) & \text{for } t \in \left[0, \frac{q}{n}\right], \\ h''(t, v) & \text{for } t \in \left[\frac{q}{n}, \frac{q+1}{n}\right], \\ x_0 & \text{for } t \in \left[\frac{q+1}{n}, 1\right], \end{cases}$$

satisfies the conditions (i)–(iv) and, moreover, $h^T(I\times\Delta_q) \subset U_{q+1}$ and $h_t^T(v) = x_0$ if $\frac{q+1}{n} \leqslant t \leqslant 1$.

3° Suppose now that $q \geqslant n$ and assume that the homotopy $h^T\colon I \times \Delta_{q-1} \to U$ is defined for each $(q-1)$-dimensional singular simplex T such that the conditions (i)–(iv) hold. Now, let $T\colon \Delta_q \to U_0$ be a q-dimensional simplex. If T is reduced, then define h^T by $h_t^T = T$ for each $t \in I$. Suppose that T is not reduced. Then, as in 2°, the homotopies $h^{T^{(i)}}$ $(i = 0, 1, \ldots, q)$ together with the mapping T yield a mapping $g\colon I \times \dot{\Delta}_q \cup \{0\} \times \Delta_q \to U$ which can be extended to a mapping $h^T\colon I\times\Delta_q \to U$, since $I \times \dot{\Delta}_q \cup \{0\} \times \Delta_q$ is a retract of $I \times \Delta_q$. The family $\{h^T\}$ of all homotopies obtained by this method satisfies the required conditions.

3.4. Lemma. *If the pair (X, x_0) is approximatively s-connected for all $s = 0, 1, \ldots, n-1$ $(n \geqslant 2)$, then for any neighbourhood U of X there is a neighbourhood U_0 of X and for any $q = 0, 1, \ldots, n+1$ there is a homomorphism $\tau_q\colon C_q(U_0) \to C_q(U)$ such that the following conditions are satisfied:*

(a) *for any $\lambda \in C_q(U_0)$ the singular chain $\tau_q(\lambda)$ is reduced,*

(b) *if the chain $\lambda \in C_q(U_0)$ is reduced, then $\tau_q(\lambda) = \lambda$,*

(c) $\partial\tau_q(\lambda) = \tau_{q-1}(\partial\lambda)$ *for any $\lambda \in C_q(U_0)$, $q = 1, 2, \ldots, n+1$,*

(d) *if the chain $a \in C_q(U_0)$ is a cycle* (mod x_0), *then the cycles a and $\tau_q(a)$ are homologous in (U, x_0).*

Proof. The set of all singular q-dimensional simplexes in U_0 is a system of generators of the group $C_q(U_0)$; hence it is sufficient to define the function τ_q on any singular q-dimensional simplex $T\colon \Delta_q \to U_0$ and then to extend it (linearly) to a homomorphism of the whole group $C_q(U_0)$. Let h^T be the homotopy obtained by Lemma 3.3. Define $\tau_q(T) = h_1^T$. The singular simplex h_1^T lies in U; therefore $\tau_q\colon C_q(U_0) \to C_q(U)$. The conditions (a) and (b) are evidently satisfied. To prove condition (c) it is sufficient to verify it for the chain $\lambda = T$, where $T\colon \Delta_q \to U_0$ is a singular q-simplex. On the one hand,

$$\partial_q \tau_q(T) = \partial h_1^T = \sum_{i=0}^{q} (-1)^i h_1^T \circ e_q^i;$$

on the other hand,

$$\tau_{q-1}(\partial_q T) = \tau_{q-1}\left(\sum_{i=0}^{q} (-1)^i T \circ e_q^i\right) = \tau_{q-1}\left(\sum_{i=0}^{q} (-1)^i T^{(i)}\right)$$
$$= \sum_{i=0}^{q} (-1)^i \tau_q(T^{(i)}) = \sum_{i=0}^{q} (-1)^i h_1^{T^{(i)}}.$$

Condition (ii) of Lemma 3.3 yields the equality $h_1^{T^{(i)}} = h_1^T \circ e_q^i$; therefore $\partial_q \tau_q(T) = \tau_{q-1}(\partial_q T)$. To verify condition (d), consider a singular q-cycle

$a = \sum_j a_j T_j$ in (U_0, x_0). For any j, the homotopy h^{T_j} is a $(q+1)$-dimensional prism in U (see [5], chap. 7, § 6–7) and

$$\partial(h^{T_j}) = h_1^{T_j} - h_0^{T_j} - \sum_{i=0}^{q} (-1)^i (h^{T_j})^{(i)} = \tau_q(T_j) - T_j - \sum_{i=0}^{q} (-1)^i h^{T_j^{(i)}}.$$

Therefore

$$\partial\Big(\sum_j a_j h^{T_j}\Big) = \tau_q(a) - a - \sum_j a_j \sum_{i=0}^{q} (-1)^i h^{T_j^{(i)}},$$

which shows that $\tau_q(a)$ and a are homologous in (U, x_0), since the prism-chain $\sum_j a_j \sum_{i=0}^{q} (-1)\, h^{T_j^{(i)}}$ lies at x_0, and the proof is complete.

Consider now a mapping T_0: $\Delta_n \to S^n$ which maps the boundary $\dot{\Delta}_n$ of Δ_n onto s_0 and is 1—1 for all other points. T_0 is a singular n-cycle in (S^n, s_0); moreover, the homology class $e = [T_0]$ of this cycle is a generator of the nth singular homology group $H_n(S^n, s_0)$.

Let U be a subset of the Hilbert-cube Q with $x_0 \in U$ and let ξ: $(S, s_0) \to (U, x_0)$ be a mapping. The composition $\xi \circ T_0$: $\Delta_n \to U$ is a reduced singular n-simplex; let it be denoted by T_ξ. Observe that the singular chain T_ξ is an n-cycle in (U, x_0) and $\xi_*(e) = [T_\xi]$, where ξ_*: $H_n(S^n, s_0) \to H_n(U, x_0)$ is the homomorphism induced by ξ.

3.5. Lemma. *Let* $[\xi^1], [\xi^2], \ldots, [\xi^m] \in \pi_n(U, x_0)$ $(n \geqslant 2)$ *be the homotopy classes of the mappings* $\xi^1, \xi^2, \ldots, \xi^m$: $(S^n s_0) \to (U, x_0)$. *If there exists a reduced* $(n+1)$-*dimensional chain* λ *in* U *such that* $\partial\lambda = \sum_{j=1}^{m} a_j T_{\xi^j}$, *then* $\sum_{j=1}^{m} a_j[\xi^j] = 0$ *in the group* $\pi_n(U, x_0)$ (compare [6], p. 527).

Proof. It is sufficient to prove the lemma under the assumption that $\lambda = T$ (where T: $\Delta_{n+1} \to U$ is a reduced singular $(n+1)$-dimensional simplex), since the group $\pi_n(U, x_0)$ is Abelian for $n \geqslant 2$. Then there exist mappings ζ_i: $(S^n, s_0) \to (U, x_0)$, $i = 0, 1, \ldots, n+1$, such that $T_{\zeta_i} = T^{(i)}$ (since T is reduced); therefore $\partial\lambda = \sum_{i=0}^{n+1} (-1)^i T_{\zeta_i}$. Let dT denote the restriction $T|_{\dot{\Delta}_{q+1}}$: $\dot{\Delta}_{q+1} \to U$, and let $(dT)_*$: $\pi_n(\dot{\Delta}_{n+1}) \to \pi_n(U, x_0)$ be the homomorphism of the homotopy groups induced by dT. There is a generator e' of the group $\pi_n(\dot{\Delta}_{n+1}) \approx Z$ such that $(dT)_*(e') = \sum_{i=0}^{n+1} (-1)^i [\zeta_i] \in \pi_n(U, x_0)$, since T is reduced. But the inclusion map j: $\dot{\Delta}_{n+1} \to \Delta_{n+1}$ induces the 0-homomorphism j_* (indeed, Δ_{n+1} is contractible) and $(dT)_*(e') = T_* j_*(e') = 0$, where T_*: $\pi_n(\Delta_{n+1}) \to \pi_n(U, x_0)$ is the homomorphism induced by T.

Proof of Theorem 3.2. Since $\underline{\varphi}$ is defined as the composition $\omega \circ \psi$ and ω is an isomorphism (see § 1), it is sufficient to show that ψ is also an isomorphism.

Let U be a neighbourhood of X in Q. Observe that if T: $\Delta_n \to U$ is a reduced singular n-simplex, then there exists a mapping ξ: $(S^n, s_0) \to (U, x_0)$ such that $T = T_\xi$. Therefore, for any reduced singular n-cycle a in (U, x_0) there exists a mapping ξ_a: $(S^n, s_0) \to (U, x_0)$ such that $\xi_{a*}(e) = [a]$.

1° We will prove that ψ is an epimorphism.

Let $\underline{a} = \{a_k\}$ be a fundamental n-cycle of (X, x_0). Let $\{U^{\varepsilon_k}\}$ be a sequence of neighbourhoods of X such that $U^{\varepsilon_{k+1}} = U_0^{\varepsilon_k}$ (see Lemma 3.4) and $0 < \varepsilon_k \to 0$. There is an infinite singular $(n+1)$-chain $\lambda = \{\lambda_k\}$ such that $\partial\lambda_k = a_k - a_{k+1} \pmod{x_0}$, since $\underline{a}$ is fundamental. Suppose that λ_k lies in U^{ε_k}. The generality of the proof is not reduced by this assumption, since instead of $\underline{a}$ we can take a suitable subsequence $\underline{a}'$ of $\underline{a}$, which is always a fundamental n-cycle homologous to $\underline{a}$ in (X, x_0) and which satisfies this assumption. Let $\beta_k = \tau_n(a_k)$ and $\varkappa_k = \tau_{n+1}(\lambda_k)$ (for $k = 2, 3, \ldots$), where τ_q: $C_q(U^{\varepsilon_k}) \to C_q(U^{\varepsilon_{k-1}})$ $(q = n, n+1)$ is the homomorphism defined in 3.4. Clearly, the sequence $\underline{\beta} = \{\beta_k\}$ is a fundamental n-cycle of (X, x_0), since $\partial\varkappa_k = \beta_k - \beta_{k+1} \pmod{x_0}$ and, moreover, $\underline{\beta}$ and $\underline{a}$ are homologous in (X, x_0).

The singular n-cycle β_k in $(U^{\varepsilon_{k-1}}, x_0)$ is reduced (for any k); hence there exists a sequence of mappings $\{\xi_k\}$, ξ_k: $(S^n, s_0) \to (U^{\varepsilon_{k-1}}, x_0)$ such that $\xi_{k*}(e) = [\beta_k]$ $(k = 2, 3, \ldots)$. We will now show that the mappings ξ_k: $(S^n, s_0) \to (Q, x_0)$ forms an approximative map of (S^n, s_0) towards (X, x_0). Let $(\xi_k - \xi_{k+1})$: $(S^n, s_0) \to (U^{\varepsilon_{k-1}}, x_0)$ denote the homotopy difference between ξ_k and ξ_{k+1} $(k = 2, 3, \ldots)$. Clearly, $(\xi_k - \xi_{k+1})^*(e) = [\beta_k - \beta_{k+1}]$. On the other hand, $\beta_k - \beta_{k+1}$ is a boundary (in $(U^{\varepsilon_{k-1}}, x_0)$) of the reduced chain $\varkappa_k$. This and Lemma 3.5 yield the homotopy $\xi_k \simeq \xi_{k+1}$ in (U^{ε_k}, x_0). Hence $\underline{\xi} = \{\xi_k, (S^n, s_0) \to (X, x_0)\}$ is an approximative map. Moreover, $\psi([\underline{\xi}]) = \underline{a}$, which proves that ψ is an epimorphism.

2° Suppose now that $\psi([\underline{\xi}]) = 0$, where $\underline{\xi} = \{\xi_k, (S^n, s_0) \to (X, x_0)\}$. There exists an infinite singular $(n+1)$-chain $\lambda = \{\lambda_k\}$ of X such that λ_k is a reduced chain in $U^{\varepsilon_k}(\varepsilon_k \to 0)$ and $\partial\lambda_k = T_{\xi_k}$, and Lemma 3.5 implies that the mapping ξ_k: $(S^n, s_0) \to (U^{\varepsilon_k}, x_0)$ is null-homotopic in (U^{ε_k}, x_0). Thus, $\underline{\xi} \simeq 0$, that is $[\underline{\xi}] = 0$, which proves that ψ is a monomorphism.

3.6. Remark. If the pointed compactum (X, x_0) is movable, then the condition $\underline{\pi}_q(X, x_0) \approx 0$ is equivalent to the assumption of the approximative q-connectedness of (X, x_0) (for $q = 0, 1, 2, \ldots$) (see [2], p. 271 and [4]).

3.7. Corollary. *If the pointed compactum* (X, x_0) *is movable and if* $\underline{\pi}_q(X, x_0) \approx 0$ *for* $q = 0, 1, \ldots, n-1$ $(n \geqslant 2)$, *then the limit Hurewicz homomorphism* $\underline{\varphi}$: $\underline{\pi}_n(X, x_0) \to \check{H}_n(X, x_0)$ *is an isomorphism.*

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.

[2] — *A note on the theory of shape of compacta*, Fund. Math. 67 (1970), pp. 265–278.

[3] — *Theory of Retracts*, Warszawa 1967.

[4] — *Some remarks concerning the shape of pointed compacta*, Fund. Math. 67 (1970), pp. 221–240.

[5] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[6] W. Hurewicz, *Beiträge zur Topologie der Deformationen* II, Proc. Ak. Amsterdam 38 (1935), pp. 521–528.

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Spaces of ANR's

by

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1. Introduction. For a finite dimensional compactum X, let 2_h^X denote the hyperspace of ANR's lying in X, with the metric ϱ_h introduced and studied by K. Borsuk [3]. Among many results established by Borsuk, we mention here that 2_h^X is complete and separable, and the topology of 2_h^X is characterized by homotopic convergence: a sequence $\{A_i\}$ converges to A in 2_h^X if and only if (1) $\{A_i\}$ converges to A in the Hausdorff sense and (2) for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each i, every subset of A_i of diameter less than δ is contractible to a point in a subset of A_i of diameter less than ε. Thus two ANR's in X which are "close" relative to the metric ϱ_h have similar homotopy properties. In particular, as was shown in [3], for each $A \in 2_h^X$, all ANR's in X which are sufficiently close to A in 2_h^X are homotopically equivalent to A.

The aim of the present paper is to investigate topological properties of the space 2_h^X, primarily for $X = S^2$.

It is evident that the subspace $\mathfrak{C}_X$ of 2_h^X consisting of all connected ANR's in X is open and closed in 2_h^X. Our attention will frequently be directed to this (complete) subspace of 2_h^X rather than to the whole space. For notational convenience, $\mathfrak{C}_{S^2}$ will be denoted simply by $\mathfrak{C}$.

We show that each pair of homotopically equivalent elements of $\mathfrak{C}$ can be joined by an arc in $2_h^{S^2}$, thus characterizing the components of $\mathfrak{C}$ as precisely the sets $[C] = \{A \in 2_h^{S^2} \mid A \underset{h}{\cong} C\}$, for $C \in \mathfrak{C}$. It is clear that S^2 is an isolated point of $2_h^{S^2}$, since no ANR properly contained in S^2 is homotopically equivalent to S^2, but there are no other isolated points in $2_h^{S^2}$. In fact, $2_h^{S^2}$ is infinite dimensional at every point of $2_h^{S^2} - \{S^2\}$, and is not locally compact at any point except S^2.

As partial answers to questions posed by Borsuk ([3], p. 201, [4], p. 221), we show that the set of polyhedra properly contained in S^2 is dense in $2_h^{S^2}$ and is of the first (Baire) category. On the other hand, the set of topological polyhedra in S^2 is of the second category (in fact, residual) in $2_h^{S^2}$.