On families of large oscillation

by

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A theorem of Erdös and Rado characterizes those pairs $\varkappa$, α of cardinals (with α regular and with $\omega \leqslant \varkappa < \alpha$) for which every collection of α sets, each with fewer than $\varkappa$ elements, has a subfamily with α members every two of which have the same intersection. We note a topological description of such pairs $\varkappa$, α, in terms of the Souslin number of certain product spaces in the $\varkappa$-box topology (Theorem 2.3); we then connect these results (as they apply to non-limit cardinals α^+) with the concept of a family of functions from α to α of $\varkappa$-large oscillation to obtain a number of conditions equivalent to the condition $\alpha = \alpha^{\underset{\sim}{\varkappa}}$ (Theorem 3.1). A result of our investigations is that the existence of a family of functions from α to α of $\varkappa$-large oscillation with α^+ elements implies the existence of such a family with 2^α elements. This dichotomy may account for the use of families of $\varkappa$-large oscillation in avoiding the generalized continuum hypothesis.

Theorem 3.1 implies the following result of Tarski: If $\alpha = \alpha^{\underset{\sim}{\varkappa}}$ then there is a family of 2^α functions from α to 2 of $\varkappa$-large oscillation. It follows that for strongly compact cardinals $\varkappa$ there are 2^{2^α} $\varkappa$-complete ultrafilters on the discrete set α (Corollary 4.2). We next apply the results of § 3 to the Rudin–Keisler order of types of ultrafilters. We prove that if $\alpha = \alpha^{\underset{\sim}{\varkappa}}$ and $\varkappa$ is regular and each $\varkappa$-complete filter on the set α extends to a $\varkappa$-complete ultrafilter, then the set of types of $\varkappa$-complete ultrafilters on α is $(2^\alpha)^+$-directed in the Rudin–Keisler order (Theorem 4.3). In particular if $(2^\alpha)^+ = 2^{2^\alpha}$ then there is in the set of types of $\beta(\alpha)$ a well-ordered, cofinal subset (with 2^{2^α} elements).

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§ 1. Definitions and terminology. The axiom of choice is assumed. An ordinal number is the set of ordinal numbers which precede it. A cardinal number is an ordinal number which is not in one-to-one correspondence with any smaller ordinal number. Ordinal numbers are denoted by ξ, ζ, η, ε, σ, and τ, and cardinals by α, β, γ, $\varkappa$, and λ; ω denotes the first infinite cardinal. The symbol $\varepsilon+1$ denotes the immediate ordinal successor of the ordinal ε, and α^+ is the smallest cardinal exceeding α. We write

$$\alpha^{\underline{\varkappa}} = \sum \{\alpha^\lambda:\ \lambda < \varkappa\}\,.$$

The cardinal α is said to be *regular* provided that the sum of fewer than α cardinals, each less than α, is itself less than α. Thus α^+ is regular whenever $\alpha \geqslant \omega$.

For each set S we write

$$\mathfrak{P}(S) = \{A:\ A \subset S\}\,.$$

For sets S and T the symbol T^S denotes the family of functions from S into T or occasionally, when the sets S and T happen to be cardinals, the cardinal number of that family. In general, the cardinal number of a set S is denoted $|S|$.

The density character of a topological space X, denoted $d(X)$, is the smallest cardinal which is equal to the cardinal number of a dense subset of X. The Souslin number $S(X)$ of X is the smallest cardinal α for which no family of pairwise disjoint, nonvoid, open subsets of X has α elements.

The $\varkappa$-box topology on a product space $\prod_{i\in I} X_i$ is that topology with a base consisting of all sets of the form $\prod_{i\in I} U_i$ where each U_i is open in X_i and $U_i = X_i$ with less than $\varkappa$ exceptions. The space $\prod_{i\in I} X_i$ with this topology is denoted $(\prod_{i\in I} X_i)_\varkappa$. The $\varkappa$-completion of a topology $\mathcal{T}$ on a space X is the smallest topology on X containing $\mathcal{T}$ and closed under the operation of intersection of fewer than $\varkappa$ elements. A topological space is said to be *$\varkappa$-complete* if it coincides with its own $\varkappa$-completion. It should be observed that unless the nonvoid spaces X_i are themselves $\varkappa$-complete then $(\prod_{i\in I} X_i)_\varkappa$ is not $\varkappa$-complete. (When $\varkappa = \omega$ of course the $\varkappa$-box topology on $\prod_{i\in I} X_i$ is the usual product topology, and is $\varkappa$-complete.)

The $\varkappa$-completion of the product topology has been considered in connection with compactness properties by a number of authors, including Parovičenko [29], Keisler and Tarski [19], and Monk and Scott [26].

§ 2. The Erdös-Rado Theorem.

DEFINITION [1]. Let α and $\varkappa$ be cardinals, with $\omega \leqslant \alpha$ and $\varkappa < \alpha$. Then *α is strongly $\varkappa$-inaccessible* provided that $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \varkappa$.

We remark that for any regular, infinite cardinal α with $\varkappa \leqslant \alpha$, α is strongly $\varkappa$-inaccessible if and only if

$$\prod \{\beta_\xi:\ \xi < \lambda\} < \alpha$$

whenever $\lambda < \varkappa$ and each $\beta_\xi < \alpha$; for this latter condition surely implies that α is strongly $\varkappa$-inaccessible (take each $\beta_\xi = \beta$), while if α is strongly $\varkappa$-inaccessible and each $\beta_\xi < \alpha$ then with

$$\beta = \sum \{\beta_\xi:\ \xi < \lambda\}$$

we have $\beta < \alpha$ (because α is regular) and hence

$$\prod \{\beta_\xi:\ \xi < \lambda\} \leqslant \beta^\lambda < \alpha\,.$$

In an earlier version of this paper we gave a detailed proof of the following theorem, and we showed its relations to Theorems I and II of the Erdös–Rado paper [12]. Subsequently A. Hajnal drew to our attention the existence of the second Erdös–Rado paper [13]. Since Lemma 1 of [13] is precisely (a) ⇒ (b) below, and (b) ⇒ (a) is a special case of Lemma 2 of [13], we give here only a short outline of our proof. The proof that (a) ⇒ (b) is related to arguments given by Całczyńska-Karłowicz [4], Marek [24], Davies [8], and Mostowski (Theorem 13.3.1 in [27]). It is different from that of Erdös and Rado, but in all likelihood it is the same as the unpublished, alternative proof of Davies to which they refer (p. 469 in [13]).

2.1. THEOREM. *If α is an infinite, regular cardinal and $\varkappa < \alpha$, then the following statements are equivalent:*

(a) *α is strongly $\varkappa$-inaccessible;*

(b) *if $\{S_\xi\}_{\xi<\alpha}$ is any family of sets with $|S_\xi| < \varkappa$ for $\xi < \alpha$, then there exist a set J and a subset A of α with $|A| = \alpha$ for which $S_\xi \cap S_\zeta = J$ whenever $\xi, \zeta \in A$ and $\xi \neq \zeta$.*

Proof. [In the statement of (b), the sets S_ξ need not be distinct for distinct subscripts ξ; and the set J to be defined may be empty.]

(a) ⇒ (b). [Outline]. One may assume that $\bigcup \{S_\xi:\ \xi < \alpha\} \subset \alpha$ and that for some fixed $\bar{\zeta}$ with $\bar{\zeta} < \lambda$ each of the sets S_ξ with $\zeta < \alpha$ is isomorphic as an ordered set to the well-ordered set $\bar{\zeta}$. Writing S_ξ

(1) It has been brought to our attention by S. Shelah that this concept has been introduced (using different terminology) and used in connection with a generalization of Łoś' ultraproduct theorem by G. Fuhrken, Languages with added quantifier "There exist at least $\aleph_\alpha$" in *The Theory of Models*, Proc. 1963 International Berkeley Symposium, Amsterdam 1965, pp. 121–131.

$=\{\eta_\xi^\zeta\}_{\zeta<\bar\zeta}$ with $\eta_\xi^\zeta<\eta_\xi^{\zeta'}$ whenever $0\leqslant\zeta<\zeta'<\bar\zeta$ and defining $\varepsilon_\zeta=\sup\{\eta_\xi^\zeta\colon \xi<\alpha\}$, one defines $\varepsilon=\sup\{\varepsilon_\zeta\colon \zeta<\bar\zeta\}$ and considers separately the cases in which ε is less than, or equal to, α. In the former case there are, immediately, α sets S_ξ whose pairwise intersections coincide. In the latter case, with $\zeta(0)$ the smallest ordinal ζ for which $\varepsilon_\zeta=\alpha$, there exists $B\subset\alpha$, with $|B|=\alpha$, for which $\sup\{\eta_\xi^{\zeta(0)}\colon \xi\in B'\}=\alpha$ whenever $B'\subset B$ and $|B'|=\alpha$; by hypothesis (a) the set B may be chosen so that, for some set J,

$$\{\eta_\xi^\zeta\colon \zeta<\zeta(0)\}\cap\{\eta_{\xi'}^\zeta\colon \zeta<\zeta(0)\}=J$$

whenever ξ and ξ' are distinct elements of B. The desired set $A=\{\psi(\sigma)\colon \sigma<\alpha\}$ is now defined recursively: $\psi(0)$ is any element of B, $\psi(1)$ is any element of B for which $\eta_{\psi(1)}^{\zeta(0)}>\sup[S_{\psi(0)}]$ and in general, if $\psi(\sigma)$ has been defined in B for $\sigma<\tau$ so that $S_{\psi(\sigma)}\cap S_{\psi(\sigma')}=J$ whenever $0\leqslant\sigma<\sigma'<\tau$, then $\psi(\tau)$ is any element of B for which

$$\eta_{\psi(\tau)}^{\zeta(0)}>\sup[\bigcup_{\sigma<\tau}S_{\psi(\sigma)}].$$

(b) $\Rightarrow$ (a). If cardinals λ and β exist for which $\lambda<\varkappa$ and $\beta<\alpha$ and $\beta^\lambda\geqslant\alpha$, then there is a family $\{f_\xi\}_{\xi<\alpha}$ of functions from λ to β for which $f_{\xi_1}\neq f_{\xi_2}$ whenever $0\leqslant\xi_1<\xi_2<\alpha$. Writing

$$G(\xi)=\{(\eta,f_\xi(\eta))\colon \eta<\lambda\}$$

we see that for no subset J of $\lambda\times\beta$ can there exist a subset A of α with $|A|=\alpha$ for which $G(\xi_1)\cap G(\xi_2)=J$ whenever ξ_1 and ξ_2 are distinct elements of A. Indeed if $|A|>\beta$ (J being given) then, since $\{f_\xi(\eta)\}_{\xi\in A}\subset\beta$ for each $\eta<\lambda$, there is for each such η an element $\varphi(\eta)$ of B with

$$\varphi(\eta)=f_{\xi_1}(\eta)=f_{\xi_2}(\eta)$$

for some pair (ξ_1,ξ_2) of distinct elements of A, hence for each such pair. Thus $|A|=1$ if $|A|>\beta$, a contradiction.

If $\varkappa=\omega$, condition (a) of Theorem 2.1 is satisfied and (b) becomes the following assertion:

If α is an uncountable, regular cardinal and $\{S_\xi\}_{\xi<\alpha}$ is a family of finite sets, then there are a set J, and a subset A of α with $|A|=\alpha$, such that $S_\xi\cap S_\zeta=J$ whenever ξ and ζ are distinct elements of A.

This result was first stated by Shanin [34] and used by him in [35] to prove a statement related to that of (c) in Theorem 2.3 below (for $\alpha=\omega$) involving the notion of caliber. It is proved explicitly by Mazur [25] and implicitly by Marczewski [23], Bockstein [3], and Solovay and Tennenbaum [36].

We note the following statement, which can be derived from Theorem 2.1 in the same way that Engelking and Karłowicz prove Theorem 1 of [11] from the Erdös–Rado theorem of [12]:

Let α be a regular, strongly $\varkappa$-inaccessible cardinal with $\omega\leqslant\alpha$ and $\varkappa<\alpha$, and let $\mathcal{A}$ and $\mathcal{B}$ be families of sets for which $|A|<\varkappa$ and $|B|<\alpha$ and $A\cap B\neq\emptyset$ whenever $A\in\mathcal{A}$ and $B\in\mathcal{B}$. Then there is a set C for which $|C|<\alpha$ such that $A\cap B\cap C\neq\emptyset$ whenever $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

We omit the details of the proof. In a preliminary version of this paper we inquired whether the converse implication was valid, i.e., whether it is true that if β and λ are cardinals for which $\lambda\leqslant\beta<\alpha$ and $\beta^\lambda\geqslant\alpha$ then there must exist families $\mathcal{A}$ and $\mathcal{B}$ of sets with $|A|=\lambda$ and $|B|=\beta$ and $A\cap B\neq\emptyset$ whenever $A\in\mathcal{A}$ and $B\in\mathcal{B}$, with the property that each set C meeting $A\cap B$ (for each A in $\mathcal{A}$ and each B in $\mathcal{B}$) must satisfy $|C|\geqslant\alpha$. The following construction, answering this question affirmatively, was furnished to us by P. Erdös and A. Hajnal and is included here with their kind permission. Let f be any one-to-one map from α into β^λ and for $\xi<\alpha$ let

$$A_\xi=\{\xi\}\cup G(f(\xi))\qquad\text{and}\qquad B_\xi=\{\xi\}\cup(\lambda\times\beta)\backslash G(f(\xi))$$

where as above $G(f(\xi))$ denotes the set $\{(\eta,f(\xi)(\eta))\colon \eta<\lambda\}$. Then the families $\mathcal{A}=\{A_\xi\}_{\xi<\alpha}$ and $\mathcal{B}=\{B_\xi\}_{\xi<\alpha}$ are as required.

One result of Erdös–Rado type not a special case of Theorem 2.1 is Lemma 10.2 of Kunen's thesis [20]. This is used by him (Lemma 10.3) to compute the Souslin number of a large product space under the topology of thin sets, a topology which is intermediate between the usual product topology and the $\varkappa$-box topology.

We introduce the following notation. Suppose that U is a subset of the product $\prod_{i\in I}X_i$ which is basic in the $\varkappa$-box topology, so that $U=\prod_{i\in I}U_i$ with each U_i open in X_i and with $U_i=X_i$ except for those i belonging to a set with fewer than $\varkappa$ elements. We denote this latter set $R(U)$ (read: the restriction set of U). For each nonvoid subset J of I we write $X_J=\prod_{i\in J}X_i$ and we denote by π_J the projection from X_I onto X_J. Using these notations, it is easy to check that for nonvoid sets $U=\prod_{i\in I}U_i$ and $V=\prod_{i\in I}V_i$ the following assertions are equivalent: (1) $U\cap V=\emptyset$; (2) $U_i\cap V_i=\emptyset$ for some i in $R(U)\cap R(V)$; (3) $R(U)\cap R(V)\neq\emptyset$ and $\pi_J[U]\cap\pi_J[V]=\emptyset$ whenever $R(U)\cap R(V)\subset J\subset I$.

2.2. Corollary. *If α is regular and strongly $\varkappa$-inaccessible with $\omega\leqslant\varkappa<\alpha$, and if $\{X_i\}_{i\in I}$ is any collection of topological spaces for which $S((X_J)_\varkappa)\leqslant\alpha$ whenever $J\subset I$ and $|J|<\varkappa$, then $S((\prod_{i\in I}X_i)_\varkappa)\leqslant\alpha$.*

Proof. If $S((\prod_{i\in I}X_i)_\varkappa)>\alpha$ there is a family $\{U_\xi\}_{\xi<\alpha}$ of pairwise disjoint, nonvoid subsets of $\prod_{i\in I}X_i$ with each U_ξ basic in the $\varkappa$-box topology. We

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have $|R(U_\xi)| < \varkappa$ for $\xi < \alpha$, so by condition (b) there exist a set J, and a subset A of α with $|A| = \alpha$, for which

$$R(U_\xi) \cap R(U_\zeta) = J$$

whenever ξ and ζ are different elements of A. It impossible in this case that $J = \emptyset$, so the sets $\pi_J[U_\xi]$ with ξ in A are nonvoid, pairwise disjoint, and open in $(X_J)_\varkappa$. (Because $|J| < \varkappa$, the space $(X_J)_\varkappa$ is X_J with the usual box topology; it has as a base all sets of the form $\prod_{i \in J} U_i$ with U_i open in X_i.) This contradicts the hypothesis $S\big((\prod_{i \in I} X_i)_\varkappa\big) \leqslant \alpha$.

2.3. THEOREM. *If α is regular and $\omega \leqslant \varkappa < \alpha$, then the following statement is equivalent to each of the statements of Theorem* 2.1:

(c) *if $d(X_i) < \alpha$ for each i in I, then $S\big((\prod_{i \in I} X_i)_\varkappa\big) \leqslant \alpha$.*

Proof. (a) $\Rightarrow$ (c). Let $J \subset I$ with $|J| < \varkappa$ and for each i in J let D_i be dense in X_i with $|D_i| < \alpha$. Then with $D = \prod_{i \in J} D_i$ we have: D is dense in $(X_J)_\varkappa$ and (as was remarked in the paragraph preceding the statement of Theorem 3.1) $|D| < \alpha$ from condition (a). Thus $S\big((X_J)_\varkappa\big) \leqslant \alpha$, so $S\big((\prod_{i \in I} X_i)_\varkappa\big) \leqslant \alpha$ by 3.2.

(c) $\Rightarrow$ (a). Let β and λ be cardinals for which $\beta < \alpha$ and $\lambda < \varkappa$. With β viewed as a discrete space, the space $(\beta^\lambda)_\varkappa$ is discrete. For any discrete space X we have $S(X) = |X|^+$, so from condition (c) it follows that $|\beta^\lambda|^+ \leqslant \alpha$, i.e., $\beta^\lambda < \alpha$.

Our derivation of condition (c) from (a) and (b) used the regularity of α. We wish to remark that, even with $\varkappa = \omega$, statement (c) becomes false in case α is not regular. To see this, we recall from Erdös and Tarski [14], Theorem 1, that for each space X the Souslin number $S(X)$, if infinite, is regular, and we suppose that α is singular, so that $\alpha = \sup_{i \in I} \alpha_i$ with $\alpha_i < \alpha$ and with $|I| < \alpha$. Then with X_i the discrete space α_i we have for each j

$$S\Big(\prod_{i \in I} X_i\Big) \geqslant S(X_j) = \alpha_j^+,$$

so that $S(\prod_{i \in I} X_i) \geqslant \alpha$. Since α is not regular we have $S(\prod_{i \in I} X_i) \geqslant \alpha^+$, so (from (c), with α replaced by α^+) it follows that $S(\prod_{i \in I} X_i) = \alpha^+$.

Suppose that the conditions of Theorem 2.3 are satisfied and let $\{X_i\}_{i \in I}$ be a collection of spaces with $w(X_i) < \alpha$ for each i. (Here $w(X)$ denotes the weight of X — i.e., the smallest cardinal which is equal to the cardinal number of an open base for X.) It is clear that if Y_i denotes the $\varkappa$-completion of X_i then $w(Y_i) < \alpha$: if $w(X_i) = \beta < \alpha$ then $w(Y_i)$ $\leqslant \sum_{\lambda < \varkappa} \beta^\lambda$ and this sum, the sum of $\varkappa$ or fewer cardinals each less than α, is less than α since α is regular. Now $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ have the same $\varkappa$-completion, namely $(\prod_{i \in I} Y_i)_\varkappa$, and from (c) above we have $S\big((\prod_{i \in I} Y_i)_\varkappa\big)$ $\leqslant \alpha$. Stated in terms of the spaces X_i, this result takes the following form.

2.4. COROLLARY. *Let $\omega \leqslant \varkappa < \alpha$ with α and $\varkappa$ regular and with α strongly $\varkappa$-inaccessible and let $\{X_i\}_{i \in I}$ be a collection of space with $w(X_i) < \alpha$ for each i. Then each family of pairwise disjoint, nonvoid sets, each of which is the intersection of fewer than $\varkappa$ open subsets of $\prod_{i \in I} X_i$, has fewer than α members.*

This corollary has been given by Engelking and Karłowicz [11], Theorem 6 (see also Engelking [10], Theorem 7) in the case that $\alpha = \mathfrak{m}^+$, $\varkappa = \mathfrak{n}^+$ and $\mathfrak{m}^\mathfrak{n} = \mathfrak{m}$.

§ 3. Families of large oscillation. According to the classical theorem of Hausdorff [16], proved earlier by Fichtenholz and Kantorovitch [15] for the cases $\alpha = \omega$ and $\alpha = 2^\omega$, if α is an infinite cardinal there is a family $\mathcal{S}$ of subsets of α with $|\mathcal{S}| = 2^\alpha$ for which: if $\mathcal{F}$ and $\mathcal{G}$ are disjoint, finite subsets of $\mathcal{S}$ then

$$\bigcap\{F\colon F \in \mathcal{F}\} \cap \bigcap\{\alpha \setminus F\colon F \in \mathcal{G}\} \neq \emptyset .$$

(Such a family $\mathcal{S}$ is said to be an *independent family of subsets of* α.) Using Hausdorff's theorem Engelking and Karłowicz [11] proved the following result: if α is an infinite cardinal there is a family $\mathcal{F}$ of functions from α to α with $|\mathcal{F}| = 2^\alpha$ satisfying the following property: given distinct functions $f_0, f_1, \ldots, f_n$ in $\mathcal{F}$ and (not necessarily distinct) elements $\xi_0, \xi_1, \ldots$ $\ldots, \xi_n$ of α, there is $\sigma < \alpha$ such that $f_k(\sigma) = \xi_k$ for $0 \leqslant k \leqslant n$. Such a family $\mathcal{F}$ of functions is said to be a *family of large oscillation*. (This terminology was given by Kenneth Kunen in an early version of his paper [22], but was later replaced by the terminology "independent family".)

DEFINITION. Let α, β and $\varkappa$ be cardinals, and let $\mathcal{F} \subset \beta^\alpha$. The family $\mathcal{F}$ is a *family of $\varkappa$-large oscillation* provided: if $\lambda < \varkappa$ and $\{f_\zeta\}_{\zeta < \lambda}$ are (distinct) elements of $\mathcal{F}$ and $\{\xi_\zeta\}_{\zeta < \lambda}$ are (not necessarily distinct) elements of α, then there is $\sigma < \alpha$ such that $f_\zeta(\sigma) = \xi_\zeta$ for $\zeta < \lambda$.

3.1. THEOREM. *Let α and $\varkappa$ be cardinals for which $\omega \leqslant \varkappa \leqslant \alpha$. Then the following statements are equivalent:*

(a) $\alpha = \alpha^{\underline{\varkappa}}$;

(b) *if $\{S_\xi\}_{\xi < \alpha^+}$ is any family of sets with $|S_\xi| < \varkappa$ for $\xi < \alpha$, then there are a set J and a subset A of α^+ with $|A| = \alpha^+$ for which $S_\xi \cap S_\zeta = J$ whenever $\xi, \zeta \in A$ and $\xi \neq \zeta$;*

(c) *if* $d(X_i) \leqslant \alpha$ *for each* i *in* I, *then* $S\big((\prod_{i\in I} X_i)_\varkappa\big) \leqslant \alpha^+$;

(d) *there exists* $\mathcal{F} \subset \alpha^\alpha$ *for which* $|\mathcal{F}| = 2^\alpha$ *and* $\mathcal{F}$ *has* $\varkappa$*-large oscillation*;

(e) *there exists* $\mathcal{F} \subset \alpha^\alpha$ *for which* $|\mathcal{F}| = \alpha^+$ *and* $\mathcal{F}$ *has* $\varkappa$*-large oscillation*;

(f) $d\big((\alpha^{(2^\alpha)})_\varkappa\big) = \alpha$;

(g) $d\big((\alpha^{(\alpha^+)})_\varkappa\big) = \alpha$.

Proof. [In conditions (f) and (g) the space α has the discrete topology; it is relative to this topology that the $\varkappa$-box topology is taken on $\alpha^{(2^\alpha)}$ and $\alpha^{(\alpha^+)}$].

We note first that in the present theorem the pair $\varkappa, \alpha^+$ satisfies the conditions imposed on $\varkappa, \alpha$ in Theorems 2.1 and 2.3. Further, the present conditions (a), (b) and (c) precisely duplicate the corresponding conditions of 2.1 and 2.3 (with α replaced by α^+). This is clear for (b) and (c), and for (a) we observe simply that α^+ is strongly $\varkappa$-inaccessible if and only if $\alpha^\lambda = \alpha$ whenever $\lambda < \varkappa$, i.e., $\alpha^{\underline{\varkappa}} = \alpha$. Thus (a), (b) and (c) of the present theorem are equivalent by Theorems 2.1 and 2.3.

We prove next that (a) $\Rightarrow$ (d). Let

$$\mathcal{S} = \bigcup\{\alpha^G \times \{F\}\colon\ F \subset \alpha,\ G \subset \mathcal{P}(F),\ |F| < \varkappa,\ |G| < \varkappa\}\,.$$

If $F \subset \alpha$ and $|F| = \lambda < \varkappa$, then the number of families G of fewer than $\varkappa$ subsets of F is $(2^\lambda)^{\underline{\varkappa}}$, a cardinal not exceeding $(\alpha^\lambda)^{\underline{\varkappa}} \leqslant \alpha^{\underline{\varkappa}} = \alpha$; and for each such family G we have

$$|\alpha^G \times \{F\}| = |\alpha^G| \leqslant \alpha^{\underline{\varkappa}} = \alpha\,.$$

Thus

$$|\mathcal{S}| \leqslant \Sigma\alpha\cdot\alpha\cdot|\{F\colon\ F \subset \alpha,\ |F| < \varkappa\}| = \alpha\cdot\alpha\cdot\alpha^{\underline{\varkappa}} = \alpha$$

and it will suffice, in order to establish (d), to find $\mathcal{F} \subset \alpha^{\mathcal{S}}$ with $|\mathcal{F}| = 2^\alpha$ for which $\mathcal{F}$ is of $\varkappa$-large oscillation.

For each subset A of α we define f_A from $\mathcal{S}$ to α as follows: If $(s, F) \in \alpha^G \times \{F\}$ with $G \subset \mathcal{P}(F)$ and $|G| < \varkappa$, then

$$f_A(s, F) = \begin{cases} s(A \cap F) & \text{if} \quad A \cap F \in G\,, \\ 0 & \text{if} \quad A \cap F \notin G\,. \end{cases}$$

To see that the functions f_A from $\mathcal{S}$ to α are distinct for different subsets A of α, and in fact to see that the family $\{f_A\}_{A\subset\alpha}$ is a family of $\varkappa$-large oscillation, let $\lambda < \varkappa$ and let $\{A_\zeta\}_{\zeta<\lambda}$ be a family of (distinct) subsets of α and let $\{\xi_\zeta\}_{\zeta<\lambda}$ be (not necessarily distinct) elements of α. For each pair (ζ, ζ') of ordinals with $0 \leqslant \zeta < \zeta' < \lambda$ we choose

$$\eta_{\zeta,\zeta'} \in (A_\zeta \backslash A_{\zeta'}) \cup (A_{\zeta'} \backslash A_\zeta)$$

and we set

$$F = \{\eta_{\zeta,\zeta'}\colon\ 0 \leqslant \zeta < \zeta' < \lambda\}\,.$$

Writing

$$G = \{A_\zeta \cap F\colon\ \zeta < \lambda\}$$

we define s from G to α by the rule

$$s(A_\zeta \cap F) = \xi_\zeta\,.$$

(This function s is well-defined because $A_\zeta \cap F \neq A_{\zeta'} \cap F$ for $0 \leqslant \zeta < \zeta' < \lambda$.) Then $(s, F) \in \mathcal{S}$, and

$$f_A(s, F) = s(A_\zeta \cap F) = \xi_\zeta$$

as required. The proof that (a) $\Rightarrow$ (d) is complete.

To see that (d) and (f) are equivalent, we note that for any family $\mathcal{F}$ of functions from α to α, $\mathcal{F}$ is of $\varkappa$-large oscillation if and only if the set $\{p(\eta)\}_{\eta<\alpha}$, with $\big(p(\eta)\big)_f = f(\eta)$, is dense in $\alpha^{\mathcal{F}}$ in the $\varkappa$-box topology. This shows also that (e) and (g) are equivalent. Since any subfamily of a family of $\varkappa$-large oscillation is another such family, we have (d) $\Rightarrow$ (e). Thus to complete the proof of the present theorem it suffices to show (g) $\Rightarrow$ (c).

To this end suppose that $d(X_i) \leqslant \alpha$ for each i in I but that $(\prod_{i\in I} X_i)_\varkappa$ has a family $\{U_\xi\}_{\xi<\alpha^+}$ of pairwise disjoint, nonvoid, basic open subsets. Writing

$$J = \bigcup\{R(U_\xi)\colon\ \xi < \alpha^+\}$$

we have: $|J| \leqslant \alpha^+\cdot\varkappa = \alpha^+$ and $\{\pi_J[U_\xi]\}_{\xi<\alpha^+}$ is a family of pairwise disjoint, nonvoid, open sets in $(X_J)_\varkappa$. For each i in J there is a (continuous) function from (the discrete space) α onto a dense subspace of X_i, and the product of these functions takes $(\alpha^{(\alpha^+)})_\varkappa$ continuously onto a dense subspace of $(X_J)_\varkappa$. Since $d\big((\alpha^{(\alpha^+)})_\varkappa\big) = \alpha$ by (g) we then have $d\big((X_J)_\varkappa\big) \leqslant \alpha$, contradicting the existence of the family $\{\pi_J[U_\xi]\}_{\xi<\alpha^+}$.

We remark that Theorem 3.1 provides characterizations of those infinite cardinals α for which $\alpha = \alpha^{\underline{\alpha}}$. As is well-known (see for example Bachmann [1]), these are the cardinals α which are regular and satisfy the equality $\alpha = 2^{\underline{\alpha}}$.

The conditions of Theorem 3.1 are, of course, (not simply equivalent but in fact) true in case $\varkappa = \omega$. We have already given several references relating to (a), (b) and (c) in this case, and condition (d) with $\varkappa = \omega$ is Theorem 2 of Engelking and Karłowicz [11] (2),(3). The first theorem of the form

$$d(X_i) \leqslant \alpha \Rightarrow S\Big(\prod_{i\in I} X_i\Big) \leqslant \alpha^+$$

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(2) In remark 3 of [11] it is asserted without proof that for arbitrary $\varkappa$ and α with $\omega \leqslant \varkappa \leqslant \alpha$ there is a family $\mathcal{F}$ of $\varkappa$-large oscillation of mappings from $\alpha^{\underline{\varkappa}}$ to $\alpha^{\underline{\varkappa}}$ with $|\mathcal{F}| = 2^\alpha$. This assertion is true in case the generalized continuum hypothesis is assumed,

is due to Szpilrajn–Marczewski [37], [23] for $\alpha = \omega$. For related results and generalizations see [32], [5], [28] and our forthcoming book [7]. Condition (f) for $\varkappa = \omega$ was given by Pondiczery [30] and Hewitt [17].

Consider the following conditions on a pair $\varkappa$, α of cardinal numbers with $\omega \leqslant \varkappa \leqslant \alpha$:

(c′) $\mathcal{S}((2^I)_\varkappa) \leqslant \alpha^+$ for each set I;

(d′) there exists $\mathcal{F} \subset 2^\alpha$ for which $|\mathcal{F}| = 2^\alpha$ and $\mathcal{F}$ is of $\varkappa$-large oscillation;

(e′) there exists $\mathcal{F} \subset 2^\alpha$ for which $|\mathcal{F}| = \alpha^+$ and $\mathcal{F}$ is of $\varkappa$-large oscillation;

(f′) $d((2^{(2^\alpha)})_\varkappa) \leqslant \alpha$;

(g′) $d((2^{(\alpha^+)})_\varkappa) \leqslant \alpha$.

Each of these primed conditions follows from its unprimed analogue in 3.1. (This is obvious in the cases of (c′), (f′) and (g′). In the cases of (d′) and (e′) it suffices to observe that if $\mathcal{F}$ is a family of $\varkappa$-large oscillation of functions from α to α, and if π is any mapping from α onto 2, then the map $f \to \pi \circ f$ is one-to-one from $\mathcal{F}$ onto a family of $\varkappa$-large oscillation of functions from α to 2.) Thus, since the conditions of 3.1 are true when $\varkappa = \omega$, the conditions (c′), (d′), (e′), (f′) and (g′) are true when $\varkappa = \omega$. We do not know ([4]) if these conditions are equivalent for arbitrary $\varkappa \leqslant \alpha$, though it is clear as in the proof of 3.1 that

$$(d') \Leftrightarrow (f') \Rightarrow (e') \Leftrightarrow (g') \Rightarrow (c') \Rightarrow \alpha \geqslant 2^{\underset{\smile}{\varkappa}}.$$

It is well known that from Hausdorff's theorem, which is the statement (d′) with $\varkappa = \omega$ and arbitrary α, it follows that a set with α elements has 2^{2^α} ultrafilters; this fact was first stated explicitly, and established by another method, by Pospíšil [31].

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or if $\varkappa$ is regular, but in general it fails in some models of set theory. Suppose for example that $2^{\aleph_n} = \aleph_{\omega+n+1}$ for $n < \omega$ (cf. Easton [9]), and set $\varkappa = \alpha = \aleph_\omega$. Then

$$\alpha^{\underset{\smile}{\alpha}} = \aleph_{\omega\cdot 2} < (\aleph_{\omega\cdot 2})^{\aleph_0} \leqslant (\alpha^{\underset{\smile}{\alpha}})^{\underset{\smile}{\alpha}}.$$

From the implication (e)$\Rightarrow$(a) of Theorem 4.1 (with α replaced by $\alpha^{\underset{\smile}{\alpha}}$) it follows that no family of functions of α-large oscillation from $\alpha^{\underset{\smile}{\alpha}}$ to $\alpha^{\underset{\smile}{\alpha}}$ has more than $\alpha^{\underset{\smile}{\alpha}}$ elements. In particular, since $2^\alpha > \alpha^{\underset{\smile}{\alpha}}$ (from the inequality $\alpha^{\underset{\smile}{\alpha}} < (\alpha^{\underset{\smile}{\alpha}})^{\underset{\smile}{\alpha}}$), no such family has 2^α elements.

([3]) Note added December 31, 1971. The equivalence (d) $\Longleftrightarrow$ (f) and the implication (a) $\Rightarrow$ (d) have been noticed independently by J. A. Ketonen, Doctoral Dissertation, University of Wisconsin, 1971; the latter implication has been noted also by S. Shelah, *Every two elementarily equivalent models have isomorphic ultrapowers*, Israel J. Math. 10 (1971), pp. 224–233.

([4]) In a letter to the authors dated December 14, 1971, S. Shelah proves the equivalence (c′)$\Longleftrightarrow \alpha \geqslant 2^{\underset{\smile}{\varkappa}}$.

§ 4. Applications to the Rudin-Keisler order of types. A filter of subsets of a set is said to be *$\varkappa$-complete* provided that the intersection of fewer than $\varkappa$ of its elements is again one of of its elements. (Thus, each filter is ω-complete.) Suppose now that $\omega \leqslant \varkappa \leqslant \alpha$ and $\alpha = \alpha^{\underset{\smile}{\varkappa}}$, so that (d) and hence (d′) are valid: there is a family $\mathcal{F}$ of $\varkappa$-large oscillation of functions from α to 2 with $|\mathcal{F}| = 2^\alpha$. For each subset $\mathcal{G}$ of $\mathcal{F}$, let

$$q_\mathcal{G} = \{f^{-1}(1)\colon f \in \mathcal{G}\} \cup \{f^{-1}(0)\colon f \in \mathcal{F}\backslash\mathcal{G}\}.$$

Because $\mathcal{F}$ has $\varkappa$-large oscillation the family $p_\mathcal{G}$ of subsets of α containing some element of $q_\mathcal{G}$ is a *$\varkappa$-complete* filter. If, further, $\mathcal{G}_1 \neq \mathcal{G}_2$, then $q_{\mathcal{G}_1}$ and $q_{\mathcal{G}_2}$ contain complementary subsets of α and so $p_{\mathcal{G}_1} \neq p_{\mathcal{G}_2}$. This proves the following theorem, which appears as Hilfsatz 3.16 in Tarski [39] (and in preliminary form as Lemme 58 in Tarski [38]).

4.1. Theorem. *Let $\omega \leqslant \varkappa \leqslant \alpha$ with $\alpha = \alpha^{\underset{\smile}{\varkappa}}$. Then there is a family $\mathcal{U}$ of $\varkappa$-complete filters on α, with $|\mathcal{U}| = 2^{2^\alpha}$, for which each pair of distinct elements of $\mathcal{U}$ contain complementary subsets of α.*

A cardinal number α is measurable if there is on α an α-complete ultrafilter which is not principal. (Thus ω is the first measurable cardinal.) Let $\Omega_\varkappa(\alpha)$ denote the family of $\varkappa$-complete ultrafilters on α (this usage differs slightly from that of [6]). It is easy to see, as in [6] or [21] for example, that if α is a measurable cardinal and $\omega \leqslant \varkappa \leqslant \alpha$ then $2^\alpha \leqslant |\Omega_\varkappa(\alpha)| \leqslant 2^{2^\alpha}$. Kunen has recently shown (Theorems 5.7 and 7.3 of [21]) that it is consistent with Zermelo–Fraenkel set theory together with the axiom that an uncountable measurable cardinal exists that the generalized continuum hypothesis hold and that $|\Omega_\alpha(\alpha)| = \alpha^+$ for some uncountable measurable cardinal α. In contrast, Corollary 4.2 shows that for cardinals α not in the Keisler–Tarski class $\mathfrak{C}_1^*$ defined in [19] — i.e., for those regular cardinals α for which every α-complete filter on α extends to an α-complete ultrafilter — we have $|\Omega_\alpha(\alpha)| = 2^{2^\alpha}$. This corollary, which follows from Hilfsatz 3.16 of Tarski [39] (a paper we did not know of when in [6] we asked for an evaluation of $|\Omega_\alpha(\alpha)|$), has also been noted by Kunen ([21], p. 205) for $\alpha = \varkappa$.

4.2. Corollary. *Let $\omega \leqslant \varkappa \leqslant \alpha$ with $\alpha = \alpha^{\underset{\smile}{\varkappa}}$, and suppose that every $\varkappa$-complete filter on α extends to a $\varkappa$-complete ultrafilter on α. Then $|\Omega_\varkappa(\alpha)| = 2^{2^\alpha}$.*

Proof. We let $\mathcal{U}$ be as given in Theorem 4.1 and extend each element u of $\mathcal{U}$ to an element u' of $\Omega_\varkappa(\alpha)$; then (since $u' \neq v'$ when $u \neq v$), we have

$$|\Omega_\varkappa(\alpha)| \geqslant |\mathcal{U}| = 2^{2^\alpha}.$$

Of course, $|\Omega_\varkappa(\alpha)| \leqslant 2^{2^\alpha}$ since there are 2^{2^α} filters on α.

An infinite cardinal $\varkappa$ is strongly compact if each $\varkappa$-complete filter on each set can be extended to a $\varkappa$-complete ultrafilter. It follows from

Corollary 4.2 that $|\Omega_\varkappa(\alpha)| = 2^{2^\alpha}$ for each strongly compact cardinal $\varkappa$ for which $\omega \leqslant \varkappa \leqslant \alpha = \alpha^{\underline{\varkappa}}$.

If p and q are elements of a space X we write $p \sim q$ if there is an auto-homeomorphism h of X for which $h(p) = q$. The $\sim$-equivalence class to which p belongs is called the *type of* p (*in* X) and is denoted $\tau(p)$; the set of all types in X is denoted $T(X)$, so that τ is the quotient map from X to $T(X)$ $(= X/\sim)$. If α is an infinite cardinal and $\beta(\alpha)$ denotes the Stone-Čech compactification of the discrete space α then for p and q in $\beta(\alpha)$ we have $p \sim q$ if and only if $\overline{\pi}(p) = q$ for some permutation π of α (where $\overline{\pi}$ denotes the continuous extension of π mapping $\beta(\alpha)$ onto $\beta(\alpha)$). Since there are 2^α permutations of α one has $|\tau(p)| \leqslant 2^\alpha$ for each p in $\beta(\alpha)$, so that (since $|\beta(\alpha)| = 2^{2^\alpha}$) $|T(\beta(\alpha))| = 2^{2^\alpha}$.

For p and q in $\beta(\alpha)$ we write $p \lesssim q$ provided that there is a function f from α to α for which $\bar{f}(q) = p$. When this occurs we have $p' \lesssim q'$ whenever $p \sim p'$ and $q \sim q'$, so the relation $\lesssim$ induces a relation, also denoted $\lesssim$, on $T(\beta(\alpha))$. This is the Rudin–Keisler order on the set of types of $\beta(\alpha)$. According to a result of M. E. Rudin [33] (cf. Theorem 2.1 in [22]), if $p \lesssim q$ and $q \lesssim p$ then $p \sim q$. Thus on $T(\beta(\alpha))$ the order $\lesssim$, which is clearly transitive and reflexive, is also anti-symmetric in the sense that $\tau(p) = \tau(q)$ whenever $\tau(p) \lesssim \tau(q)$ and $\tau(q) \lesssim \tau(p)$.

We now let $U(\alpha)$ denote the set of uniform ultrafilters over α, i.e., those elements p of $\beta(\alpha)$ for which $|A| = \alpha$ whenever $A \in p$; and we say that a partially ordered set S is *γ-directed* provided that each subset of S with fewer than γ elements has an upper bound (in S). The following results concerning the Rudin–Keisler order are known:

(a) $\tau[U(\alpha)]$ contains a family of 2^α mutually incomparable elements (Theorem 2.7 in Kunen [22]);

(b) $T(\beta(\alpha))$ is ω-directed (III.B. 3 in M. E. Rudin [33]; Theorem 3.5 in Booth [2]).

To prove (a), Kunen uses a family of 2^α functions from α to α of ω-large oscillation, and M. E. Rudin proves (b) by considering what is in effect a pair of functions with ω-large oscillation. Because (a) employs transfinite induction, it does not appear susceptible to generalization by the methods of the present paper.

4.3. THEOREM. *Let $\varkappa$ be a regular cardinal for which $\omega \leqslant \varkappa \leqslant \alpha = \alpha^{\underline{\varkappa}}$, and suppose that each $\varkappa$-complete filter on α extends to a $\varkappa$-complete ultrafilter. Then $\tau[\Omega_\varkappa(\alpha)]$ is $(2^\alpha)^+$-directed.*

Proof. [We remark that according to Corollary 4.2 we have $|\Omega_\varkappa(\alpha)| = 2^{2^\alpha}$.]

It will suffice to show that there is a continuous function g from $\Omega_\varkappa(\alpha)$ onto $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$ for which $g[\alpha] \subset \alpha^{(2^\alpha)}$, for then, given elements q_η, $\eta < 2^\alpha$, in $\Omega_\varkappa(\alpha)$, we need only set $q = \langle q_\eta \rangle_{\eta < 2^\alpha}$ in $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$ and find p in $\Omega_\varkappa(\alpha)$ for which $g(p) = q$; then $\tau(q_\eta) \lesssim \tau(p)$ for each η, since $(\pi_\eta \circ g)(p) = q_\eta$.

According to the implication (a) $\Rightarrow$ (f) of Theorem 3.1, there is a map f from α onto a dense subspace of the space $(\alpha^{(2^\alpha)})_\varkappa$. Viewed as a map from α to $\alpha^{(2^\alpha)}$, the function f admits a continuous extension $\bar{f}$ from $\beta(\alpha)$ onto $(\beta(\alpha))^{(2^\alpha)}$. We denote by g the restriction of $\bar{f}$ to $\Omega_\varkappa(\alpha)$.

To verify that g takes $\Omega_\varkappa(\alpha)$ into $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$, fix p in $\Omega_\varkappa(\alpha)$ and $\eta < 2^\alpha$ and set $q_\eta = (\pi_\eta \circ g)(p)$. Then if $\{A_\xi\}_{\xi<\lambda}$ is a collection of elements of the ultrafilter q_η (with $\lambda < \varkappa$), so that $(\pi_\eta \circ g)^{-1}(A_\xi) \in p$ for each ξ, there is an element B of p for which

$$B \subset \bigcap_{\xi<\lambda} (\pi_\eta \circ g)^{-1}(A_\xi);$$

then $(\pi_\eta \circ g)[B] \subset \bigcap_{\xi<\lambda} A_\xi$. Further, $(\pi_\eta \circ g)[B] \in q_\eta$: for $p \in \mathrm{cl}_{\beta(\alpha)} B$, so that

$$q_\eta = (\pi_\eta \circ g)(p) \in \mathrm{cl}_{\beta(\alpha)} (\pi_\eta \circ g)[B].$$

This shows that q_η is $\varkappa$-complete, i.e., that $q_\eta \in \Omega_\varkappa(\alpha)$.

We note that the space $\Omega_\varkappa(\alpha)$ is $\varkappa$-complete, in the sense that the intersection of fewer than $\varkappa$ open-and-closed subsets of $\Omega_\varkappa(\alpha)$ is open-and-closed. Indeed, given $p \in \bigcap_{\xi<\lambda} U_\xi$ with each U_ξ open-and-closed in $\Omega_\xi(\alpha)$ and $\lambda < \varkappa$, there is $A_\xi \in p$ for which

$$(\mathrm{cl}_{\beta(\alpha)} A_\xi) \cap \Omega_\varkappa(\alpha) \subset U_\xi;$$

let $A \subset \bigcap_{\xi<\lambda} A_\xi$ with $A \in p$; then $(\mathrm{cl}_{\beta(\alpha)} A) \cap \Omega_\varkappa(\alpha)$ is a neighborhood in $\Omega_\varkappa(\alpha)$ of p, and a subset of $\bigcap_{\xi<\lambda} A_\xi$.

It remains to show that g takes $\Omega_\varkappa(\alpha)$ onto $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$. Let

$$q = \langle q_\eta \rangle_{\eta<2^\alpha} \in (\Omega_\varkappa(\alpha))^{(2^\alpha)}$$

and let $\mathcal{U}$ be the family of $\varkappa$-box neighborhoods in $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$ of q. Because $|R(U)| < \varkappa$ for each element U of $\mathcal{U}$ and $\varkappa$ is regular, the intersection of fewer than $\varkappa$ elements of $\mathcal{U}$ again belongs to $\mathcal{U}$ and therefore, because $g[\alpha]$ is dense in $((\Omega_\varkappa(\alpha))^{(2^\alpha)})_\varkappa$, each such intersection meets $g[\alpha]$. Thus $\{g^{-1}(U) \cap \alpha \colon U \in \mathcal{U}\}$ is a $\varkappa$-complete filter on α which, accordingly, extends to an element p of $\Omega_\varkappa(\alpha)$. We claim that $g(p) = q$. For each neighborhood U of q in $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$ we have $g^{-1}(U) \cap \alpha \in p$, i.e.,

$$p \in \mathrm{cl}_{\beta(\alpha)}\big(g^{-1}(U) \cap \alpha\big),$$

and therefore, because g is continuous from $\Omega_\varkappa(\alpha)$ to $(\Omega_\varkappa(\alpha))^{(2^\alpha)}$, we have

$$g(p) \in \mathrm{cl}_{\beta(\alpha)}(U \cap \alpha^{(2^\alpha)})$$

for each such U. Thus $g(p) = q$ and the proof is complete.

As special cases of Theorem 4.3 we have the following statements.

4.4. COROLLARY. *Let* $\alpha \geqslant \omega$.

(a) *The set* $T(\beta(\alpha))$ *is* $(2^\alpha)^+$-*directed;*

(b) *if* $\varkappa$ *is a strongly compact cardinal and* $\alpha = \alpha^{\underline{\varkappa}}$, *then* $\tau[\Omega_\varkappa(\alpha)]$ *is* $(2^\alpha)^+$-*directed;*

(c) *if* $\alpha \notin \mathrm{C}_1^*$, *then* $\tau[\Omega_\alpha(\alpha)]$ *is* $(2^\alpha)^+$-*directed.*

We have seen already that $|\tau[\Omega_\varkappa(\alpha)]| \leqslant 2^{2^\alpha}$ and that each element of $\tau[\Omega_\varkappa(\alpha)]$ has at most 2^α predecessors. Each linearly ordered set has a well-ordered, cofinal subset, and thus we have the following consequence of Theorem 4.3.

4.5. COROLLARY. *Let* $\varkappa$ *be a regular cardinal for which* $\omega \leqslant \varkappa \leqslant \alpha = \alpha^{\underline{\varkappa}}$, *and suppose that each* $\varkappa$-*complete filter on* α *extends to a* $\varkappa$-*complete ultrafilter. If* $(2^\alpha)^+ = 2^{2^\alpha}$, *then* $\tau[\Omega_\varkappa(\alpha)]$ *has a well-ordered, cofinal subset with* 2^{2^α} *elements.*

Now let $\mathfrak{F}_\omega(\alpha)$ denote the set of nonempty, finite subsets of α. Following Keisler [18] we say that an ultrafilter p on α is α^+-*good* if for each map f from $\mathfrak{F}_\omega(\alpha)$ to p, with $f(F_1) \supset f(F_2)$ whenever $F_1 \subset F_2$, there is a function g from $\mathfrak{F}_\omega(\alpha)$ to p for which $g(F_1) \subset f(F_1)$ and $g(F_1 \cup F_2) = g(F_1) \cap g(F_2)$ whenever $F_1 \in \mathfrak{F}_\omega(\alpha)$ and $F_2 \in \mathfrak{F}_\omega(\alpha)$. The set of α^+-good ultrafilters on α is denoted $G(\alpha)$.

The set of countably incomplete ultrafilters on α—i.e., those ultrafilters which are not ω^+-complete—is denoted $I(\alpha)$.

4.6. LEMMA. *Let* $\alpha \geqslant \omega$.

(a) $\tau[U(\alpha)]$ *is cofinal in* $T(\beta(\alpha))$;

(b) $\tau[I(\alpha) \cap G(\alpha)]$ *is cofinal in* $\tau[I(\alpha)]$.

Proof. Let $p \in \beta(\alpha)$ and let $\{A_\xi\}_{\xi<\alpha}$ be a family of pairwise disjoint subsets of α, each with α elements, for which $\bigcup_{\xi<\alpha} A_\xi = \alpha$. There is, for $\xi < \alpha$, a one-to-one mapping f_ξ from A_ξ onto α, and for each $\xi < \alpha$ there is a unique element p_ξ in $\mathrm{cl}_{\beta(\alpha)} A_\xi$ for which $\bar{f}_\xi(p_\xi) = p$ (here $\bar{f}_\xi$ is the continuous extension of f_ξ which takes $\mathrm{cl}_{\beta(\alpha)} A_\xi$ onto $\beta(\alpha)$).

Now in (a) let q be any element of $\beta(\alpha)$ which is uniform over $\{p_\xi\}_{\xi<\alpha}$ (i.e., $q \in \beta(\alpha)$ and each neighborhood in $\beta(\alpha)$ of q contains α of the ultrafilters p_ξ) and in (b), it being assumed that p is countably incomplete, let q be any element of $\beta(\alpha)$ which is α^+-good over $\{p_\xi\}_{\xi<\alpha}$ (the existence of such an ultrafilter without the generalized continuum hypothesis is proved by Kunen in [22], by another application of the existence of families of large oscillation). Evidently $\bar{f}(q) = p$, where f is the map from α to α which agrees with f_ξ on A_ξ, so that $\tau(p) \precsim \tau(q)$. In (a) we have $q \in U(\alpha)$ because q was chosen uniform over $\{p_\xi\}_{\xi<\alpha}$. And in (b) the ultrafilter q, chosen to be α^+-good over $\{p_\xi\}_{\xi<\alpha}$, is in fact α^+-good over α by Theorem 6 of Keisler [18]; that q is in $I(\alpha)$ follows from the facts that $p \in I(\alpha)$ and $p \precsim q$.

4.7. THEOREM. *Let* $\alpha \geqslant \omega$.

(a) *The sets* $\tau[U(\alpha)]$, $\tau[I(\alpha)]$, *and* $\tau[I(\alpha) \cap G(\alpha)]$ *are* $(2^\alpha)^+$-*directed;*

(b) *if* α *is smaller than the first uncountable measurable cardinal and* $(2^\alpha)^+ = 2^{2^\alpha}$, *then* $T(\beta(\alpha))$ *contains a well-ordered, cofinal subset of* 2^{2^α} *types of* α^+-*good ultrafilters.*

Proof. That $\tau[U(\alpha)]$ is $(2^\alpha)^+$-directed follows from 4.4(a) and 4.6(a); that $\tau[I(\alpha)]$ is $(2^\alpha)^+$-directed follows from 4.4(a) and that fact that $q \in I(\alpha)$ whenever $p \precsim q$ and $p \in I(\alpha)$; the rest of (a) then follows from 4.6(b).

It is well-known that if neither α nor any of its predecessors is measurable, then each non-principal ultrafilter on α is countably incomplete. Thus $G(\alpha) \subset U(\alpha) \subset I(\alpha)$, so assertion (b) follows from (a) and 4.5 and 4.6(b).

References

[1] H. Bachmann, *Transfinite Zahlen*, Berlin 1955.
[2] D. Booth, *Ultrafilters on a countable set*, Annals of Math. Logic 2 (1970), pp. 1–24
[3] M. Bockstein, *Un théorème de séparabilité pour les produits topologiques*, Fund Math. 35 (1948), pp. 242–246.
[4] M. Całczyńska-Karłowicz, *Theorem on families of finite sets*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 12 (1964), pp. 87–89.
[5] W. W. Comfort and A. W. Hager, *Estimates for the number of real-valued continuous functions*, Trans. Amer. Math. Soc. 150 (1970), pp. 619–631.
[6] — and S. Negrepontis, *Some topological properties associated with measurable cardinals*, Fund. Math. 69 (1970), pp. 191–205.
[7] — — *The Theory of Ultrafilters*, to appear.
[8] R. O. Davies, *An intersection theorem of Erdös and Rado*, Proc. Cambridge Phil. Soc. 63 (1967), pp. 995–996.
[9] W. B. Easton, *Powers of regular cardinals*, Annals of Math. Logic 1 (1970), pp. 139–178.
[10] R. Engelking, *Cartesian products and dyadic spaces*, Fund. Math. 57 (1965), pp. 287–304.
[11] — and M. Karłowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. 57 (1965), pp. 275–285.
[12] P. Erdös and R. Rado, *Intersection theorems for systems of sets*, J. London Math. Soc. 35 (1960), pp. 85–90.
[13] — — *Intersection theorems for systems of sets II*, J. London Math. Soc. 44 (1969), pp. 467–479.
[14] — and A. Tarski, *On families of mutually exclusive sets*, Annals of Math. 44 (1943), pp. 315–329.
[15] G. Fichtenholz and L. Kantorovitch, *Sur les opérations linéaires dans l'espace des fonctions bornées*, Studia Math. 5 (1935), pp. 69–98.
[16] F. Hausdorff, *Über zwei Sätze von G. Fichtenholz und L. Kantorovitch*, Studia Math. 6 (1936), pp. 18–19.

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[17] E. Hewitt, *A remark on density characters*, Bull. Amer. Math. Soc. 52 (1946), pp. 641–643.
[18] H. J. Keisler, *Ideals with prescribed degree of goodness*, Annals of Math. 81 (1965), pp. 112–116.
[19] — and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math. 52 (1962), pp. 225–308.
[20] K. Kunen, *Inaccessibility properties of cardinals*, Doctoral Dissertation, Stanford University, 1968.
[21] — *Some applications of iterated ultrapowers in set theory*, Annals of Math. Logic 1 (1970), pp. 179–227.
[22] — *Ultrafilters and independent sets*, to appear.
[23] E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fund. Math. 34 (1947), pp. 127–143.
[24] W. Marek, *On families of sets*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 12 (1964), pp. 443–448.
[25] S. Mazur, *On continuous mappings in product spaces*, Fund. Math. 39 (1952), pp. 229–238.
[26] D. Monk and D. Scott, *Additions to some results of Erdös and Tarski*, Fund. Math. 53 (1964), pp. 335–343.
[27] A. Mostowski, *Constructible sets with applications*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1969.
[28] N. Noble and M. Ulmer, *Factoring functions on Cartesian products*, Trans. Amer. Math. Soc. 163 (1972), pp. 329–340.
[29] I. I. Parovičenko, *The branching hypothesis and the correlation between local weight and power to topological spaces*, Soviet Math. Doklady 8 (1967), pp. 589–591.
[30] E. S. Pondiczery, *Power problems in abstract spaces*, Duke Math. J. 11 (1944), pp. 835–837.
[31] B. Pospíšil, *Remark on bicompact spaces*, Annals of Math. 38 (1937), pp. 845–846.
[32] K. A. Ross and A. H. Stone, *Products of separable spaces*, American Math. Monthly 71 (1964), pp. 398–403.
[33] M. E. Rudin, *Partial orders on the types in* βN, Trans. Amer. Math. Soc. 155 (1971), pp. 353–362.
[34] N. A. Shanin, *A theorem from the general theory of sets*, Comptes Rendus (Doklady) Acad. Sci. URRS 53 (1946), pp. 399–400.
[35] — *On intersection of open subsets in the product of topological spaces*, Comptes Rendus (Doklady) Acad. Sci. URRS 53 (1946), pp. 499–501.
[36] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, to appear.
[37] E. Szpilrajn, *Remarque sur les produits Cartésiens d'espaces topologiques*, Comptes Rendus (Doklady) Acad. Sci. URRS 31 (1941), pp. 525–527.
[38] A. Tarski, *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math. 16 (1930), pp. 181–304.
[39] — *Ideale in vollständingen Mengenkörpern* I, Fund. Math. 32 (1939), pp. 45–63.

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Menger's Theorem for topological spaces

by

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§ 1. Introduction. Menger's Theorem [1] for graphs has been generalized by Nöbeling [3] to locally connected compact metric spaces. In this paper we generalize Menger's Theorem to Hausdorff topological spaces with no other global conditions on the space, but with local conditions on the two subsets involved.

THEOREM 1.1. *Let A and B be disjoint open subsets of a Hausdorff topological space X. Suppose that the maximal number of disjoint arcs from A to B is finite. Then this number is equal to the minimal number of points that have to be removed from X to separate A and B into different arc components.*

When we restrict X to be a graph, our proof of Theorem 1.1 reduces essentially to Ore's proof of Menger's Theorem [4], Chapter 12.

COROLLARY 1.2 (Menger's Theorem). *Let A and B be disjoint sets of vertices of a finite or infinite graph X. Suppose that there is no edge with one vertex in A and the other in B. Then the maximal number of disjoint arcs from A to B is equal to the minimal number of vertices that have to be removed from X to separate A and B into different components.*

Sections 2 and 3 are devoted to proving Theorem 1.1. In section 4 we show, by example, that some of the conditions of Theorem 1.1 and Nöbeling's result cannot be weakened. Finally in section 5 we discuss the case where the maximal number of arcs is infinite.

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§ 2. Definitions. Let A and B be subsets of a topological space X. Let $I = [0, 1]$ be the closed unit interval and $\mathring{I} = (0, 1)$ be the open unit interval. An *arc* λ from A to B in X is an injective map $\lambda\colon I \to X$ such that $\lambda(0) \in A$ and $\lambda(1) \in B$. The family of arcs $\{\lambda_q\}$ is said to be *disjoint* if

$$\lambda_q(\mathring{I}) \cap \lambda_r(\mathring{I}) = \emptyset$$

for all arcs λ_q, λ_r in the family with $q \neq r$.

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