A reduced free product of lattices

by

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1. The purpose of this note is to provide a generalization of the Basic Lemma of [1]. The new form is more general and easier to apply than the form given in [1].

To state the result we need some notation. For a lattice K with 0 and 1 let $\mathcal{C}(K)$ denote the set of complemented pairs, that is,

$$\mathcal{C}(K) = \big\{\{x, y\}|\ x, y \in K, x\wedge y = 0,\ x\vee y = 1\big\}.$$

K is called a lattice *with no comparable complements* if $\{x, y\}, \{x, z\} \in \mathcal{C}(K)$, and $y \geqslant z$, imply $y = z$.

Let L_λ, $\lambda \in \Lambda$, be pairwise disjoint lattices with 0 and 1. Let $\mathcal{C}$ be a set of two element subsets of $\bigcup(L_\lambda|\ \lambda \in \Lambda)$ such that if $\{x, y\} \in \mathcal{C}$ then for some $\lambda, \mu \in \Lambda$, $x \in L_\lambda$, $y \in L_\mu$, $x \neq 0, 1$, $y \neq 0, 1$, and $\lambda \neq \mu$.

THEOREM 1. *Let L_λ, $\lambda \in \Lambda$, and $\mathcal{C}$ be given as described above. Assume that all L_λ are lattices with more than one element and with no comparable complements, and that $\mathcal{C}$ satisfies the following condition:*

(P) *if* $\{x_1, y_1\}, \{x_2, y_2\} \in \mathcal{C}$, $x_1, x_2 \in L_\lambda$, $y_1, y_2 \in L_\nu$ $(\lambda, \nu \in \Lambda)$, $x_1 \leqslant x_2$, $y_1 \leqslant y_2$, *then* $x_1 = x_2$, *and* $y_1 = y_2$.

Then there exists a lattice L with 0 and 1 satisfying the following conditions:

(i) *L contains all L_λ as $\{0, 1\}$-sublattices;*

(ii) *L is generated by $\bigcup(L_\lambda|\ \lambda \in \Lambda)$;*

(iii) $\mathcal{C}(L) = \bigcup\big(\mathcal{C}(L_\lambda)|\ \lambda \in \Lambda\big) \cup \mathcal{C}$.

A lattice L satisfying Theorem 1 can be described using free products. Let K be the free product of the lattices L_λ, $\lambda \in \Lambda$. (Note that K has neither 0 nor 1 if Λ is infinite.) In terms of $\mathcal{C}$ we define a congruence relation $\Theta(\mathcal{C})$ on K:

$\Theta(\mathcal{C})$ is the smallest congruence relation satisfying the following conditions:

(a) if 0_λ is the zero of L_λ, and $x \leqslant 0_\lambda$, then $x \equiv 0_\lambda\big(\Theta(\mathcal{C})\big)$;

* Research supported by the National Research Council of Canada.

(b) if 1_λ is the unit of L_λ, and $1_\lambda \leqslant x$, then $1_\lambda \equiv x(\Theta(\mathfrak{C}))$;

(c) if $\{x, y\} \in \mathfrak{C}$, and $x \vee y \leqslant z$, then $x \vee y \equiv z(\Theta(\mathfrak{C}))$;

(d) if $\{x, y\} \in \mathfrak{C}$, and $z \leqslant x \wedge y$, then $z \equiv x \wedge y(\Theta(\mathfrak{C}))$.

The quotient lattice $K/\Theta(\mathfrak{C})$ will be called the $\mathfrak{C}$-*reduced free product of the* L_λ, $\lambda \in \Lambda$.

THEOREM 2. *Let L be the $\mathfrak{C}$-reduced free product of the L_λ, $\lambda \in \Lambda$. Then L satisfies conditions* (i), (ii), *and* (iii) *of Theorem* 1.

A few remarks are in order. Firstly, note that of the three conditions of Theorem 1, (i) and (iii) are the important ones. Secondly, observe that apart from a few pathological cases (e.g., $|\Lambda| = 1$, or for $\lambda \neq \mu$, $|L_\lambda| = 1$, or $\mathfrak{C} = \{\{x, y\} |\ x \in L_\lambda,\ y \in L_\mu,\ x \neq 0, 1,\ y \neq 0, 1\}$, i.e. the full graph, and so on) if the conclusions of Theorem 1 hold, then so do the hypotheses. In other words, apart from a few pathological exceptions, Theorem 1 is best possible. Thirdly, Theorem 2 follows from Theorem 1, although we shall proceed the other way around. Nevertheless, Theorem 2 is the more important result in some applications.

The Basic Lemma of [1] is a special case of Theorem 1: one L_λ is arbitrary and all the others are three element chains. It is easily seen that the Basic Lemma does not imply Theorem 1. Theorems 1 and 2 were made possible by the solution of the word problem for free products of lattices given in [3].

The results of [3] needed in the proof are summarized in § 2. Theorem 1 is proved in § 3 and Theorem 2 is proved in § 4. Some applications are given in § 5, a further application will be given in [6].

2. In this section let L_λ, $\lambda \in \Lambda$, be lattices, assumed to be pairwise disjoint. Set $Q = \bigcup(L_\lambda |\ \lambda \in \Lambda)$.

DEFINITION 1 (Lattice polynomials over Q).

(i) If $x \in Q$ then x is a lattice polynomial of length 1; we write $l(x) = 1$.

(ii) If A_0, A_1 are lattice polynomials of length l_0, l_1, respectively, then $A_0 \vee A_1$ and $A_0 \wedge A_1$ are lattice polynomials of length $l_0 + l_1$; $l(A_0 \vee A_1) = l(A_0 \wedge A_1) = l(A_0) + l(A_1)$.

(iii) The only lattice polynomials over Q are those obtained from a finite sequence of applications of (i) and (ii). The set of all lattice polynomials over Q is denoted by $P(Q)$. It may help the reader to observe that $P(Q)$ with the operations $\wedge$ and $\vee$ is the absolutely free algebra over Q.

DEFINITION 2 (Upper and lower λ-cover). For each $A \in P(Q)$ and each $\lambda \in \Lambda$, existence and value of the *upper* λ-*cover*, $A^{(\lambda)}$, and the *lower* λ-*cover*, $A_{(\lambda)}$, are defined as follows:

(i) If $A \in L_\lambda$ then $A_{(\lambda)}$ and $A^{(\lambda)}$ exist, and they are both equal to A; $A_{(\mu)}, A^{(\mu)}$ do not exist for $\mu \neq \lambda$.

(ii) If $A = B \vee C$ then $A^{(\lambda)}$ exists if and only if $B^{(\lambda)}$ and $C^{(\lambda)}$ both exist and in this event $A^{(\lambda)} = B^{(\lambda)} \vee C^{(\lambda)}$ (the join is in L_λ, of course). Furthermore, $A_{(\lambda)}$ exists if and only if at least one of $B_{(\lambda)}, C_{(\lambda)}$ exists; $A_{(\lambda)} = B_{(\lambda)}$ (respectively $C_{(\lambda)}$) if only $B_{(\lambda)}$ (respectively $C_{(\lambda)}$) exists, and $A_{(\lambda)} = B_{(\lambda)} \vee C_{(\lambda)}$ if both $B_{(\lambda)}, C_{(\lambda)}$ exist.

(iii) If $A = B \wedge C$ then $A_{(\lambda)}$ exists if and only if $B_{(\lambda)}$ and $C_{(\lambda)}$ both exist and in this event $A_{(\lambda)} = B_{(\lambda)} \wedge C_{(\lambda)}$. $A^{(\lambda)}$ exists if and only if at least one of $B^{(\lambda)}, C^{(\lambda)}$ exists; $A^{(\lambda)} = B^{(\lambda)}$ (respectively $C^{(\lambda)}$) if only $B^{(\lambda)}$ (respectively $C^{(\lambda)}$) exists, and $A^{(\lambda)} = B^{(\lambda)} \wedge C^{(\lambda)}$ if both $B^{(\lambda)}, C^{(\lambda)}$ exist.

DEFINITION 3 (Quasi ordering on $P(Q)$). For any $A, B \in P(Q)$ we define by induction on $l(A) + l(B)$ the relation $A \subseteq B$ to hold if and only if at least one of the conditions (1) to (6) below holds:

(1) $A = B$;

(2) there is a $\lambda \in \Lambda$ such that $A^{(\lambda)}, B_{(\lambda)}$ exist and $A^{(\lambda)} \leqslant B_{(\lambda)}$ in L_λ;

(3) $A = A_0 \vee A_1$, where $A_0 \subseteq B$ and $A_1 \subseteq B$;

(4) $A = A_0 \wedge A_1$, where $A_0 \subseteq B$ or $A_1 \subseteq B$;

(5) $B = B_0 \vee B_1$, where $A \subseteq B_0$ or $A \subseteq B_1$;

(6) $B = B_0 \wedge B_1$, where $A \subseteq B_0$ and $A \subseteq B_1$.

Set $A \cong B$ if $A \subseteq B$ and $B \subseteq A$.

THEOREM 3 (The structure theorems of free products of lattices [3]).

(i) *The relation $\subseteq$ is a quasi-order (that is, $\subseteq$ is reflexive and transitive) and thus $\cong$ is an equivalence relation.*

(ii) *Given $A \in P(Q)$ let $\langle A \rangle$ denote the equivalence class of A under $\cong$, and let $L = \{\langle A \rangle |\ A \in P(Q)\}$. Define the binary relation $\leqslant$ on L by $\langle A \rangle \leqslant \langle B \rangle$ if and only if $A \subseteq B$. Then $\leqslant$ is a partial order on L with respect to which L is a lattice. Moreover, $\langle A \rangle \vee \langle B \rangle = \langle A \vee B \rangle$ and $\langle A \rangle \wedge \langle B \rangle = \langle A \wedge B \rangle$.*

(iii) *For each $\lambda \in \Lambda$ the mapping φ_λ: $L_\lambda \to L$, given by $\varphi_\lambda(x) = \langle x \rangle$, is a 1-1 lattice homomorphism, and $\langle (\varphi_\lambda |\ \lambda \in \Lambda); L \rangle$ is the free product of the family $(L_\lambda |\ \lambda \in \Lambda)$.*

(iv) *For each $\lambda \in \Lambda$ and $A \in P(Q)$, $A_{(\lambda)}$ exists if and only if $\{x \in L_\lambda |\ \langle x \rangle \leqslant \langle A \rangle\} \neq \emptyset$ and in this event $A_{(\lambda)} = \bigvee(x \in L_\lambda |\ \langle x \rangle \leqslant \langle A \rangle)$, and dually for $A^{(\lambda)}$. Therefore, if both $A_{(\lambda)}$ and $A^{(\lambda)}$ exist, then $A_{(\lambda)} \leqslant A^{(\lambda)}$.*

(v) *For $\lambda, \mu \in \Lambda$ and $A \in P(Q)$, if both $A_{(\lambda)}$ and $A^{(\mu)}$ exist, then $\lambda = \mu$.*

3. In this section let L_λ, $\lambda \in \Lambda$ and $\mathfrak{C}$ be given as in Theorem 1. We denote by 0_λ and 1_λ the zero and unit of L_λ. Set $Q = \bigcup(L_\lambda |\ \lambda \in \Lambda)$ as in § 2. The following definition contains the idea of the proof of Theorem 1.

DEFINITION 4. A subset $R(Q)$ of $P(Q)$ is defined by induction on the length of the polynomial:

(i) if $A \in Q$, then $A \in L_\lambda$ for exactly one $\lambda \in \Lambda$; $A \in R(Q)$ iff A is not 0_λ or 1_λ;

(ii) if $A = B \vee C$, then $A \in R(Q)$ iff $B, C \in R(Q)$ and the following two conditions are satisfied:

(ii_1) $1_\lambda \subseteq A$, for no $\lambda \in \Lambda$;

(ii_2) $x \subseteq B$, $y \subseteq C$, for no $\{x, y\} \in \mathfrak{C}$;

(iii) if $A = B \wedge C$, then $A \in R(Q)$ if $B, C \in R(Q)$ and the following two conditions are satisfied:

(iii_1) $A \subseteq 0_\lambda$, for no $\lambda \in \Lambda$;

(iii_2) $B \subseteq x$, $C \subseteq y$, for no $\{x, y\} \in \mathfrak{C}$.

Now we are ready to construct L:

$$L = \{0, 1\} \cup \{\langle A\rangle \mid A \in R(Q)\}\,,$$

partially ordered by

$$0 < \langle A\rangle < 1 \quad \text{for all} \quad A \in R(Q)\,,$$
$$\langle A\rangle \leqslant \langle B\rangle \quad \text{if} \quad A \subseteq B\,.$$

In other words, $L - \{0, 1\}$ is a subset of the free product; the partial ordering on $L - \{0, 1\}$ is the same as on the free product. Thus L is obviously a partially ordered set.

To show that L is a lattice, take $X, Y \in L$; we have to find $X \vee Y$. If X or $Y \in \{0, 1\}$ this is obvious. So let $X, Y \notin \{0, 1\}$; then $X = \langle B\rangle$, $Y = \langle C\rangle$, $B, C \in R(Q)$. Set $A = B \vee C$. We claim that $X \vee Y = \langle A\rangle$ if A satisfies (ii_1) and (ii_2), and $X \vee Y = 1$ otherwise. This follows from the observation that if $A, A_1 \in P(Q)$, $A \subseteq A_1$, and A violates (ii_1) or (ii_2), then so does A_1. The dual argument now proves that L is a lattice.

For $a \in L_\lambda$, $a \neq 0_\lambda, 1_\lambda$, identify a with $\langle a\rangle$; identify 0_λ with 0 and 1_λ with 1. This makes L_λ a $\{0, 1\}$-sublattice of L. ($\langle a\rangle = \langle b\rangle$ implies $a = b$ follows from (2) of Definition 3; the identification preserves meets and joins in view of the discussion in the previous paragraph.) Thus (i) of Theorem 1 has been verified. (ii) of Theorem 1 is obvious from (ii) and (iii) of Definition 4.

Finally, we verify (iii) of Theorem 1. It follows from (i) of Theorem 1 that $\mathfrak{C}(L_\lambda) \subseteq \mathfrak{C}(L)$. Let $\{x, y\} \in \mathfrak{C}$; ($\mathrm{ii}_2$) and ($\mathrm{iii}_2$) of Definition 4 yield $x \vee y = 1$ and $x \wedge y = 0$ in L, hence $\{x, y\} \in \mathfrak{C}(L)$. This proves $\supseteq$ in (iii) of Theorem 1.

To prove the converse, let $\{X, Y\} \in \mathfrak{C}(L)$, that is $X \wedge Y = 0$, $X \vee Y = 1$. We can assume that $\{X, Y\} \neq \{0, 1\}$, hence $X = \langle A\rangle$, $Y = \langle B\rangle$, $A, B \in R(Q)$. Therefore $A \vee B$ violates (ii_1) or (ii_2) and $A \wedge B$ violates (iii_1) or (iii_2) of Definition 4. The four cases that arise are handled separately.

Case 1. $A \vee B$ violates (ii_1) and $A \wedge B$ violates (iii_1). Hence for some $\lambda, \nu \in \Lambda$, $1 \subseteq A \vee B$, $A \wedge B \subseteq 0_\nu$. Thus by (iv) of Theorem 3, $1_\lambda = (A \vee B)_{(\lambda)}$, and $(A \wedge B)^{(\nu)} = 0_\nu$. Note that if $A_{(\lambda)}$ exists but $B_{(\lambda)}$ does not exist then by (ii) of Definition 2, $A_{(\lambda)} = 1_\lambda$, and so $A \supseteq 1_\lambda$ by (2) of Definition 3, hence $A \notin R(Q)$, contradicting our assumptions. Three more similar arguments and (ii), (iii) of Definition 2 give that $A_{(\lambda)}$, $B_{(\lambda)}$, $A^{(\nu)}$, and $B^{(\nu)}$ exist. Hence, by (v) of Theorem 3, we conclude that $\lambda = \mu$. Since $A_{(\lambda)} \leqslant A^{(\lambda)}$ and $B_{(\lambda)} \leqslant B^{(\lambda)}$ the above equations mean that both $B_{(\lambda)}$ and $B^{(\lambda)}$ are complements of $A_{(\lambda)}$ in L_λ. By the assumption that L_λ has no comparable complements we conclude that $B_{(\lambda)} = B^{(\lambda)}$. Similarly, $A_{(\lambda)} = A^{(\lambda)}$. Therefore, $X = \langle A\rangle = A_{(\lambda)} = A^{(\lambda)} \in L_\lambda$, $Y = \langle B\rangle = B_{(\lambda)} = B^{(\lambda)} \in L_\lambda$, and $\{X, Y\} \in \mathfrak{C}(L_\lambda)$, which was required to prove.

Case 2. $A \vee B$ violates (ii_1) and $A \wedge B$ violates (iii_2). Hence, for some $\lambda \in \Lambda$, $1_\lambda \subseteq A \vee B$, and for some $\{x, y\} \in \mathfrak{C}$, $A \subseteq x$ and $B \subseteq y$. Let $x \in L_\nu$, $y \in L_\mu$ ($\nu, \mu \in \Lambda$, $\nu \neq \mu$). Just as in Case 1 we conclude that $A_{(\lambda)} \wedge B_{(\lambda)} = 1_\lambda$; furthermore, $A^{(\nu)} \leqslant x$, $B^{(\mu)} \leqslant y$. Since $\nu \neq \mu$ we have $\lambda \neq \nu$ or $\lambda \neq \mu$. If, say, $\lambda \neq \nu$ then the existence of both $A_{(\lambda)}$ and $A^{(\nu)}$ is a contradiction. Thus Case 2 cannot occur.

Case 3. $A \vee B$ violates (ii_2) and $A \wedge B$ violates (iii_1). This leads to a contradiction just as Case 2 did.

Case 4. $A \vee B$ violates (ii_2) and $A \wedge B$ violates (iii_2). Let $\{x_1, y_1\}$, $\{x_2, y_2\} \in \mathfrak{C}$, $x_1 \subseteq A$, $y_1 \subseteq B$, $A \subseteq x_2$, $B \subseteq y_2$. Then $x_1 \subseteq x_2$, $y_1 \subseteq y_2$, hence by condition (P) of Theorem 1, $x_1 = x_2$ and $y_1 = y_2$. Hence $x_1 = A$, $y_1 = B$ and so $\{X, Y\} = \{\langle A\rangle, \langle B\rangle\} \in \mathfrak{C}$. This concludes the proof of Theorem 1.

4. To prove Theorem 2 we shall verify that the L constructed in § 3 is isomorphic to $K/\Theta(\mathfrak{C})$. Since $L - \{0, 1\} \subseteq K$, and in view of the behaviour of $\wedge$ and $\vee$ in L, it suffices to show that every congruence class of K modulo $\Theta(\mathfrak{C})$ except $[1_\lambda]\Theta(\mathfrak{C})$ and $[0_\lambda]\Theta(\mathfrak{C})$ contains exactly one element of $L - \{0, 1\}$.

Let $A \in P(Q)$. We prove by induction on the length of A that if $A \not\equiv 1_\lambda(\Theta(\mathfrak{C}))$ and $A \not\equiv 0_\lambda(\Theta(\mathfrak{C}))$, then there exists a $D \in R(Q)$ such that $A \equiv D(\Theta(\mathfrak{C}))$.

Let $A \in Q$; by assumption $A \neq 0_\lambda$ and $A \neq 1_\lambda$ for any $\lambda \in \Lambda$, hence we can take $D = A$. Let $A = A_1 \vee A_2$ and let $A_i \equiv D_i(\Theta(\mathfrak{C}))$, $D_i \in R(Q)$, $i = 1, 2$. Then $A \equiv D_1 \vee D_2(\Theta(\mathfrak{C}))$. If $D_1 \vee D_2 \in R(Q)$, then the proof is finished. Otherwise, by (ii) of Definition 4, $D_1 \vee D_2 \equiv 1_\lambda(\Theta(\mathfrak{C}))$, for any $\lambda \in \Lambda$, hence $A \equiv 1_\lambda(\Theta(\mathfrak{C}))$, contrary to assumption. The dual argument completes the proof.

Since K is the free product of the L_λ, $\lambda \in \Lambda$, there is a homomorphism $\varphi\colon K \to L$ which is the identity map on all L_λ, $\lambda \in \Lambda$. Let Φ denote the congruence of K induced by φ. Obviously, for $A \in R(Q)$, $\langle A\rangle\varphi = \langle A\rangle$.

Hence if $A, B \in R(Q)$, $\langle A\rangle \neq \langle B\rangle$, then $\langle A\rangle \not\equiv \langle B\rangle(\Phi)$. Since $\Theta(\mathfrak{C}) \leqslant \Phi$ is obvious, we conclude that $\langle A\rangle \not\equiv \langle B\rangle(\Theta(\mathfrak{C}))$, showing that every congruence class modulo $\Theta(\mathfrak{C})$ other than $[0_\lambda]\Theta(\mathfrak{C})$ and $[1_\lambda]\Theta(\mathfrak{C})$ contains exactly one element of $L-\{0,1\}$.

5. The concept of $\{0,1\}$-free product of lattices is the same as that of free product of lattices, except that it is applied only to lattices with 0 and 1, and homomorphism is replaced by $\{0,1\}$-homomorphism. Let us make two observations. First, the construction of L in § 3 and the proof that L satisfies (i) and (ii) of Theorem 1 made no use of the assumptions of Theorem 1. Second, the proof of $K/\Theta(\mathfrak{C}) \cong L$ in § 4 is again independent of the assumptions of Theorem 1. Hence this isomorphism holds for L_λ arbitrary and $\mathfrak{C} = \emptyset$, showing that L is the $\{0,1\}$-free product of the L_λ, $\lambda \in \Lambda$. Since the word problem in L is solved we conclude:

THEOREM 4. *The word problem of $\{0,1\}$-free product of lattices L_λ, $\lambda \in \Lambda$, is solved relative to the L_λ, $\lambda \in \Lambda$.*

This result is not new, as it can also be concluded from a result of [5].

Next we specialize Theorem 1 to $\mathfrak{C} = \emptyset$; this result appears to be new.

THEOREM 5. *Let L_λ, $\lambda \in \Lambda$, be lattices with 0 and 1, $0 \neq 1$, and with no comparable complements. Let L be the $\{0,1\}$-free product of the L_λ, $\lambda \in \Lambda$. For $a, b \in L$, a is a complement of b iff for some $\lambda \in \Lambda$, $a, b \in L_\lambda$, and a is a complement of b in L_λ.*

As a further application we prove the following result of R. P. Dilworth [2]:

THEOREM 6. *Every lattice M can be embedded in a uniquely complemented lattice.*

Proof. Let $\Lambda = \{0, 1, 2, \ldots\}$; let L_0 be M with a new zero and unit. Let X_i, $i = 1, 2, \ldots$, be pairwise disjoint infinite sets, $|X_i| = \max\{\aleph_0, |M|\}$; let L_i be the lattice freely generated by X_i with zero and unit added. Since $|X_1| \geqslant |M|$ we can define a function $f_0\colon M \to X_1$ which is one-to-one; set $\mathfrak{C}_0 = \{\{x, y\} \mid y = f_0(x),\ x \in M\}$. Let M_1 be the $\mathfrak{C}_0$-reduced free product of L_0 and L_1. Assuming then M_n has been defined, let f_n be a one-to-one map from the set of non-complemented elements of M_n into X_{n+1}, $\mathfrak{C}_n = \{\{x, y\} \mid x \in M_n,\ y = f_n(x)\}$, and let M_{n+1} be the $\mathfrak{C}_n$-reduced free product of M_n and L_{n+1}. Then $M \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n$; the lattice $L = \bigcup (M_n \mid n = 1, 2, \ldots)$ is the uniquely complemented lattice containing M.

The generalizations of Theorem 6 given in [1] can also be proved in a similar fashion. The present proof of Theorem 6 is equivalent to the proof given in [1].

Finally, we give an application of Theorem 2 which is crucial in some applications that are given in [4]:

THEOREM 7. *Let L_λ, $\lambda \in \Lambda$, $\mathfrak{C}$ and L_λ^1, $\lambda \in \Lambda$ and $\mathfrak{C}^1$ be given as in Theorem 1. For every $\lambda \in \Lambda$ let φ_λ be a $\{0,1\}$-homomorphism of L_λ into L_λ^1 such that if $\{x, y\} \in \mathfrak{C}$, $x \in L_\lambda$, $y \in L_\nu$, then $\{x\varphi_\lambda, y\varphi_\nu\} \in \mathfrak{C}^1$. Let L be the $\mathfrak{C}$-reduced free product of the L_λ, $\lambda \in \Lambda$, and L^1 the $\mathfrak{C}^1$-reduced free product of the L_λ^1, $\lambda \in \Lambda$. Then there exists a $\{0,1\}$-homomorphism φ of L into L^1 such that φ restricted to L_λ is φ_λ, for all $\lambda \in \Lambda$.*

Proof. Let K and K^1 be the free product of the L_λ, $\lambda \in \Lambda$, and the L_λ^1, $\lambda \in \Lambda$, respectively. Since φ_λ maps L_λ into $L_\lambda^1 \subseteq K^1$, by the free product property, there exists a homomorphism ψ of K into K^1, such that ψ restricted to L_λ is φ_λ for all $\lambda \in \Lambda$. Set $L = K/\Theta(\mathfrak{C})$, and $L^1 = K^1/\Theta(\mathfrak{C}^1)$, and let α and α^1 denote the natural homomorphisms. Then $\psi\alpha^1\colon K \to L^1$ is a homomorphism; let Θ be the congruence relation of K induced by $\psi\alpha^1$. We claim that $\Theta(\mathfrak{C}) \leqslant \Theta$. This follows from the assumption that $\{x, y\} \in \mathfrak{C}$ implies $\{x\psi, y\psi\} \in \mathfrak{C}^1$. The computation is based on (a)–(d) of the definition of $\Theta(\mathfrak{C})$, the easy details are left to the reader. Hence there is a natural homomorphism φ from $L = K/\Theta(\mathfrak{C})$ into $K/\Theta \subseteq L^1$. Since α is the identity map on L_λ, α^1 is the identity map on L_λ^1, and ψ restricted to L_λ is φ_λ, the relation $\alpha\varphi = \psi\alpha^1$ implies that φ restricted to L_λ is φ_λ, completing the proof of Theorem 7.

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Reçu par la Rédaction le 24. 3. 1970