On the sum and difference of two sets in topological vector spaces

by

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§ 1. In 1920 H. Steinhaus [9] proved his classical theorem which says that for any subsets A, B of the set of real numbers with a positive Lebesgue measure, the sets $A+B$ and $A-B$ contain an interval. This theorem was then generalized by S. Kurepa [6] and J. H. B. Kemperman [2] (cf. also [3]) to the case of n-dimensional Euclidean spaces.

A topological analogue of H. Steinhaus's theorem has been proved by S. Piccard ([8], pp. 187–188). She has proved that if $A \subset R$ is of the second category and has the Baire property, then the set $A+A$ contains an interval.

The object of the present paper is to prove a topological analogue of H. Steinhaus's theorem for arbitrary sets A and B (second category Baire sets) contained in a topological vector space. This is done in § 3. In § 4 we shall show that M. R. Mehdi's theorem [7] about convex functions is a direct consequence of our theorem.

§ 2. We shall start from some definitions. Let X be an arbitrary topological space. The set $A \subset X$ is of the first category if and only if A is a countable sum of nowhere-dense sets. The set $B \subset X$ is of the second category if and only if it is not of the first category. The set $S \subset X$ has the Baire property if $S = (G \setminus P) \cup R$, where G is open and P and R are first category sets. The set $S \subset X$ is of the first category at a point s if there exists a neighbourhood G_s of s such that $G_s \cap S$ is of the first category.

By $D(S)$ we shall denote the set of all points of the space X at which S is not of the first category.

LEMMA 1 ([5], p. 51–52).

(a) *$S \subset X$ is of the first category if and only if $D(S) = \emptyset$.*

(b) *$D(S)$ is closed.*

(c) $D(S_1 \cup S_2) = D(S_1) \cup D(S_2)$.

(d) $D(S) \subset \overline{S}$.

(e) *If* $S_1 \subset S_2$, *then* $D(S_1) \subset D(S_2)$.

(f) $D(S) = \overline{\operatorname{Int}[D(S)]}$.

A topological field is any set K with the structure of a field and with a topology, which is a T_1-space with respect to this topology and the functions $f_1(x, y) = x+y$, $f_2(x) = -x$, $f_3(x, y) = xy$, $f_4(x) = x^{-1}$ define continuous transformations f_1: $K \times K \to K$, f_3: $K \times K \to K$, f_2: $K \to K$, f_4: $K \setminus \{0\} \to K \setminus \{0\}$, respectively.

A topological vector space X over a topological field K is a set X with the structure of a vector space and with a topology such that X is a T_1-space with respect to this topology and the functions $f_1(x, y) = x+y$, $f_2(x) = -x$, $f_3(x) = \lambda x$ define continuous transformations f_1: $X \times X \to X$, f_2: $X \to X$, f_3: $K \times X \to X$, respectively.

Lemma 2. *Let X be a topological vector space over a topological field K and let $A \subset X$. Then for arbitrary $a \in A$ and $\lambda \in K$ we have*

$$D(\lambda A + a) = \lambda D(A) + a .$$

This follows easily from the formula (cf. M. R. Mehdi [7])

$$V \cap (\lambda A + a) = \lambda \left[\frac{V-a}{\lambda} \cap A \right] + a .$$

§ 3. In this section we shall prove our main result. At first we shall prove the following lemma:

Lemma 3. *Let X be a topological space. If $S \subset X$ is of the second category and has the Baire property, then $D(S) \cap [D(S')]'$ contains a non-empty open set.*

Proof. Since S has the Baire property, it has the form $S = (G \setminus P) \cup R$, where G is an open set and P, R are of the first category. According to Lemma 1 (a) and (c) we have

$$\emptyset \neq D(S) = D[(G \setminus P) \cup R] = D(G \setminus P) = D[(G \setminus P) \cup P] = D(G \cup P) = D(G), \tag{1}$$

and similarly

$$D(S') = D[(G' \cup P) \setminus R] = D(G') . \tag{2}$$

By Lemma 1 (f) we have

$$\emptyset \neq \operatorname{Int} D(G) \subset D(G) . \tag{3}$$

Further, we obtain from Lemma 1 (d)

$$\operatorname{Int} D(G) \subset \operatorname{Int} \bar{G} .$$

Hence and from the fact that G is dense in $\operatorname{Int}\bar{G}$ we have

$$G \cap \operatorname{Int} D(G) \neq 0 . \tag{4}$$

It follows from Lemma 1 (d)

$$D(G') \subset \bar{G}' = G',$$

and hence

$$G \subset [D(G')]' . \tag{5}$$

Relations (3), (4) and (5) imply

$$\emptyset \neq G \cap \operatorname{Int} D(G) \subset [D(G')]' \cap D(G) ,$$

whence by (1) and (2)

$$\emptyset \neq G \cap \operatorname{Int} D(G) \subset [D(S')]' \cap D(S) .$$

This completes the proof.

Theorem 1. *Let X be a topological vector space over a topological field K. If $A, B \subset X$ are second category Baire sets, then both $A+B$ and $A-B$ contain non-empty open set.*

Proof. From Lemma 3 we infer that there exist non-empty open sets V_1 and V_2 satisfying the following conditions:

$$V_1 \subset D(A) \cap [D(A')]', \tag{6}$$

$$V_2 \subset D(B) \cap [D(B')]' . \tag{7}$$

We shall show that

$$V_1 + p \subset A + B , \tag{8}$$

where p is an arbitrary element of V_2.

Suppose that this is not true. Then there exists a d such that

$$d \in V_1 + p \quad \text{and} \quad d \notin A + B .$$

The last condition implies that

$$(-A) \cap (B-d) = \emptyset . \tag{9}$$

Hence and from Lemma 1 (e) we obtain

$$\{D[(-A)']\}' \subset [D(B-d)]' . \tag{10}$$

Since $d \in V_1 + p$ and $p \in V_2$, we have $p - d \in -V_1$ and $p - d \in V_2 - d$. Thus

$$(-V_1) \cap (V_2 - d) \neq \emptyset . \tag{11}$$

From formula (6) and from Lemma 2 we obtain

$$(-V_1) \subset D(-A) \cap \{D[(-A)']\}' , \tag{12}$$

whereas (7) and Lemma 2 imply

$$(V_2 - d) \subset D(B-d) . \tag{13}$$

According to (10) and (12) we have

$$(-V_1) \subset [D(B-d)]',$$

whence we obtain in view of (13)

$$(-\dot{V}_1) \cap (V_2 - d) = O,$$

which contradicts (11).

This contradiction proves formula (8).

So $A+B$ contains the non-empty open set V_1+p. Replacing the set B by $-B$, we obtain the theorem for the set $A-B$, which completes the proof.

The following generalization of the theorem of S. Piccard [8] is direct consequence of the above theorem.

THEOREM 2. *Let X be a topological vector space over a field K. If $S \subset X$ is a second category set with Baire property, then $S+S$ contains a non-empty open set.*

§ 4. Let X be a topological vector space over the field of real numbers. A real-valued function f defined on a convex set $\Delta \subset X$ is called convex if for any $x, y \in \Delta$ we have

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}.$$

The conditions implying the continuity of f can be divided into two classes:

I. the class of measure conditions,

II. the class of topological conditions.

The strongest result from the class I is M. Kuczma's theorem [4] saying that every convex function bounded on a set T such that $m_i J(T) > 0$, is continuous. Here m_i denotes the inner Lebesgue measure and the set $J(T)$ is defined as follows: $T_0 = T$, $T_{n+1} = \frac{1}{2}(T_n + T_n)$, $J(T) = \bigcup_{n=0}^{\infty} T_n$.

The most general theorem from the class II is M. R. Mehdi's theorem [7], namely:

THEOREM 3. *Let Δ be a non-empty open convex set in a topological vector space X, and let f be a convex function on Δ. If f is bounded above on a second category Baire subset S of Δ, then f is continuous on Δ.*

Now, this theorem follows directly from our Theorem 2 and from the fact that every convex function bounded from above on an open set is continuous ([1], p. 116–117).

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Note added in proof. Analogous results for Banach spaces were also obtained by W. Orlicz and Z. Ciesielski (*Some remarks on the convergence of functionals on bases*, Studia Math. 16 (1958), pp. 335-352).

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