A general invariant metrization theorem for compact spaces

by

Karl H. Hofmann (New Orleans, La)

0. Introduction and background

The question of metrizability of uniform spaces may be considered to be more than 50 years old; it is true uniform spaces were introduced in 1937 by A. Weil [18] who immediately established their general properties (including the one that a uniform space is isomorphic to a metric uniform space if and only if its uniform structure has a countable basis); but as early as 1917 Chittenden [5] established what in suitable interpretation might be considered as the first metrization theorem for uniform spaces. Nowadays these questions are treated as standard in any topology text (such as N. Bourbaki's *Topologie Générale*, 1948 and J. L. Kelley's *General Topology*, 1955). The standard method in a special case appears in a paper by Birkhoff ([3], 1936). In fact this latter paper produces a metric on a topological group which is compatible with the topology and is such that all left translations (say) are isometries. A necessary and sufficient condition for existence of such a metric is the existence of a countable neighborhood basis for the identity. Independently this was at the same time noted by Kakutani [13] who even stresses the invariance of the metric. The proof has found entrance into the texts about topological groups, such as Montgomery and Zippin's *Topological Transformation Groups*, 1955; the version in Hewitt and Ross' *Abstract Harmonic Analysis*, 1963, is a slightly different variant. In fact, for groups with a countable basis for their topology the existence of invariant metrization has been established by van Dantzig ([6]) in 1928 who used a process which is based on the ideas of Urysohn of 1925 ([17]). For Lie groups, invariant metrization was probably known much earlier as would be indicated in the 1930 monograph by E. Cartan ([4]). A slightly more general question is the metrization of a topological space X one which a topological group G operates on the left as topological transformation group, whereby one wants a metric relative to which the group operations are isometries. Such metrics should be called invariant. If G is a Lie group and X a quotient space modulo a compact subgroup, then the metrizability as indicated was known to Cartan (op. cit.) 1930. The predecessor of most of the work in this direction, however, is a crucial paper by Eilenberg ([8], 1937), in which the invariant metrizability is established for metrizable X and compact G. On this result rests the observation made by Arens in 1946 ([1]) that invariant metrization is possible for a uniform X with a countable basis for its uniformity and for an equicontinuous G. This in turn is used by Hudson [12] to prove invariant metrizability for the coset space of a compact group (which part of Hudson's results is contained in Eilenberg's) thereby obtaining what Kristensen [14] had proved independently in 1958 (apparently unaware of the generality of Eilenberg's result). More recently, in the investigation of compact semigroups (which generally is considered as one of the youngest branches of topological algebra), several authors have established the existence of metrics d on a compact metrizable topological semigroup S such that $d(sx, ty) \leqslant d(x, y)$ for all $s, t, x, y \in S$ (Friedberg [10], 1968, Hofmann and Mostert [11], 1966, Schneperman [16], 1966); however Hofmann and Mostert are not right in surmising that this was new at the time (loc. cit. p. 60) because Eilenberg's paper contains this result and in fact the more general one that on a metrizable space X on which a compact topological semigroup S operates continuously on the left, there is a metric d with $d(s\cdot x, t\cdot y) \leqslant d(x, y)$ for $s, t \in S$, $x, y \in X$. (Metrizability is assumed also for S but is only used in talking about sequences.)

After this historic outline (which does not claim to be exhaustive) one wonders what there might be left to be desired in the topic of invariant metrization of compact spaces. Yet let us consider the compact additive group G of the ring of p-adic integers. If $|x|_p$ denotes the p-adic norm of the element x then one has $|x+y|_p \leqslant \max\{|x|_p, |y|_p\}$. If one defines a metric d on G by $d(x, y) = |x-y|_p$, then $d(a+x, b+y) \leqslant \max\{d(a, b), d(x, y)\}$. In agreement with current terminology in number theory we call d an *ultrametric*. More generally, if G is a (multiplicatively written) topological group or semigroup, then an *ultrametric* on G is a metric d which defines the topology of G and satisfies

(U) $d(ax, by) \leqslant \max\{d(a, b), d(x, y)\}$.

Such a metric is much more special than an invariant group (or semigroup) metric. One naturally asks the question: What topological groups are ultrametric? The answer is a comparatively simple exercise:

1. *A topological Hausdorff group G has an ultrametric if and only if it has a countable neighborhood basis for the identity consisting of subgroups.*

Since any subgroup with an interior point is open closed, this means that only zero dimensional groups have an ultrametric. Conversely if G is locally compact and zero dimensional, then the neighborhood filter of the identity has a basis of compact open subgroups. Thus for groups even the question of ultrametrics does not provide any difficulties. The situation, however, is strangely different for topological semigroups, even compact ones. While the invariant metrization theorem for compact semigroups follows without much additional effort from the standard metrization construction of uniform spaces (if one does not want to go back to Eilenberg's construction, which is different), this is definitely not the case for ultrametrics. An example of an ultrametric on a semigroup which is not a group is the ordinary distance $d(x, y) = |x-y|$ on the unit interval I relative to the (semilattice) multiplication $(x, y) \mapsto \min\{x, y\}$. None of the multitude of slightly different constructions seems to be of help. Let us look at the problem this way: For a metric d on a space X and any real number $r \geqslant 0$ let $U(d, r) = \{(x, y) : d(x, y) \leqslant r\}$. If X is a semigroup, then d is an ultrametric if and only if all $U(d, r)$ are subsemigroups of $X \times X$. Let us call a semigroup S *ultra-uniform*, if it has a uniform structure with a basis consisting of subsemigroups in $S \times S$. (In such a semigroup multiplication is clearly uniformly continuous.) It is in fact possible to observe this property in a variety of compact semigroups. The metrization process which we are going to describe is specifically designed to yield the following result:

2. *A topological Hausdorff compact semigroup S has an ultrametric if and only if it is ultra-uniform and has a countable basis for its (unique) uniform structure.*

In fact, our metrization process is so general that it contains all the other invariant or subinvariant metrization processes known to me as long as compact spaces are involved.

There are some applications which are of a purely topological nature. For instance it is not hard to derive from our Main Theorem that

3. *On a compact connected locally connected metrizable space there is a metric relative to which all open and closed balls are connected.*

This is in fact a special case of a much stronger theorem proved in 1949 independently by Bing [2] and Moise [15] saying that every space as described in 3 allows a convex, metric where a metric d is called convex if for any pair x, z of elements there is an element y such that $2d(x, y) = 2d(y, z) = d(x, z)$. (This result solves a classical problem posed by Menger in 1928).

Unfortunately, I do not see how our main construction could be carried on to uniform spaces which are essentially more general than compact ones.

In the first section we formulate the definitions and the main results and draw the essential conclusions. As usual, everything carries over to pseudometrics.

The bulk of the proof is given in Section 2. It is completely algebraic and in fact semigroup theoretical. However, Urysohn's classical ideas still shine through. Section 3 finishes the proof of the main theorem and Section 4 contains an outlook on some applications which emphasize the significance of the concept of ultrametrics in a compact semigroup theory.

I am grateful to Jim Rogers for pointing out references [2] and [15] to me.

1. Definitions and main results

1. Let X be a set. A set $A \subset X \times X$ is called a *relation* on X. If A and B are relations on X, the *composition* $A \circ B$ is the relation defined by $\{(a, b)$: there is an $x \in X$ with $(a, x) \in A$ and $(x, b) \in B\}$; the *converse relation* $A^{(-1)}$ of A *is* $\{(x, y)\colon (y, x) \in A\}$. The *diagonal* Δ_X of $X \times X$ (or *equality* on X) is the set $\{(x, x)\colon x \in X\}$. The set $\mathfrak{R}(X)$ of all relations on X containing the equality is a semigroup relative to composition with identity Δ_X and zero $X \times X$; and the function $A \mapsto A^{(-1)}$ is an involution of $\mathfrak{R}(X)$ satisfying $(A \circ B)^{(-1)} = B^{(-1)} \circ A^{(-1)}$. A subset $\mathfrak{A}$ of $\mathfrak{R}(X)$ is called *involutive* if $A \in \mathfrak{A}$ implies $A^{(-1)} \in \mathfrak{A}$. The containment relation $\subset$ on $\mathfrak{R}(X)$ is a partial order such that $A \subset B$ implies $C \circ A \subset C \circ B$ and $A \circ C \subset B \circ C$, $A^{(-1)} \subset B^{(-1)}$, and $A \subset (A \circ C) \cap (C \circ A)$ for all A, B, C in $\mathfrak{R}(X)$.

2. Recall that a *uniform structure* on X is a involutive filter $\mathfrak{U} \subset \mathfrak{R}(X)$ on $X \times X$ with the following property:

For any $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ with $V \circ V \subset U$. The pair $(X, \mathfrak{U})$ is called a *uniform space*. We recall that the neighborhood filter $\mathfrak{U}$ of the diagonal Δ_X in $X \times X$ for a compact space X is a uniform structure and in fact the only one compatible with the topology.

3. Lemma. *Let X be a compact Hausdorff space and $\mathfrak{U}$ its uniform structure. Let A, B be closed subsets of $X \times X$. Then the following condition is satisfied:*

$S(A, B)$: *For all $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ such that*

$$A \circ V \circ B \subset U \circ A \circ B \circ U .$$

Proof. First we observe that $U \circ A \circ B \circ U$ is a neighborhood of $A \circ B$; hence $U \circ A \circ B \circ U$ contains an open neighborhood W of $A \circ B$. If $S(A, B)$ were false then for any $V \in \mathfrak{U}$ there would exist elements $(a_V, u_V) \in A$, $(u_V, b_V) \in V$, $(v_V, b_V) \in B$ with $(a_V, b_V) \notin W$. Since X is compact, the net $V \mapsto (a_V, u_V, v_V, b_V)$ has a cluster point (a, u, v, b). Since $(u, v) \in \bigcap \mathfrak{U} = \Delta$, we have $u = v$. Since A and B are closed, then $(a, u) \in A$ and $(v, b) \in B$, whence (a, b) is in $A \circ B$. On the other hand, (a, b) is a cluster point of the net (a_V, b_V) which is outside the open set W, whence $(a, b) \notin W$, so in particular $(a, b) \notin A \circ B$. This contradiction finishes the proof.

Now we define the crucial concept of a property of a uniform space.

4. Definition. A *property* of a uniform space $(X, \mathfrak{U})$ is an involutive subsemigroup $\mathfrak{P}$ of $\mathfrak{R}(X)$ containing the zero $X \times X$ and satisfying the following condition:

$$\mathfrak{U} \cap \mathfrak{P} \text{ is a basis of } \mathfrak{U} .$$

If all sets in $\mathfrak{P}$ are closed and condition $S(A, B)$ of Lemma 3 is satisfied for all $A, B \in \mathfrak{P}$, we say that $\mathfrak{P}$ is a *smooth* property. If any $\subset$-totally ordered subcollection of $\mathfrak{P}$ which has a lower bound in $\mathfrak{U}$ has its intersection in $\mathfrak{P}$, then $\mathfrak{P}$ is called *complete*.

Note that $\mathfrak{R}(X)$ is a property. If $\mathfrak{R}_c(X)$ denotes the set of all closed members in $\mathfrak{R}(X)$ then $\mathfrak{R}_c(X)$ is a smooth complete property whenever X is compact Hausdorff. In fact all properties contained in $\mathfrak{R}_c(X)$ are then automatically smooth by Lemma 3. Less trivial examples will follow below.

5. If X is a set, then a *pseudometric* on X is a function $d\colon X \times X \to \boldsymbol{R}^+$ (the set of non-negative reals) such that

(a) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(b) $d(x, y) + d(y, z) \leqslant d(x, z)$ for all $x, y, z \in X$.

It is called a *metric*, if moreover

(c) $d(x, y) = 0$ implies $x = y$.

If d is a pseudometric, we set $U(d; r) = \{(x, y) \in X \times X\colon d(x, y) \leqslant r\}$ and $\mathfrak{B}(d) = \{U(d; r)\colon 0 < r\}$. If $\mathfrak{U}(d)$ is the filter generated by the filter-basis $\mathfrak{B}(d)$, then $\mathfrak{U}(d)$ is a uniform structure on X. If $\mathfrak{D}$ is a set of pseudometrics, then $\mathfrak{U}(\mathfrak{D}) = \bigcap \{\mathfrak{U}(d)\colon d \in \mathfrak{D}\}$ is called the *uniform structure associated with* $\mathfrak{D}$. The pair $(X, \mathfrak{D})$ is called a *pseudometric space*, and if d is a metric, then (X, d) is called a *metric space*.

To the classic results of Weil's [17] about uniform spaces belongs the one that for any uniform structure $\mathfrak{U}$ on a set there is a set $\mathfrak{D}$ of pseudometrics with $\mathfrak{U} = \mathfrak{U}(\mathfrak{D})$.

We now define the concept of a property of a pseudometric space.

6. Definition. A *property* of a pseudometric space $(X, \mathfrak{D})$ is an involutive subsemigroup $\mathfrak{P}$ of $\mathfrak{R}(X)$ containing the zero $X \times X$ and satisfying

$$\mathfrak{B}(d) \subset \mathfrak{P} \text{ for all } d \in \mathfrak{D} .$$

Note that any pseudometric space has property $\mathfrak{R}(X)$.

Now let us proceed to consider further examples (and in fact the ones motivating the preceding definitions):

7. Let X be a set and S a semigroup operating on X on the left. (I.e. there is a function $(s,x)\mapsto s\cdot x$: $S\times X\to X$ with $(st)\cdot x = s\cdot(t\cdot x)\cdot$). Then S acts on $X\times X$ under $s\cdot(x,y) = (s\cdot x, s\cdot y)$. The set $\mathfrak{P}$ of all $A\subset X\times X$ such that $S\cdot A\subset A$ is an involutive subsemigroup of $\mathfrak{R}(X)$ containing $X\times X$ and being closed under intersections.

Proof. Let $A, B\in\mathfrak{P}$, and take $(a,b)\in A\circ B$. Then there is an $x\in X$ such that $(a,x)\in A$ and $(x,b)\in B$. Now let $s\in S$. Then $(s\cdot a, s\cdot x) = s\cdot(a,x)\in A$ and $(s\cdot x, s\cdot b) = s\cdot(x,b)\in B$. Hence $s\cdot(a,b) = (s\cdot a, s\cdot b)\in A\circ B$. Thus $S\cdot(A\circ B)\subset A\circ B$. The remaining assertions are even more trivial.

The following are special cases of 7.

7 (a). Let G be a semigroup. Let $X = S = G$ and $s\cdot x = sx$.

7 (b). Let G be a semigroup and G' the semigroup on the set on which G is based with the multiplication $g\circ h = hg$. Set $X = G$, $S = G\times G'$ and $(g,h)\cdot x = gxh$.

If $(X,\mathfrak{R})$ [resp., $(X,\mathfrak{D})$] is a uniform [resp., pseudometric] space having property $\mathfrak{P}$ of 7, then $\mathfrak{U}$ [resp. $\mathfrak{D}$] is called *invariant*. In Case 7 (a) the term *left subinvariant* has been used.

8. Let S be a semigroup. The set $\mathfrak{P}$ of all subsemigroups of $S\times S$ containing the diagonal is an involutive subsemigroup of $\mathfrak{R}(X)$ containing $X\times X$ and being closed under intersections. The proof is not more complicated than the one of 7 and is omitted.

If $\mathfrak{U}$ [resp. $\mathfrak{D}$] is a uniform structure [resp. a set of pseudometrics] on S such that $(X,\mathfrak{U})$ [resp. $(X,\mathfrak{D})$] has property $\mathfrak{P}$ of 8, then $(S,\mathfrak{U})$ is called an *ultra-uniform* semigroup [resp. $(S,\mathfrak{D})$ is called an *ultra-pseudometric semigroup*]. Multiplication in an ultra-uniform semigroup is uniformly continuous. An ultra-pseudometric d on S is characterized by the condition

$$d(ax, by)\leqslant \max\{d(a,b), d(x,y)\}\quad \text{for}\quad a,b,x,y\in S.$$

A metric which is an ultra-pseudometric is called an *ultrametric*.

Now we formulate the general pseudometrization theorem.

9. Main Theorem. *Let $(X,\mathfrak{U})$ be a uniform space with a smooth and complete property $\mathfrak{P}$. Then there is a set $\mathfrak{D}$ of pseudometrics such that $(X,\mathfrak{D})$ has property $\mathfrak{P}$ and $\mathfrak{U}(\mathfrak{D}) = \mathfrak{U}$. If $\mathfrak{U}$ has a countable basis then there is a pseudometric d having property $\mathfrak{P}$ and satisfying $\mathfrak{U}(d) = \mathfrak{U}$. If in addition $\mathfrak{U}$ is separated, then d is a metric.*

With $\mathfrak{P} = \mathfrak{R}_c(X)$ this includes the standard pseudometrization theorem of compact spaces; however, because of the smoothness requirement it does not generalize the general pseudometrization theorem for uniform spaces. In fact it includes the left invariant metrization theorem for first countable topological groups only if the group has a basis of the identity neighborhoods each of which is invariant under inner automorphisms. This covers the abelian case. However, virtually all known results referring to compact spaces are included, e.g. the two sided invariant pseudometrization theorem for compact semigroups (Hofmann and Mostert [11] p. 49) and groups (van Dantzig [6]), the invariant metrization theorem of Eilenberg for compact semigroups S operating continuously on a compact metrizable space X [8] (Eilenberg's space need not be compact, however!) and therefore also the invariant metrization theorems of Hudson [12] and Kristensen [14].

For the sake of completeness we formulate a version of Eilenberg's theorem:

10. Corollary. *Let S be a compact semigroup operating continuously on a compact space X. Then there is a family $\mathfrak{D}$ of pseudometrics defining the topology of X such that*

$$d(s\cdot x, s\cdot y)\leqslant d(x,y)\quad \text{for all}\quad s\in S,\ x,y\in X,\ d\in\mathfrak{D}.$$

If S is effective (i.e. if $s\cdot x = t\cdot x$ for all $x\in X$ implies $s = t$) then the following two conditions are equivalent:

(i) S is a group whose identity acts as identity on X.

(ii) All S-operations are $\mathfrak{D}$-isometries (i.e. $d(s\cdot x, s\cdot y) = d(x,y)$ for all $d\in\mathfrak{D}$).

Proof. We apply the Main Theorem with the property defined in 7. If (i) is satisfied, then

$$d(x,y)\leqslant d(s^{-1}\cdot x, s^{-1}y)\leqslant d\big(s\cdot(s^{-1}x), s\cdot(s^{-1}\cdot y)\big) = d(x,y)\quad \text{for all } d\in\mathfrak{D}.$$

Hence (ii). Suppose that (ii) is satisfied. We may identify S with a compact subsemigroup of the semigroup of all continuous self maps of X with the topology of uniform convergence. If φ is a $\mathfrak{D}$-isometry then s is injective since φ defines the topology. But $\varphi(X)$ is a closed subspace since X is compact; again s cannot be a $\mathfrak{D}$-isometry if $\varphi(X)\neq X$. Thus φ is bijective and hence a homeomorphism. The group of all $\mathfrak{D}$-isometries is equicontinuous and closed in the pointwise topology. By the theorem of Ascoli it is compact. Thus S is a compact subsemigroup of a compact group and is, therefore, a group ([11], p. 77).

It follows, that under the conditions of Corollary 10 the group of units in S is exactly the set of $\mathfrak{D}$-isometries, if S acts effectively.

If the smoothness condition could be eliminated, then Theorem 9 would unquestionably be the most general in the area.

However, Theorem 9 has consequences which do not follow from any of the known metrization theorems.

11. Corollary. *Let S be a compact Hausdorff topological semigroup. If the (unique) uniform structure $\mathfrak{U}$ of S has a basis of subsemigroups in $S\times S$, then the topology of S is defined by a set $\mathfrak{D}$ of ultra-pseudometrics. If the space of S is metrizable, then S has an ultrametric.*

Proof. Let $\mathfrak{P}$ be the set of all closed subsemigroups of $S\times S$ containing the diagonal. Then $(S, \mathfrak{U})$ has the smooth and complete property $\mathfrak{P}$. The result follows from Theorem 9.

12. Corollary. *Every ultra-uniform compact Hausdorff semigroup is the projective limit of ultrametric compact semigroups.*

The proof follows routinely from Corollary 10 (see [11], p. 48 ff.) Some remarks as to the significance of Corollary 12 will be given in Section 4.

There are, however, purely topological applications of the Main Theorem. The following is an example.

13. Corollary. *Let X be a compact connected, locally connected [resp. locally arcwise connected] space. Then there is a set of pseudometrics $\mathfrak{D}$ defining the topology of X such that all balls $\{x\colon d(x,y)\leqslant r\}$ and $\{x\colon d(x,y)<r\}$, $0<r$, $y\in X$, $d\in\mathfrak{D}$ are connected [resp. arcwise connected].*

In particular, if X is metrizable we may find a metric d defining the topology and having this property.

Proof. Let $\mathfrak{P}$ be the set of all closed $A\subset X\times X$, $\Delta_X\subset A$ such that $A(x)$ and $A^{(-1)}(x)$ is connected [resp. arcwise connected], where $A(x) = \{a\in X\colon (x,a)\in A\}$. Let $\mathfrak{U}$ be the uniform structure of X and $U\in\mathfrak{U}$. By the compactness and local connectivity of X, the space is covered by a finite union of open connected sets V_i, $i=1,\ldots,n$, with $V^*\times V_i^*\subset U$. Then $W=\bigcup(V_i^*\times V_i^*)\in\mathfrak{U}$ and $W\subseteq U$. Hence $\mathfrak{U}\cap\mathfrak{P}$ is a basis for $\mathfrak{U}$. Moreover, suppose that $A, B\in\mathfrak{P}$. Then for any $x\in X$ we have $(A\circ B)(x) = \bigcup\{B(y)\colon y\in A(x)\}$; since $A(x)$ and $B(y)$ are always connected, so is $(A\circ B)(x)$. Similarly, $(A\circ B)^{(-1)}(x) = (B^{(-1)}\circ A^{(-1)})(x)$ is connected. [These arguments maintain in the case of arcwise connectivity.] Thus $\mathfrak{P}$ is a subsemigroup of $\mathfrak{R}(X)$. It is clear that $A\in\mathfrak{P}$ implies that $A^{(-1)}$ is in $\mathfrak{P}$. Moreover, the intersection of a $\subset$-totally ordered subset of is again in $\mathfrak{P}$. Hence $\mathfrak{P}$ is a smooth and complete property of (X, U). By Theorem 9 there is a family of pseudometrics with property $\mathfrak{P}$ defining the topology. If $d\in\mathfrak{D}$ and $U(d;r)=\{(x,y)\colon d(x,y)\leqslant r\}$ for $0<r$, then $U(d;r)(y) = \{x\colon d(x,y)\leqslant r\}$ for all $y\in Y$; since $U(d;r)$ is contained in $\mathfrak{P}$, then all closed d-balls are connected [resp. arcwise connected]. Since any d-ball $\{x\colon d(x,y)<r\}$ is the union of the closed d-balls $\{x\colon d(x,y)\leqslant s\}$ with $0<s<r$, all open d-balls are connected [resp. arcwise connected].

For related and in some sense much stronger results see [2], [15].

2. Involutive partially ordered semigroups

1. A semigroup S is said to be *partially ordered relative to* $\leqslant$, if $\leqslant$ is an antisymmetric transitive relation on S satisfying

(1) $a\leqslant b$ implies $ac\leqslant bc$ and $ca\leqslant cb$ for all $a,b,c\in S$,

(2) $a\leqslant ab$ and $a\leqslant ba$ for all $a,b\in S$.

2. A semigroup S is said to be *involutive*, if there is a function $a\mapsto a^*\colon S\to S$ such that

(1) $a^{**}=a$ for all $a\in S$,

(2) $(ab)^*=b^*a^*$.

An element $a\in S$ is called *hermitean* (or *symmetric*) if $a^*=a$. An *involutive partially ordered semigroup* is a partially ordered semigroup with involution $*$ satisfying

(3) $a\leqslant b$ implies $a^*\leqslant b^*$ for all $a,b\in S$.

Remark. It is readily seen that in an involutive semigroup any element of the form a^*a is symmetric and that the identity and zero (if they exist) are symmetric. If a and b are symmetric, then aba is symmetric.

Examples. Any group is an involutive semigroup with involution $a^*=a^{-1}$. The multiplicative semigroup of any involutive algebra (such as a C^*-algebra) is an involutive semigroup. For any set X, the semigroup $\mathfrak{R}(X)$ is an involutive partially ordered semigroup relative to composition of relations and $A\to A^{(-1)}$ as involution and $\subset$ as partial order (Section 1).

3. A *unisemigroup* is an involutive partially ordered semigroup U satisfying the following conditions:

(1) $u\in U$ and $v\in U$ imply the existence of a $w\in U$ with $w\leqslant u$ and $w\leqslant v$.

(2) $u\in U$, $v\in U$ and $u\leqslant v$ imply $v^*\leqslant u^*$.

(3) $u\in U$ implies the existence of a $v\in U$ with $v^2\leqslant u$.

We will call the unisemigroup *smooth*, if in addition it satisfies the following condition:

(4) $a,b,u\in U$ imply the existence of a $v\in U$ with $avb\leqslant uabu$.

Example. If $(X, \mathfrak{U})$ is a uniform space, then $\mathfrak{U}$ is a unisemigroup. More generally, if $(X,\mathfrak{U})$ has a property $\mathfrak{P}$ [resp. a smooth property $\mathfrak{P}$ consisting of closed sets (Section 1)] then $\mathfrak{U}\cap\mathfrak{P}$ is a unisemigroup [resp. a smooth unisemigroup].

The following properties are direct consequences of the definitions; we give the proofs to indicate the general working of the concepts:

3 (a). Let U be a unisemigroup. Then the following statements hold:

(5) $u \in U$ implies the existence of a $v \in U$ with $v^*v \leqslant u$.

(6) $u \in U$ implies the existence of a symmetric w with $w \leqslant u$.

Proof. By 3 (3) there is a $p \in U$ with $p^2 \leqslant u$. By 3 (2) we have $p^* \in U$, and by 3 (1) there is a $w \in U$ with $w \leqslant p$ and $w \leqslant p^*$. From 1 (1) and 2 (2) we deduce $w^*w \leqslant p^{**}\, p = p^2 \leqslant u$. Thus we have (4), and (5) is an immediate consequence thereof.

In the next step we introduce a partial order which generally refines $\leqslant$ properly.

4. Definition. Let U be a unisemigroup. We define $a \ll b$ if and only if there is a $u \in U$ such that $uau \leqslant b$.

Remark. The definition is clearly equivalent to the stipulation that there be elements $u, v \in U$ with $uav \leqslant b$.

The following assertions are straightforward in the spirit of the proofs given above.

5. (1) On a unisemigroup U the relation $\ll$ is transitive and refines $\leqslant$, i.e. $a \ll b$ implies $a \leqslant b$.

(2) If $p \leqslant a \ll b \leqslant q$, then $p \ll q$.

(3) If $a \ll b$ then there is an m with $a \ll m \ll b$.

(4) If a, b are symmetric and $a \ll b$, then there is a symmetric m such that $a \ll m \ll b$.

The following lemmas are used in the main induction step in the proof of the principal theorem.

6. Let I be a finite set. Suppose that x, a_i, b_i are symmetric elements of a smooth unisemigroup U satisfying

$$xa_i,\ a_ix \ll b_i\,, \qquad i \in I\,.$$

Then there is a symmetric y such that

(i) $x \ll y$,

(ii) $ya_i,\ a_iy \leqslant b_i$, $\quad i \in I$.

(This is the only place where smoothness is used in the entire discussion.)

Proof. By 3 (1), (6), and by (1) it suffices to produce y if I is singleton. Therefore suppose that we have symmetric elements x, a, b, c with $xa,\ ax \ll b$. This means that there is a $w \in U$ with $wxaw,\ waxw \leqslant b$. Then we pick a $u \in U$ with $u^2 \leqslant w$. Since U is smooth, we find a symmetric $v \in U$ with $v \leqslant u$ and $xva \leqslant uxau$, $avx \leqslant uaxu$. Let $y = vxv$. Then y is symmetric and $ya = vxva \leqslant vuxau \leqslant u^2xau^2 \leqslant wxaw \leqslant b$. Similarly $ay \leqslant b$.

7. Let x, a be symmetric elements of a smooth unisemigroup. If $x^2 \ll a$, then there is a symmetric y with $x \ll y$ such that $y^2 \leqslant a$.

Proof. There is a $w \in U$ with $wx^2w \leqslant a$. We find $u \in U$ with $u^2 \leqslant w$. Since U is smooth, there is a $v \in U$ such that $xvx \leqslant ux^2u$. Finally we find a symmetric $z \in U$ such that $z^2 \leqslant v$ and $z \leqslant u$. Now let $y = zxz$. Then $x \ll y$ and $y^2 = zxz^2xz \leqslant zxvxz \leqslant zux^2 \leqslant u^2x^2u^2 \leqslant wx^2w \leqslant a$.

For the formulation of the following main result we introduce the following convention: If I is any interval of real numbers, then I_d denotes the set of all dyadic rational numbers $m/2^n$ which are contained in I.

8. Theorem. *Let U be a smooth unisemigroup and $u_0 \geqslant u_1 \dots$ a non-increasing sequence in U. Then there is a function $f\colon\,]0, 1]_d \to U$ satisfying the following conditions:*

(1) $f(r)^* = f(r)$, $r \in\,]0, 1]_d$.

(2) $f(r)f(s) \leqslant f(r+s)$, $r, s, r+s \in\,]0, 1]_d$.

(3) $f(r) \ll f(s)$ *whenever* $r < s$; $r, s \in\,]0, 1]_d$.

(4) $f(1/2^n) \leqslant u_n$, $\ f\big((2q+1)/2^{n+1}\big) \leqslant u_nf(q/2^n)u_n$ *for* $q = 1, \dots, 2^n-1$, *and* $n = 0, 1, \dots$

Proof. We define $f(p/2^n)$, $p = 1, 2, \dots$, $n = 0, 1, \dots$, by induction proceeding as follows:

(a) We let $f(1) \leqslant u_0$.

(b) If $f(p/2^n)$ is defined then, of course, we have

$$f\left(\frac{2^mp}{2^{m+n}}\right) = f\left(\frac{p}{2^n}\right), \qquad m = 0, 1, 2, \dots$$

(c) If $f(p/2^n)$ is defined for $n = 1, \dots, 2^n$, $n = 0, 1, \dots, N$, we produce inductively $f\big(r(k)\big)$, $k = 1, 2, \dots, 2^N$, with $r(k) = (2^{N+1}-2k+1)2^{-(N+1)}$. This will finish the induction.

Now suppose, that indeed $f(r)$ is defined for $r \in J_N = \{p/2^n : p = 1, \dots, 2^n,\ n = 0, 1, \dots, N\}$ such that the following properties are satisfied:

(1_N) $f(r)^* = f(r)$ for $r \in J_N$,

(2_N) $f(r)f(s) \leqslant f(r+s)$ for $r, s, r+s \in J_N$,

(3_N) $f(r) \ll f(s)$ for $r < s$, $r, s \in J_N$

(4_N) $f(1/2^n) \leqslant u_n^2$ for $n = 0, 1, \dots, N$,

$$f\left(\frac{2q+1}{2^{n+1}}\right) \leqslant u_nf\left(\frac{q}{2^n}\right)u_n \qquad \text{for} \qquad q = 1, \dots, 2^n-1,\ n = 1, \dots, N-1\,.$$

Note that $f(2p/2^{N+1})$ is already defined by Remark (b) for $p = 1, \dots, 2^N$. The arguments $r(k)$ are exactly the ones missing in J_{N+1}. By 5 (4) we find a symmetric $f\big(r(1)\big)$ such that $f(1-1/2^n) \ll f\big(r(1)\big) \ll f(1)$. Now we suppose that $f(r(k))$ is defined for $k=1,\dots,K<2^N$, so that conditions (1)–(4) are satisfied for all $r, s, r+s, (2q+1)/2^{n+1} \in J_N \cup \{r(k): k=1,\dots,K\}$. Call this last set $J_{N,K}$.

Let $s=(2^{N+1}-2K-2)/2^{N+1}$; then

$$r(K+1)=\frac{2^{N+1}-2(K+1)+1}{2^{N+1}}=s+1/2^{N+1}.$$

For convenience we let $f(0)$ be an identity associated to U. By induction hypothesis we have

$$f(t)f(s),f(s)f(t)\leqslant f(s+t) \quad \text{for} \quad t\leqslant 1-s,\ t\in J_{N,K}.$$

But $f(s+t)\ll f(s+t+1/2^{N+1})$, whenever $t<1-s$, $t\in J_{N,K}$. Then

$$f(t)f(s),f(s)f(t)\ll f\big(t+r(K+1)\big) \quad \text{for} \quad t\in J_{N,K},\ t+r(K+1)\leqslant 1.$$

Now we apply Lemma 6, Lemma 5 (4) and Definition 3 (1) to obtain a symmetric element $f\big(r(K+1)\big)\in U$ such that

$$f(s)\ll f\big(r(K+1)\big)\ll f\left(\frac{2^N-K}{2^N}\right)$$

and that

$$f(t)f\big(r(K+1)\big),f\big(r(K+1)\big)f(t)\leqslant f\big(t+r(K+1)\big) \quad \text{for}$$
$$t+r(K+1)\leqslant 1,\ t\in J_{N,K}.$$

We also observe that

$$f(s)^2\leqslant f(2s)\ll f\left(2s+\frac{1}{2^N}\right)=f\big(2r(K+1)\big) \quad \text{for} \quad 2s<1.$$

By Lemma 7 and 3 (1) we may then also assume that we have selected $r(K+1)$ in such a fashion that $f\big(r(K+1)\big)^2\leqslant f\big(2r(K+1)\big)$. Finally, we find a symmetric $v\in U$ with $v^2\in u_{N+1}$. Then

$$vf(s)v\leqslant v^2f(s)v^2\leqslant u_{N+1}f(s)u_{N+1}, \quad \text{whence} \quad f(s)\ll u_{N+1}f(s)u_{N+1}.$$

So there is a symmetric $w\in U$ with $f(s)\ll w\ll u_{N+1}f(s)u_{N+1}$. By 3 (1) and 3 (5) we may assume that in fact $f\big(r(K+1)\big)\leqslant w$. Thus $f\big(r(k)\big)$ is defined for $k=1,\dots,K+1\leqslant 2^N$, so that conditions (1)–(4) are satisfied for all $r, s, r+s, (2q+1)/2^{n+1}\in J_{N,K+1}$.

This finishes part (c) of the induction and thereby finishes the proof.

9. Corollary. *Let U be a smooth unisemigroup in which every $\leqslant$-totally ordered set with a lower bound has a greatest lower bound. Let $u_0\geqslant u_1\geqslant\dots$ be a non-increasing sequence in U. Let f be defined as in Theorem 10 and define functions F, G: $]0,1[\to U$ by $F(r)=\inf f([r,1]_d)$, $G(r)=\inf f(]r,1]_d)$. Then the following conditions are satisfied:*

(1) *$F(r)^*=F(r)$, $G(r)^*=G(r)$ for all $r\in]0,1[$.*

(2) *$F(r)F(s)\leqslant F(r+s)$, $G(r)G(s)\leqslant G(r+s)$ for $r, s, r+s\in]0,1[$.*

(3) *$F(r)\leqslant G(r)\ll F(s)\leqslant G(s)$ whenever $0<r<s\leqslant 1$.*

(4) *$G(r)=\inf G(]r,1[)=\inf F(]r,1[)$ for $0<r<1$; $f(r)=F(r)\leqslant G(r)$ for $r\in]0,1[_d$ and $F(r)=G(r)$ for $r\in]0,1[\setminus]0,1[_d$.*

(5) *$F(1/2^n)\leqslant u_n$, $n=0,1,2,\dots$, and $G(1/2^n)\leqslant u_{n-1}$, $n=0,1,2,\dots$*

Proof. The relation $F(r)\leqslant G(r)$ is trivial.

(1) If $S\subset U$ satisfies $S^*=S$, and if $s=\inf S$, then $s^*=\inf S^*$ by 2 (3), hence $s^*=\inf S^*=\inf S=s$.

(2) Let $r, s, r+s\in]0,1[$. Then there are dyadic rationals r', s' with $r<r'$, $s<s'$, $r'+s'<1$. Then we have $F(r)F(s)\leqslant G(r)G(s)\leqslant f(r')f(s')\leqslant f(r'+s')$ by 1 (1) and 8 (2).

But $\inf\{f(r'+s')\colon r'\in[r,1]_d,\ s'\in[s,1]_d,\ r'+s'<1\}=F(r+s)$ and

$$\inf\{f(r'+s')\colon r'\in]r,1]_d,\ s'\in]s,1]_d,\ r'+s'<1\}=G(r+s).$$

Hence (2) is satisfied.

(3) Let $0<r<s\leqslant 1$. Find dyadic rationals r', s' with $r<r'<s'<s$. Then $F(r)\leqslant G(r)\leqslant f(r')\ll f(s')\leqslant F(s)\leqslant G(s)$.

(4) Let $0<r<1$. We have

$$\inf G(]r,1])=\inf\{\inf f(]s,1]_d)\colon r<s<1\}=\inf f(]r,1]_d)=G(r),$$

but also

$$\inf F(]r,1])=\inf\{\inf f([s,1]_d)\colon r<s<1\}=\inf f(]r,1]_d)=G(r).$$

If r is dyadic rational, then clearly $f(r)=F(r)$, if r is otherwise, then $[r,1]_d=]r,1]_d$, whence $F(r)=G(r)$.

(5) By (4) above, $F(1/2^n)=f(1/2^n)$, and by 8 (4), $f(1/2^n)\leqslant u_n$ for all $n=0,1,\dots$ Further, by (3) above, we have $G(1/2^n)\ll F(1/2^{n-1})$ for $n=1,2,\dots$; hence (5).

3. The proof of the Main Theorem

Let $(X,\mathfrak{U})$ be a uniform space and $\mathfrak{P}$ a smooth and complete property of $(X,\mathfrak{U})$. Then $U=\mathfrak{U}\cap\mathfrak{P}$ is a smooth unisemigroup in which every $\subset$-totally ordered subset with a lower bound has a greatest lower bound. If $U_0\supset U_1\supset\dots$ is a sequence of entourages, then we may in fact assume that all of the U_n are in U by the definition of a property.

Now Corollary 9 of Section 2 applies and gives a function $G\colon]0,1[\to U$ with the properties indicated there.

We define a function $d\colon X\times X\to \boldsymbol{R}^+$ by

$$d(x,y) = \inf\big(\{r\in]0,1[\colon (x,y)\in G(r)\}\cup\{1\}\big).$$

Thus $d(x,y)=0$ if and only if $(x,y)\in\bigcap\{G(r)\colon r\in]0,1[\}$ and $d(x,y)=1$ if and only if $(x,y)\notin\bigcup\{G(r)\colon r\in]0,1[\}$. In all other cases we have in fact $d(x,y)=\min\{r\in]0,1[\colon (x,y)\in G(r)\}$ (8 (4) of Section 2). It follows in particular that

(a) $G(r)=\{(x,y)\in X\times X\colon d(x,y)\leqslant r\}$ for $r\in]0,1[$, since trivially, $(x,y)\in G(r)$ implies $d(x,y)\leqslant r$, and $d(x,y)\leqslant r$ means $\min\{s\colon (x,y)\in G(s)\}\leqslant r$ and therefore implies $(x,y)\in G(r)$.

(Note that this conclusion would fail in general if we had used the function F in place of G.) Since all $G(r)$ are symmetric, we have

(b) $d(x,y)=d(y,x)$ for all $x,y\in X$.

We claim that

(c) $d(x,z)\leqslant d(x,y)+d(y,z)$ for all $x,y,z\in X$. If $d(x,y)+d(y,z)\geqslant 1$, then there is nothing to prove.

Now suppose that $d(x,y)+d(y,z)<1$. If $r,s,r+s\in]0,1[$ are chosen so that $(x,y)\in G(r)$ and $(y,z)\in G(s)$ then 8 (2) of Section 2 implies that $(x,z)\in G(r)\circ G(s)\subset G(r+s)$, whence $d(x,z)\leqslant r+s$. Passing to the greatest lower bounds of the sets of admissible r and s we obtain (c).

By 9 (5) we observe that

(d) $d(x,y)\leqslant 1/2^n$ implies $(x,y)\in U_{n-1}$ for $n=1,2,\ldots$

Conditions (a), (b) and (c) show that d is a pseudometric with property $\mathfrak{P}$. There is a set $\mathfrak{F}$ of families $\{U_n\colon n=0,1,\ldots\}$ such that all families are decreasing and that $\bigcup\mathfrak{F}$ is a basis for $\mathfrak{U}$. Hence there is a set $\mathfrak{D}$ of pseudometrics with property $\mathfrak{P}$ such that $\mathfrak{U}=\mathfrak{U}(\mathfrak{D})$. The remainder of Theorem 9 of Section 1 is clear.

4. Applications of ultrametrization

The question of ultrametrization naturally plays an important role in compact group theory and in number theory; but these aspects do not particularly depend on our ultrametrization theorem. However, there is a class of compact semigroups which in some sense is diametrically opposite to the class of groups within the category of compact semigroups, namely the category of compact semilattices (i.e. semigroups with a commutative idempotent multiplication). We had mentioned that the unit interval semilattice is an ultra-uniform semilattice. All totally

disconnected compact semilattices are ultra-uniform. One observes without too much difficulty:

1. *The category of ultra-uniform compact semigroups is complete and closed under the forming of quotients.*

In particular it follows that all subsemilattices of a semilattice $\boldsymbol{I}^X$ where $\boldsymbol{I}$ is the unit interval semilattice and X is any set are ultrauniform. It is less obvious and may be proved with results of Lawson's and our ultrametrization theorem that:

2. For a compact semilattice S the following statements are equivalent:

(a) *S is isomorphic to a subsemigroup of some $\boldsymbol{I}^X$ (i.e. the morphisms $S\to\boldsymbol{I}$ separate the points of S).*

(b) *S is ultrauniform.*

(c) *S is a projective limit of ultrametric compact semilattices.*

(d) *Every point in S has a neighborhood basis consisting of compact subsemilattices.*

(e) *Every point in S has a neighborhood basis consisting of open subsemilattices.*

(f) *S is a quotient of a totally disconnected semigroup.*

It is noteworthy that (e) does not follow straight-forwardly from (d), but does via the ultrauniformity. Also there are semilattices which do not satisfy the conditions of 2. Thus ultrametrization yields an important contribution to the classification of compact semilattices.

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Reçu par la Rédaction le 24. 3. 1969

Finitely generated semigroups of continuous functions on [0,1]

by

Sam W. Young (Salt Lake City, Ut.)

1. Introduction.

Definition 1.1. Let C denote the topological semigroup of continuous function of $[0,1]$ into $[0,1]$ employing the composition product and uniform topology. We will use the norm notation for the uniform metric $\|f-g\| = \sup_{0\leqslant x\leqslant 1} |f(x)-g(x)|$.

Definition 1.2. Let C_0 denote the subsemigroup of C consisting of those elements of C which map $[0,1]$ onto $[0,1]$.

In [2], the authors show that there exist two elements of C which together generate a dense subsemigroup of C. One of the functions is $g(x) = \frac{1}{2}+\frac{1}{2}x$ and the other contains "copies" of elements of a countable dense subset of C. The main result of this paper is Theorem 3.6 which asserts that there are two fairly elementary elements of C_0 which together generate a dense subsemigroup of C_0. The techniques of proof in this paper are entirely different from those in [2].

The motivation for this work comes from the theory of inverse limit spaces. One would like to choose the minimum number of functions and the simplest possible functions as bonding maps in an inverse limit system. In this regard, the corollaries following Theorem 3.6 may be useful. [1] and [5] are applications of [2] to inverse limit spaces.

Without specific reference, all of the functions in this paper are assumed to be in C_0.

2. The prime functions.

Definition 2.1. A function $f \in C_0$ is called *prime* if f is not a homeomorphism and $f = f_1 f_2$ for $f_1, f_2 \in C_0$ implies that either f_1 or f_2 is a homeomorphism.

Definition 2.2. Let PM denote the subsemigroup of C_0 consisting of those functions which are made up of a finite number of strictly monotone pieces. That is, $f \in \mathrm{PM}$ if there exists a partition $0 = a_0 < a_1 < \ldots$