Extensions of congruence relations on infinitary partial algebras A problem of G. Grätzer

by

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§ 1. Introduction. The result due to G. Grätzer and E. T. Schmidt [2], [1] that every partial universal algebra $\mathfrak{A}$ can be embedded into a "largest" universal algebra $\mathfrak{F}^{(\alpha)}(t)/\theta_{\bar{a}}$ generated by $\mathfrak{A}$ (i.e. into a certain factor algebra of the free algebra on α generators over the class of all algebras of the type of $\mathfrak{A}$) such that every congruence relation θ on $\mathfrak{A}$ can be extended to a congruence relation on $\mathfrak{F}^{(\alpha)}(t)/\theta_{\bar{a}}$ is well known.

Various weaker or stronger forms of this result and less or more involved proofs thereof (see e.g. [1], [2], [3]) are known and its applications (as a.o. in the theory of free algebras generated by a partial algebra over a class of algebras [1] or in the representation theory, e.g. [1], [2], [5]) justify its special consideration. The crucial point of the extension theorem is the suitable definition of the congruence relation $\theta_{\bar{a}}$ which is obvious in case $\mathfrak{A}$ is a universal algebra and owes its less obvious definition in case of a partial universal algebra $\mathfrak{A}$ to G. Grätzer [1]. The latter definition of $\theta_{\bar{a}}$ (¹) is not applicable if we switch from partial universal algebras to infinitary partial universal algebras is seen by the following example:

Let $F = \{f_0, \ldots, f_\gamma, \ldots\}_{\gamma<\beta}$ be a non-empty set of ν_γ-ary operations, $\nu_\gamma \geqslant \omega_0$. If $\mathfrak{A} = \langle A;\, F\rangle$ is a partial algebra of the same type τ, $A = \{a_0, \ldots, a_\delta, \ldots\}_{\delta<\alpha}$, $\alpha \geqslant \omega_0$, $D(f_\gamma, \mathfrak{A}) = \emptyset$ for all $f_\gamma \in F$ where $D(f_\gamma, \mathfrak{A})$ denotes the domain of f_γ in $\mathfrak{A}$, then obviously $\theta_{\bar{a}} = \omega$ (i.e. if $\mathfrak{p}, \mathfrak{q} \in P^{(\alpha)}(\tau)$ then $\mathfrak{p} \equiv \mathfrak{q}\,(\theta_{\bar{a}})$ if and only if $\mathfrak{p} = \mathfrak{q}$) and $\mathfrak{F}^{(\alpha)}(\tau)/\theta_{\bar{a}} \cong \mathfrak{F}^{(\alpha)}(\tau)$. Let $\mathfrak{p} \in P^{(\alpha)}(\tau)$ be built up from $\aleph_0$ different projection symbols $\mathfrak{x}_\gamma$. Then $\mathfrak{p} \equiv \mathfrak{p}\,(\theta_{\bar{a}})$ would (according to (¹)) imply the existence of $0 \leqslant k < \omega_0$, $\mathfrak{r} \in P^{(k)}(\tau)$ and $\mathfrak{p}_i, \mathfrak{q}_i \in P^{(\alpha)}(\tau)$, $0 \leqslant i < k$, such that $\mathfrak{p} = \mathfrak{r}(\mathfrak{p}_0, \ldots, \mathfrak{p}_{k-1})$ and $p_i(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$

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(¹) $\mathfrak{p} \equiv \mathfrak{q}\,(\theta_{\bar{a}})$ holds if and only there are $0 \leqslant k < \omega_0$, $\mathfrak{r} \in P^{(k)}(\tau)$ and $\mathfrak{p}_i, \mathfrak{q}_i \in P^{(\alpha)}(\tau)$, $0 \leqslant i < k$, such that $\mathfrak{p} = \mathfrak{r}(\mathfrak{p}_0, \ldots, \mathfrak{p}_{k-1})$, $\mathfrak{q} = \mathfrak{r}(\mathfrak{q}_0, \ldots, \mathfrak{q}_{k-1})$ and $p_i(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha} = q_i(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$ ([1], theorem II.2.1).

is defined for $i = 0, \ldots, k-1$. Hence, by our choice of $\mathfrak{A}$, $\mathfrak{p}_i = \mathfrak{q}_{\mu_i}$ for all $i = 0, 1, \ldots, k-1$; i.e. $\mathfrak{p} = r(\mathfrak{x}_{\mu_0}, \ldots, \mathfrak{x}_{\mu_{k-1}})$ is built up from $< \aleph_0$ different symbols $\mathfrak{x}_\gamma$. This contradiction proves our point. The problem poses itself: Can we define a congruence relation $\theta_{\bar{a}}$ associated with the infinitary partial algebra $\mathfrak{A}$ on $P^{(\alpha)}(\tau)$ such that we get the same useful extension theorem which we obtained in the finitary case? In other words: Can we find $\theta_{\bar{a}}$ such that $\mathfrak{A}$ is (isomorphic to) a relative subalgebra of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$, every congruence on $\mathfrak{A}$ can be extended to $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ and the characterization of strong congruence relations as given in [3] can still be realized within $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ (where τ it the type of $\mathfrak{A}$)? Can we, moreover, define $\theta_{\bar{a}}$ in such a way that it coincides with the definition of $\theta_{\bar{a}}$ as given e.g. in [1] in case where the type is finitary? Even more: Can we give a description of $\theta_{\bar{a}}$ which is a natural extension of the neat description in the finitary case? This paper will answer the questions in the affirmative and thereby settle research-problem II.4 as proposed by G. Grätzer in [1]. It accomplishes a bit more though: We know ([3]) that the strong congruence relations on a partial universal algebra are exactly those which can be extended to a congruence relation θ' on an algebra $\mathfrak{B}$ containing $\mathfrak{A}$ in such a manner that $A = [A]\theta'$ where $[A]\theta' = \bigcup([a]\theta';\ a \in A)$ and $[a]\theta'$ denotes the congruence block of a modulo θ'. G. Grätzer [1] observed that $\mathfrak{B}$ can always be chosen to be $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$. We add that, in the other extreme, we can extend every congruence relation θ on $\mathfrak{A}$ to θ' on $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ in such a way that $[A]\theta' = P^{(\alpha)}(\tau)/\theta_{\bar{a}}$; this holds for both infinitary and finitary types. Finally, the author believes that not only the generalizations of the theorems but also the proofs given are of interest, for they successfully avoid the computational difficulties tied up with the proof of the transitivity of $\theta_{\bar{a}}$ as presented till now. This yields in particular a simpler proof for the extension theorem even in the finitary case. (The reader will observe that our proof of the latter theorem becomes even simpler if we start out with finitary algebras, since then finite induction on the rank of the polynomial symbols involved is applicable.) As has doubtless become clear from the style of this introduction we presuppose the reader's familiarity with the basic concepts connected with (universal) algebras, infinitary algebras and partial (infinitary) algebras (see, e.g., [1], [4], [6]). The terminology is in accordance with [1].

§ 2. The definition of $\theta_{\bar{a}}$. Let $\mathfrak{A} = \langle A; F\rangle$ be an infinitary partial algebra, $A = \{a_0, a_1, \ldots, a_\gamma, \ldots\}_{\gamma<\alpha}$, $\bar{a} = \langle a_0, a_1, \ldots, a_\gamma, \ldots\rangle_{\gamma<\alpha}$, $F = \{f_0, f_1, \ldots, f_\delta, \ldots\}_{\delta<\beta}$. If $\mathfrak{P}^{(\alpha)}(\tau)$ is the algebra of polynomial symbols of type τ determined by $\mathfrak{A}$, then we call all polynomial symbols $\notin X$ which are used in the build-up of the polynomial symbol $\mathfrak{p}$ (including $\mathfrak{p}$ itself) the "components" of $\mathfrak{p}$. We define the algebra $\mathfrak{T}(\mathfrak{A}) = \langle T(\mathfrak{A}); F\rangle$ of type τ as follows:

(i) The set of symbols $X' = \{\mathfrak{x}'_0, \mathfrak{x}'_1, \ldots, \mathfrak{x}'_\gamma, \ldots\}_{\gamma<\alpha}$ is in $T(\mathfrak{A})$.

(ii) If $\mathfrak{p} \in P^{(\alpha)}(t)\setminus X$, then $\mathfrak{p}' \in T(\mathfrak{A})$ if and only if $q(a_0, \ldots, a_\gamma, \ldots)_{\gamma<\alpha}$ is undefined for every component $\mathfrak{q}$ of $\mathfrak{p}$.

(iii) $T(\mathfrak{A})$ consists precisely of the elements described in (i), (ii), and equality is formal equality.

(iv) If $f_\gamma \in F$, then $f_\gamma(\mathfrak{x}'_{\mu_0}, \ldots, \mathfrak{x}'_{\mu_\delta}, \ldots)_{\delta<\nu_\gamma} = \mathfrak{x}'_\varepsilon$ if and only if

$$f_\gamma(a_{\mu_0}, \ldots, a_{\mu_\delta}, \ldots)_{\delta<\nu_\gamma} = a_\varepsilon .$$

(v) If $f_\gamma \in F$, $\{\mathfrak{p}'_0, \ldots, \mathfrak{p}'_\delta, \ldots\}_{\delta<\nu_\gamma} \subseteq T(\mathfrak{A})$ and case (iv) does not apply, then we define

$$f_\gamma(\mathfrak{p}'_0, \ldots, \mathfrak{p}'_\delta, \ldots)_{\delta<\nu_\gamma} = \big(f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}\big)' .$$

The definitions of $T(\mathfrak{A})$ and of the phrase "$p(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$ is defined" make it clear that (iv) and (v) turn every $f_\gamma \in F$ into a well-defined operation on $T(\mathfrak{A})$, and $\mathfrak{T}(\mathfrak{A}) = \langle T(\mathfrak{A}); F\rangle$ is an algebra of type τ. Therefore $\pi\colon\ P^{(\alpha)}(\tau) \to T(\mathfrak{A})$ defined by $p\pi = p(\mathfrak{x}'_0, \ldots, \mathfrak{x}'_\gamma, \ldots)_{\gamma<\alpha}$ is an epimorphism with kernel $\ker(\pi)$ and $\mathfrak{P}^{(\alpha)}(\tau)/\ker(\pi) \cong \mathfrak{T}(\mathfrak{A})$.

Definition 1. The congruence relation $\ker(\pi)$ on $\mathfrak{P}^{(\alpha)}(\tau)$ is denoted by $\theta_{\bar{a}}$ and called the *congruence relation induced by* $\bar{a} = (a_0, \ldots, a_\gamma, \ldots)_{\gamma<\alpha}$.

Remark. If τ is a finitary type then $\theta_{\bar{a}}$ as defined above and as defined by Grätzer in [1], Theorem II.2.1 (see footnote (1)) coincide. We omit a formal proof since it is a consequence of footnote (1) and the remark following Theorem 2. We give a final definition which we need in Theorem 2:

Definition 2. $\mathfrak{p} \in P^{(\alpha)}(\tau)$ is called $\mathfrak{A}$-*irreducible* if $\mathfrak{p}' \in T(\mathfrak{A})$.

§ 3. The extension theorem and a generalization to infinitary algebras of G. Grätzer's description of $\theta_{\bar{a}}$.

Theorem 1. *Let τ be an infinitary type and $\mathfrak{A}$ a partial algebra of type τ. Then $\mathfrak{A}$ is (up to isomorphism) a relative subalgebra of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ if*

$$\bar{a} = \langle a_0, a_1, \ldots, a_\gamma, \ldots\rangle_{\gamma<\alpha} \quad \text{and} \quad A = \{a_0, a_1, \ldots, a_\gamma, \ldots\}_{\gamma<\alpha} .$$

(1) *If θ is a congruence relation on $\mathfrak{A} = \langle A; F\rangle$, then we can extend it to a congruence relation θ' on $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ such that $[A]\theta' = P^{(\alpha)}(\tau)/\theta_{\bar{a}}$ where $[A]\theta' = \{x;\ x \in P^{(\alpha)}(\tau)/\theta_{\bar{a}} \wedge x \equiv a(\theta')$ for some $a \in A\}$.*

(2) *θ is a strong congruence relation on $\mathfrak{A}$ if and only if there is an extension θ' of θ to $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ such that $[A]\theta' = A$.*

Remark. $\langle A; F\rangle$ is a relative subalgebra of $\langle B; F\rangle$ if

$$D(f_\gamma, \mathfrak{A}) = \{(y_0, \ldots, y_\delta, \ldots)_{\delta<\nu_\gamma} \in A^{\nu_\gamma} \cap D(f_\gamma, \mathfrak{B});\ f_\gamma(y_0, \ldots, y_\delta, \ldots)_{\delta<\nu_\gamma} \in A\} .$$

Proof. (1) Clearly, $\mathfrak{A}$ is isomorphic to the relative subalgebra $\mathfrak{X}_1 = \langle\{[\mathfrak{x}_\gamma]\theta_{\bar{a}};\ 0 \leqslant \gamma < \alpha\};\ F\rangle$ of $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ and to the relative subalgebra

$\mathfrak{X}' = \langle X'; F\rangle$ of $\mathfrak{T}(\mathfrak{A})$. So θ induces a congruence relation θ_1 on $\mathfrak{X}_1$, and we have to extend θ_1 to $\mathfrak{P}^{(\alpha)}(\tau)/\theta_{\bar{a}}$ in order to settle the proof. For proof-technical reasons we distinguish between $\mathfrak{X}_1$, $\mathfrak{X}'$ and $\mathfrak{A}$ by choosing pairwise disjoint carrier sets X_1, X', A. Let $C_0, \ldots, C_\gamma, \ldots, \gamma < \xi$, be the congruence blocks of θ in A, $C_0^1, \ldots, C_\gamma^1, \ldots, \gamma < \xi$ the corresponding blocks of θ_1 in X_1. If $S_\gamma = \{C_{\mu_0} \times \ldots \times C_{\mu_\delta} \times \ldots;\ \delta < \nu_\gamma, 0 \leqslant \mu_{\mu_\delta} < \xi\}$, $0 \leqslant \gamma < \beta$, then we consider all intersections $\Delta = R \cap D(f_\gamma, \mathfrak{A})$ with $R \in S_\gamma$. If $\Delta = \emptyset$, then we associate some fixed $a_\Delta \in A$ with Δ; if $\Delta \neq \emptyset$, then we associate some fixed $(x_{\mu_0}^{\Delta}, \ldots, x_{\mu_\delta}^{\Delta}, \ldots)_{\delta<\nu_\gamma} \in \Delta$ with Δ. Since every ν_γ-tuple $(y_0, \ldots, y_\delta, \ldots)_{\delta<\nu_\gamma} \in A^{\nu_\gamma}$ is in some uniquely determined $R \in S_\gamma$, we can extend $D(f_\gamma, \mathfrak{A})$ to A^{ν_γ} as follows:

(i) $f_\gamma(y_0, \ldots, y_\delta, \ldots)_{\delta<\nu_\gamma}$ is defined as before if $(y_0, \ldots, y_\delta, \ldots)_{\delta<\nu_\gamma} \in R \cap D(f_\gamma, \mathfrak{A})$.

(ii) $f_\gamma(y_0, \ldots, y_\delta, \ldots)_{\delta<\nu_\gamma} = f_\gamma(x_{\mu_0}^{\Delta}, \ldots, x_{\mu_\delta}^{\Delta}, \ldots)_{\delta<\nu_\gamma}$ [resp., a_Δ] if $(y_0, \ldots$ $\ldots, y_\delta, \ldots)_{\delta<\nu_\gamma} \in R$ and $\Delta = R \cap D(f_\gamma, \mathfrak{A}) \neq \emptyset$ [resp., $\Delta = \emptyset$] and if case (i) does not apply.

These definitions turn $\mathfrak{A}$ into an infinitary algebra $\mathfrak{A}_1$ of type τ upon which θ is still a congruence relation.

Lemma 1. *φ: $T(\mathfrak{A}) \to A/\theta$ mapping* $\mathfrak{p}'$ *into* $[p(a_0, a_1, \ldots, a_\gamma, \ldots)_{\gamma<a}]\theta$ *defines a homomorphism from* $\mathfrak{T}(\mathfrak{A})$ *into* $\mathfrak{A}_1/\theta$.

Proof. φ is obviously a well-defined mapping.

(a) If $f_\gamma(\mathfrak{x}'_{\mu_0}, \ldots, \mathfrak{x}'_{\mu_\delta}, \ldots)_{\delta<\nu_\gamma} = \mathfrak{x}'_\varepsilon$, then $f_\gamma(a_{\mu_0}, \ldots, a_{\mu_\delta}, \ldots)_{\delta<\nu_\gamma} = a_\varepsilon$ is defined in $\mathfrak{A}$; hence,

$$f_\gamma(\mathfrak{x}'_{\mu_0}, \ldots, \mathfrak{x}'_{\mu_\delta}, \ldots)_{\delta<\nu_\gamma}\varphi = \mathfrak{x}'_\varepsilon\varphi = [a_\varepsilon]\theta = [f_\gamma(a_{\mu_0}, \ldots, a_{\mu_\delta}, \ldots)_{\delta<\nu_\gamma}]\theta$$
$$= f_\gamma([a_{\mu_0}]\theta, \ldots, [a_{\mu_\delta}]\theta, \ldots)_{\delta<\nu_\gamma} = f_\gamma(\mathfrak{x}'_{\mu_0}\varphi, \ldots, \mathfrak{x}'_{\mu_\delta}\varphi, \ldots)_{\delta<\nu_\gamma}.$$

(b) If $f_\gamma(\mathfrak{p}'_0, \ldots, \mathfrak{p}'_\delta, \ldots)_{\delta<\nu_\gamma} \notin X'$, then, by definition, $f_\gamma(\mathfrak{p}'_0, \ldots, \mathfrak{p}'_\delta, \ldots)_{\delta<\gamma}$ $= \big(f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}\big)'$. Hence,

$$f_\gamma(\mathfrak{p}'_0, \ldots, \mathfrak{p}'_\delta, \ldots)_{\delta<\nu_\gamma}\varphi = \big(f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}\big)'\varphi$$
$$= [f_\gamma(p_0, \ldots, p_\delta, \ldots)_{\delta<\nu_\gamma}(a_0, \ldots, a_\delta, \ldots)_{\delta<a}]\theta$$
$$= \big[f_\gamma\big(p_0(a_0, \ldots, a_\delta, \ldots)_{\delta<a}, \ldots, p_\varrho(a_0, \ldots, a_\delta, \ldots)_{\delta<a}, \ldots\big)_{\varrho<\nu_\gamma}\big]\theta$$
$$= f_\gamma\big([p_0(a_0, \ldots, a_\delta, \ldots)_{\delta<a}]\theta, \ldots, [p_\varrho(a_0, \ldots, a_\delta, \ldots)_{\delta<a}]\theta, \ldots\big)_{\varrho<\nu_\gamma}$$
$$= f_\gamma(\mathfrak{p}'_0\varphi, \ldots, \mathfrak{p}'_\delta\varphi, \ldots)_{\delta<\nu_\gamma}.$$

This proves the lemma. Thus, $\pi\varphi$: $P^{(a)}(\tau) \to A$ is a homomorphism from $\mathfrak{P}^{(a)}(\tau)$ into $\mathfrak{A}_1$, $\ker(\pi\varphi) \geqslant \ker\pi = \theta_{\bar{a}}$, and the second isomorphism theorem yields

$$\mathfrak{P}^{(a)}(\tau)/\ker(\pi\varphi) \cong \mathfrak{P}^{(a)}(\tau)/\theta_{\bar{a}}/\ker(\pi\varphi)/\theta_{\bar{a}}.$$

Therefore, $[\mathfrak{x}_\gamma]\theta_{\bar{a}} \equiv [\mathfrak{x}_\delta]\theta_{\bar{a}}\big(\ker(\pi\varphi)/\theta_{\bar{a}}\big)$ is equivalent to $\mathfrak{x}_\gamma \equiv \mathfrak{x}_\delta\big(\ker(\pi\varphi)\big)$, i.e. to $[a_\gamma]\theta = [a_\delta]\theta$, i.e. to $a_\gamma \equiv a_\delta(\theta)$ or, finally, to $[\mathfrak{x}_\gamma]\theta_{\bar{a}} \equiv [\mathfrak{x}_\delta]\theta_{\bar{a}}(\theta_1)$. Hence, $\ker(\pi\varphi)/\theta_{\bar{a}}$ restricted to X_1 equals θ_1, and $\theta'_1 = \ker(\pi\varphi)/\theta_{\bar{a}}$ is the desired extension of θ_1 to $P^{(a)}(\tau)/\theta_{\bar{a}}$. Since $P^{(a)}(\tau)\pi\varphi \subseteq A$, we conclude that $[X_1]\ker(\pi\varphi)/\theta_{\bar{a}} = P^{(a)}(\tau)/\theta_{\bar{a}}$ finishing up the proof of part (1) of the theorem.

To see part (2), we assume θ to be strong, choose the blocks $D_0, D_1, \ldots, D_\gamma, \ldots, \gamma < \xi$, of $\ker(\pi\varphi)/\theta_{\bar{a}}$ such that $D_\gamma \supseteq C_\gamma^1$ and define a new equivalence relation θ' on $P^{(a)}(\tau)/\theta_{\bar{a}}$ by the blocks $C_0^1, D_0 \setminus C_0^1, \ldots, C_\gamma^1, D_\gamma \setminus C_\gamma^1, \ldots, \gamma < \xi$. If we study the corresponding partition in the isomorphic algebra $\mathfrak{T}(\mathfrak{A})$, then we see immediately that the new equivalence θ' is actually a congruence relation (we skip the easy details) since θ_1 is strong. Hence, $[X_1]\theta' = X_1$ and θ' extends θ. The converse statement is trivial (as in the finitary case), and the proof is complete.

We now turn to the description of $\theta_{\bar{a}}$ in case of infinitary algebras which together with the example in § 1 settles Problem II.4 [1]. First we prove a lemma which corresponds to another lemma due to the author of this paper in case of finitary types (see Lemma II. 2.1 in [1]).

Lemma 2. *Let* $\mathfrak{p} = f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}$, $\mathfrak{q} = f_\zeta(\mathfrak{q}_0, \ldots, \mathfrak{q}_\delta, \ldots)_{\delta<\nu_\zeta} \in P^{(a)}(\tau)$. $\mathfrak{p} \equiv \mathfrak{q}(\theta_{\bar{a}})$ *holds if and only if* (i) $p(a_0, \ldots, a_\delta, \ldots)_{\delta<a} = q(a_0, \ldots, a_\delta, \ldots)_{\delta<a}$ *or* (ii) $\gamma = \zeta$ *and* $\mathfrak{p}_\delta \equiv \mathfrak{q}_\delta(\theta_{\bar{a}})$, $0 \leqslant \delta < \nu_\gamma$.

Proof. Since the $\mathfrak{A}$-irreducible elements form a complete and irredundant system of representatives of the congruence classes modulo $\theta_{\bar{a}}$ in $P^{(a)}(\tau)$, we can associate a unique $\mathfrak{A}$-irreducible polynomial $\mathfrak{S}_{\mathfrak{p}}$ with every $\mathfrak{p} \in P^{(a)}(\tau)$ such that $\mathfrak{S}_{\mathfrak{p}} = \mathfrak{S}_{\mathfrak{q}}$ holds if and only if $\mathfrak{p} \equiv \mathfrak{q}(\theta_{\bar{a}})$. Thus, $\mathfrak{p} \equiv \mathfrak{q}(\theta_{\bar{a}})$ implies $\mathfrak{p} \equiv \mathfrak{S}_{\mathfrak{p}} = \mathfrak{S}_{\mathfrak{q}} \equiv \mathfrak{q}(\theta_{\bar{a}})$, i.e. $\mathfrak{p}\pi = \mathfrak{q}\pi = \mathfrak{S}'_{\mathfrak{p}} = \mathfrak{S}'_{\mathfrak{q}}$. If $\mathfrak{S}'_{\mathfrak{p}} = \mathfrak{x}'_\varepsilon$ for some ε, then

$$p(\mathfrak{x}'_0, \ldots, \mathfrak{x}'_\delta, \ldots)_{\delta<a} = \mathfrak{x}'_\varepsilon = q(\mathfrak{x}'_0, \ldots, \mathfrak{x}'_\delta, \ldots)_{\delta<a}$$

and

$$p(a_0, \ldots, a_\delta, \ldots)_{\delta<a} = a_\varepsilon = q(a_0, \ldots, a_\delta, \ldots)_{\delta<a}.$$

Otherwise,

$$\mathfrak{p}\pi = f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}\pi = f_\gamma(\mathfrak{S}'_{\mathfrak{p}_0}, \ldots, \mathfrak{S}'_{\mathfrak{p}_\delta}, \ldots)_{\delta<\nu_\gamma}$$

and

$$\mathfrak{q}\pi = f_\zeta(\mathfrak{q}_0, \ldots, \mathfrak{q}_\delta, \ldots)_{\delta<\nu_\zeta}\pi = f_\zeta(\mathfrak{S}'_{\mathfrak{q}_0}, \ldots, \mathfrak{S}'_{\mathfrak{q}_\delta}, \ldots)_{\delta<\nu_\zeta}.$$

$\mathfrak{p}\pi = \mathfrak{q}\pi$ and formal equality shows that $\zeta = \gamma$ and $\mathfrak{S}'_{\mathfrak{p}_\delta} = \mathfrak{S}'_{\mathfrak{q}_\delta}$, $0 \leqslant \delta < \nu_\gamma$. Hence $\zeta = \gamma$ and $\mathfrak{S}_{\mathfrak{p}_\delta} = \mathfrak{S}_{\mathfrak{q}_\delta}$, i.e. $\mathfrak{p}_\delta \equiv \mathfrak{q}_\delta(\theta_{\bar{a}})$ for all $\delta < \nu_\gamma$. This proves the lemma.

Theorem 2. *Let* $\mathfrak{p}, \mathfrak{q} \in P^{(a)}(\tau)$. *Then* $\mathfrak{p} \equiv \mathfrak{q}(\theta_{\bar{a}})$ *holds if and only if there exist* $\psi < \theta = \dim(\mathfrak{A})$, $\mathfrak{r} \in P^{(\psi)}(\tau)$, $\mathfrak{p}_\delta, \mathfrak{q}_\delta \in P^{(a)}(\tau)$, $0 \leqslant \delta < \psi$, *such that* $\mathfrak{p} = r(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\psi}$, $\mathfrak{q} = r(\mathfrak{q}_0, \ldots, \mathfrak{q}_\delta, \ldots)_{\delta<\psi}$ *and* $p_\delta(a_0, \ldots, a_\varrho, \ldots)_{\varrho<a}$ $= q_\delta(a_0, \ldots, a_\varrho, \ldots)_{\varrho<a}$ *for all* $0 \leqslant \delta < \psi$.

Remark. Since $\dim(\mathfrak{A}) \leqslant \omega_0$ in case of finitary types, our description coincides with that given by Grätzer (see footnote (1)) in the finitary case.

Proof of Theorem 2. Let $\mathfrak{p} \equiv \mathfrak{q}(\theta_{\bar{a}})$. We prove the theorem by transfinite induction on the indices of the Borel sets $B_\lambda = B_\lambda(X)$, $0 \leqslant \lambda < \theta$, building up $P^{(\alpha)}(\tau)$ and containing $\mathfrak{p}$ (see [6]).

If $\mathfrak{p} \in B_0$, i.e. $\mathfrak{p} = \mathfrak{x}_\varepsilon$, then, by Lemma 2, $p(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha} = q(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$ and (with $\mathfrak{x}_0 \in P^{(1)}(\tau)$) $\mathfrak{p} = x_0(\mathfrak{p})$, $\mathfrak{q} = x_0(\mathfrak{q})$; this proves the induction beginning. We assume the theorem to be true for all Borel sets $B_\varkappa$ with $\varkappa < \lambda$ and choose $\mathfrak{p} \in B_\lambda = \bigcup(B_\varkappa;\ \varkappa < \lambda) \cup \bigcup\big(f_\gamma(\bigcup(B_\varkappa;\ \varkappa < \lambda));\ 0 \leqslant \gamma < \beta\big)$. If $\mathfrak{p} \in \bigcup(B_\varkappa;\ \varkappa < \lambda)$, then the induction hypothesis applies immediately. If $\mathfrak{p} \in \bigcup\big(f_\gamma(\bigcup(B_\varkappa; \varkappa < \lambda)); 0 \leqslant \gamma < \beta\big)$, say $\mathfrak{p} \in f_\gamma\big(\bigcup(B_\varkappa;\ \varkappa < \lambda)\big)$ for some $0 \leqslant \gamma < \beta$, then $\mathfrak{p} = f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}$ with $\mathfrak{p}_\delta \in \bigcup(B_\varkappa;\ \varkappa < \lambda)$. If $p(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$ is defined, then (by Lemma 2) $q(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$ is defined and $\mathfrak{p} = x_0(\mathfrak{p})$, $\mathfrak{q} = x_0(\mathfrak{q})$ proves our point again. If $p(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$ is undefined, then (by Lemma 2) $\mathfrak{q} = f_\gamma(\mathfrak{q}_0, \ldots, \mathfrak{q}_\delta, \ldots)_{\delta<\nu_\gamma}$, $\mathfrak{q}_\delta \in P^{(\alpha)}(\tau)$, and $\mathfrak{p}_\delta \equiv \mathfrak{q}_\delta(\theta_{\bar{a}})$ for all $\delta < \nu_\gamma$. By induction hypothesis, there are $\varphi_\delta < \theta$, $\mathfrak{r}_\delta \in P^{\varphi_\delta}(\tau)$ and $\mathfrak{p}_\mu^\delta, \mathfrak{q}_\mu^\delta \in P^{(\alpha)}(\tau)$, $0 \leqslant \delta < \nu_\gamma$, $0 \leqslant \mu < \varphi_\delta$, such that

$$\mathfrak{p}_\delta = r_\delta(\mathfrak{p}_0^\delta, \ldots, \mathfrak{p}_\mu^\delta, \ldots)_{\mu<\varphi_\delta}, \qquad \mathfrak{q}_\delta = r_\delta(\mathfrak{q}_0^\delta, \ldots, \mathfrak{q}_\mu^\delta, \ldots)_{\mu<\varphi_\delta}$$

and

$$p_\mu^\delta(a_0, \ldots, a_\varrho, \ldots)_{\varrho<\alpha} = q_\mu^\delta(a_0, \ldots, a_\varrho, \ldots)_{\varrho<\alpha}.$$

We define

$$\psi = \varphi_0 + \varphi_1 + \ldots + \varphi_\delta + \ldots, \qquad \delta < \nu_\gamma$$

and observe that $\psi < \theta$ since $\theta = \dim(\mathfrak{A})$ is not cofinal with any ν_γ, $0 \leqslant \gamma < \beta$. Now we define the ψ-ary polynomials $\mathfrak{r}_\delta^* \in P^{(\psi)}(\tau)$ by

$$r_\delta^*(z_0^0, \ldots, z_{\mu_0}^0, \ldots; \ldots; z_0^\varepsilon, \ldots, z_{\mu_\varepsilon}^\varepsilon, \ldots; \ldots)_{0\leqslant\varepsilon<\nu_\gamma; 0\leqslant\mu_\varepsilon<\varphi_\varepsilon}$$
$$= r_\delta(z_0^\delta, \ldots, z_{\mu_\delta}^\delta, \ldots)_{\mu_\delta<\varphi_\delta}, \qquad 0 \leqslant \delta < \nu_\gamma.$$

Then

$$\begin{aligned}\mathfrak{p} &= f_\gamma(\mathfrak{p}_0, \ldots, \mathfrak{p}_\delta, \ldots)_{\delta<\nu_\gamma}\\ &= f_\gamma\big(r_0(\mathfrak{p}_0^0, \ldots, \mathfrak{p}_\mu^0, \ldots)_{\mu<\varphi_0}, \ldots, r_\delta(\mathfrak{p}_0^\delta, \ldots, \mathfrak{p}_\mu^\delta, \ldots)_{\mu<\varphi_\delta}, \ldots\big)_{\delta<\nu_\gamma}\\ &= f_\gamma(\mathfrak{r}_0^*, \ldots, \mathfrak{r}_\delta^*, \ldots)_{\delta<\nu_\gamma}(\mathfrak{p}_0^0, \ldots, \mathfrak{p}_{\mu_0}^0, \ldots; \ldots; \mathfrak{p}_0^\varepsilon, \ldots, \mathfrak{p}_{\mu_\varepsilon}^\varepsilon, \ldots; \ldots)_{0\leqslant\varepsilon<\nu_\gamma; 0\leqslant\mu_\varepsilon<\varphi_\varepsilon};\end{aligned}$$

similarly

$$\mathfrak{q} = f_\gamma(\mathfrak{r}_0^*, \ldots, \mathfrak{r}_\delta^*, \ldots)_{\delta<\nu_\gamma}(\mathfrak{q}_0^0, \ldots, \mathfrak{q}_{\mu_0}^0, \ldots; \ldots; \mathfrak{q}_0^\varepsilon, \ldots, \mathfrak{q}_{\mu_\varepsilon}^\varepsilon, \ldots; \ldots)_{0\leqslant\varepsilon<\nu_\gamma; 0\leqslant\mu_\varepsilon<\varphi_\varepsilon}.$$

Thus,

$$\mathfrak{p} = r(\mathfrak{p}_0^0, \ldots, \mathfrak{p}_{\mu_0}^0, \ldots; \ldots; \mathfrak{p}_0^\varepsilon, \ldots, \mathfrak{p}_{\mu_\varepsilon}^\varepsilon, \ldots; \ldots)_{0\leqslant\varepsilon<\nu_\gamma; 0\leqslant\mu_\varepsilon<\varphi_\varepsilon},$$

$$\mathfrak{q} = r(\mathfrak{q}_0^0, \ldots, \mathfrak{q}_{\mu_0}^0, \ldots; \ldots; \mathfrak{q}_0^\varepsilon, \ldots, \mathfrak{q}_{\mu_\varepsilon}^\varepsilon, \ldots; \ldots)_{0\leqslant\varepsilon<\nu_\gamma; 0\leqslant\mu_\varepsilon<\varphi_\varepsilon},$$

and

$$p_{\mu_\varepsilon}^\varepsilon(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha} = q_{\mu_\varepsilon}^\varepsilon(a_0, \ldots, a_\delta, \ldots)_{\delta<\alpha}$$

for all $0 \leqslant \varepsilon < \nu_\gamma$, $0 \leqslant \mu_\varepsilon < \varphi_\varepsilon$ where $\mathfrak{r} = f_\gamma(\mathfrak{r}_0^*, \ldots, \mathfrak{r}_\delta^*, \ldots)_{\delta<\nu_\gamma} \in P^{(\psi)}(\tau)$, $\psi < \theta$. Hence, the proof by induction is complete.

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