3. *$(S, \mathcal{F})$ satisfies condition* (ii). Suppose $|p-q| < f_a^{-1}(x)$. Choose $a' = a$ and x' so that $0 < 2f_a^{-1}(x') < f_a^{-1}(x) - |p-q|$.

4. *Finally, $T(a, b) = 0$ for $0 \leqslant a, b < 1$.* Suppose a, b, c given. Choose x so that $f_c^{-1}(2x) < f_a^{-1}(x) + f_b^{-1}(x)$ and then p, q, r such that $|p-q| < f_a^{-1}(x)$, $|q-r| < f_b^{-1}(x)$ and $|p-r| \geqslant f_c^{-1}(2x)$.

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Sequents in many valued logic II *

by

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The notions of validity in classical and intuitionistic logic may be defined semantically by the methods of Tarski [5] and Kripke [2] respectively. If we replace the two truth-values occurring in these definitions by a system of M truth-values, we obtain what may be referred to as classical M-valued logic and intuitionistic M-valued logic respectively. Gentzen [1] gives sequent calculi LK and LJ for classical and intuitionistic logic. The present work is concerned with the many valued analogues of these calculi. We shall limit our attention here to propositional logic; some remarks about predicate logic will be made at the end of the paper. We show that for each choice of M-valued truth-functions there exist corresponding sequent calculi LK_M and LJ_M for classical M-valued logic and intuitionistic M-valued logic respectively. The relation between these calculi is similar to that between LK and LJ. We note that the calculus LK_M differs from the sequent calculus constructed in [3] (§1) in that the notion of sequent is more restricted.

We take $\boldsymbol{M} = \{0, 1, \ldots, M-1\}$ $(M \geqslant 2)$ as the set of truth-values and consider a fixed system of M-valued truth-functions f_k: $\boldsymbol{M}^{r_k} \to \boldsymbol{M}$ $(k = 1, \ldots, u)$. We also choose a set $\mathfrak{A}$ of atomic statements and connectives F_k of degree r_k $(k = 1, \ldots, u)$, thus determining the set $\mathfrak{S}$ of statements. We denote statements by the letters $\alpha, \beta, \gamma, \ldots$, and finite sets of statements by $\Gamma, \Delta, \ldots$

A sequent is an expression of the form

$$\Gamma_0 | \Gamma_1 | \ldots | \Gamma_{M-2} | \Gamma_{M-1}\,, \tag{1}$$

where for each $\alpha \in \mathfrak{S}$ the set $\{m\colon \alpha \in \Gamma_m\}$ is the complement of an interval of $\boldsymbol{M}$. Thus if $\alpha \in \Gamma_{m'}$ then either $\alpha \in \Gamma_{m'}$ for all $m' < m$ or $\alpha \in \Gamma_{m'}$ for all $m' > m$. Sequents will be denoted by the letters $\Pi, \Sigma, \ldots, \Omega$. We observe that the notion of sequent as here defined coincides with that used in [3] only in the case $M = 2$.

* This paper is a sequel to [3]. We note that p. 32 line 18 of [3] should read: $\alpha, \Gamma^{[m]}\gamma = ((J_m \alpha \supset \gamma) \supset \Gamma^{[m]}\gamma)$. It is simpler however to make the correction in the way suggested in [4].

If Π is the sequent (1) then the mth place of Π is the set $\Pi[m] = \Gamma_m$. If Π and Σ are sequents then there exists a sequent $\Pi\Sigma$ whose mth place is $\Pi[m] \cup \Sigma[m]$ for all $m \in \boldsymbol{M}$. If $\Pi[m] \subseteq \Sigma[m]$ for all $m \in \boldsymbol{M}$ we write $\Pi \subseteq \Sigma$. If R is the complement of an interval of $\boldsymbol{M}$ then for each Γ there is a sequent $|\Gamma|_R$ whose mth place is Γ if $m \in R$ and empty otherwise. In particular, if R is the set $[0, m]$ (resp. $[m, M-1]$), then we denote $|\Gamma|_R$ by $|\Gamma|_m^-$ (resp. $|\Gamma|_m^+$).

By a classical valuation we mean a map $v\colon \mathfrak{A} \to \boldsymbol{M}$ which assigns a truth-value to each atomic statement. We define $v^*(a)$ by induction as follows: (i) if $a \in \mathfrak{A}$ then $v^*(a) = v(a)$; (ii) if $a_1, \ldots, a_r \in \mathfrak{S}$ and F_k is of degree $r = r_k$ then

$$v^*(F_k a_1 \ldots a_r) = f_k\big(v^*(a_1), \ldots, v^*(a_r)\big) .$$

If Λ is any partially ordered set then a family $(v_\lambda)_{\lambda\in\Lambda}$ of mappings $v_\lambda\colon \mathfrak{A} \to \boldsymbol{M}$ will be called an intuitionistic valuation if for each $a \in \mathfrak{A}$ we have for all $\lambda, \mu \in \Lambda$

$$v_\lambda(a) \leqslant v_\mu(a) \quad \text{whenever} \quad \lambda \leqslant \mu .$$

We define $v_\lambda^*(a)$ by induction as follows: (i) if $a \in \mathfrak{A}$ then $v_\lambda^*(a) = v_\lambda(a)$; (ii) if $a_1, \ldots, a_r \in \mathfrak{S}$ and F_k is of degree $r = r_k$ then

(2) $$v_\lambda^*(F_k a_1 \ldots a_r) = \inf_{\mu \geqslant \lambda} f_k\big(v_\mu^*(a_1), \ldots, v_\mu^*(a_r)\big) .$$

It is clear that for each $a \in \mathfrak{S}$ we have for all $\lambda, \mu \in \Lambda$

(3) $$v_\lambda^*(a) \leqslant v_\mu^*(a) \quad \text{whenever} \quad \lambda \leqslant \mu .$$

We say that a mapping $h\colon \mathfrak{S} \to \boldsymbol{M}$ satisfies the sequent (1) if $m \in h(\Gamma_m)$ for some $m \in M$. A classical valuation v is said to satisfy the sequent Π if v^* satisfies Π; an intuitionistic valuation $(v_\lambda)_{\lambda\in\Lambda}$ is said to satisfy the sequent Π if v_λ^* satisfies Π for each $\lambda \in \Lambda$. If a sequent is satisfied by every classical (resp. intuitionistic) valuation then it is said to be classically (resp. intuitionistically) valid.

LEMMA. *Let f_k be a truth-function of degree $r = r_k$, and let m be any truth-value. Then there exist families of sets*

$$R_j^i (i \in I^-;\ j = 1, \ldots, r) \quad \text{and} \quad R_j^i (i \in I^+;\ j = 1, \ldots, r)$$

such that each R_j^i is the complement of an interval of $\boldsymbol{M}$ and such that

(4) $$f_k(x_1, \ldots, x_r) \leqslant m \iff \bigwedge_{i\in I^-} [x_1 \in R_1^i \vee \ldots \vee x_r \in R_r^i] ,$$

(5) $$f_k(x_1, \ldots, x_r) \geqslant m \iff \bigwedge_{i\in I^+} [x_1 \in R_1^i \vee \ldots \vee x_r \in R_r^i] .$$

We can suppose the number of conjuncts to be at most $[\frac{1}{2}(M^r+1)]$, and this bound is best possible.

Proof. Any subset S of $\boldsymbol{M}^r$ can be expressed as the union of at most $[\frac{1}{2}(M^r+1)]$ Cartesian products of intervals of $\boldsymbol{M}$. Indeed, if M is even then $\boldsymbol{M}^r$ is the sum of $\frac{1}{2}M^r$ two-element products of intervals, and if M is odd then $\boldsymbol{M}^r$ is the sum of $\frac{1}{2}(M^r-1)$ two-element products of intervals together with a single one-element set; in either case $\boldsymbol{M}^r$ is the sum of $[\frac{1}{2}(M^r+1)]$ one- or two-element products of intervals; we obtain the desired representation of S by forming the intersections of S with each of these one- or two-elements products of intervals, since each such intersection is obviously a product of intervals. The first part of the lemma now follows by an application of this remark to the sets

$$S = \{(x_1, \ldots, x_r) \in \boldsymbol{M}^r\colon f_k(x_1, \ldots, x_r) > m\}$$

and

$$S = \{(x_1, \ldots, x_r) \in \boldsymbol{M}^r\colon f_k(x_1, \ldots, x_r) < m\}$$

respectively. The set $S = \{(m_1, \ldots, m_r)\colon m_1 + \ldots + m_r \equiv 0 \pmod 2\}$ has $[\frac{1}{2}(M^r+1)]$ elements but includes no Cartesian product of intervals with more than one element; from this we deduce that the bound $[\frac{1}{2}(M^r+1)]$ is best possible.

Let f_k be a truth-function of degree $r = r_k$ and let m be a truth-value. For any $a_1, \ldots, a_r \in \mathfrak{S}$ the sequent

$$|a_1|_{R_1^i} \ldots |a|_{R_r^i}$$

will be denoted by $\Pi_i^-(a_1, \ldots, a_r)$ if $i \in I^-$ or by $\Pi_i^+(a_1, \ldots, a_r)$ if $i \in I^+$.

We now describe the rules of the sequent calculi LK_M and LJ_M. Both calculi have the following "weakening" rule:

(Wk) $$\frac{\Sigma}{\Pi\Sigma} .$$

In addition both calculi have the following introduction rules for each F_k and m:

$(F_k, m)^-$ $$\frac{\Pi\Pi_i^-(a_1, \ldots, a_r)\,(i \in I^-)}{\Pi|F_k a_1 \ldots a_r|_m^-}$$

$(F_k, m)^+$ $$\frac{\Pi\Pi_i^+(a_1, \ldots, a_r)\,(i \in I^+)}{\Pi|F_k a_1 \ldots a_r|_m^+}$$

The only difference between the two calculi lies in the fact that in LK_M the rule $(F_k, m)^+$ may be applied unrestrictedly whereas in LJ_M we require that

(6) $$\Pi[0] \supseteq \Pi[1] \supseteq \ldots \supseteq \Pi[M-1] .$$

A sequent Π is said to be fundamental if there exist a statement a such that a occurs in every place of Π. In either calculus the provable sequents are those obtainable from fundamental sequents by repeated application of the rules. We note that the rules $(F_k, M-1)^-$ and $(F_k, 0)^+$ are redundant since their conclusions are always fundamental sequents.

The completeness of the classical sequent calculus can be proved as in [3]:

THEOREM 1. *A sequent is provable in* LK_M *if and only if it is classically valid.*

In order to prove the completeness of the intuitionistic sequent calculus, we first show that every sequent provable in LJ_M is intuitionistically valid.

Clearly every fundamental sequent is (intuitionistically) valid, and any sequent obtainable from a valid sequent by rule (Wk) is valid. Hence it suffices to show that the introduction rules $(F_k, m)^-$ and $(F_k, m)^+$ preserve validity.

Let $(v_\lambda)_{\lambda\in\Lambda}$ be an arbitrary intuitionistic valuation, and suppose that v_λ^* satisfies the premisses of $(F_k, m)^-$. If v_λ^* does not satisfy Π, then v_λ^* satisfies the sequent $\Pi_i^-(a_1, \ldots, a_r)$ for each $i \in I^-$, and so by (4)

$$f_k\big(v_\lambda^*(a_1), \ldots, v_\lambda^*(a_r)\big) \leqslant m ;$$

applying (2) we obtain

$$v_\lambda^*(F_k a_1 \ldots a_r) \leqslant m ,$$

so that v_λ^* satisfies $|F_k a_1 \ldots a_r|_m^-$. Thus we have proved that if v_λ^* satisfies the premisses of $(F_k, m)^-$ then it also satisfies the conclusion. Hence $(F_k, m)^-$ preserves validity.

Suppose now that the premisses of $(F_k, m)^+$ are satisfied by the intuitionistic valuation $(v_\lambda)_{\lambda\in\Lambda}$. Let λ be any element of Λ. If v_λ^* does not satisfy Π, then by virtue of (3) and the restriction (6) on Π, we see that v_μ^* does not satisfy Π for any $\mu \geqslant \lambda$. Hence, for all $\mu \geqslant \lambda$, v_μ^* satisfies $\Pi_i^+(a_1, \ldots, a_r)$ for each $i \in I^+$, and so by (5)

$$f_k\big(v_\mu^*(a_1), \ldots, v_\mu^*(a_r)\big) \geqslant m ;$$

by (2) it follows that

$$v_\lambda^*(F_k a_1 \ldots a_r) \geqslant m ,$$

so that v_λ^* satisfies $|F_k a_1 \ldots a_r|_m^+$. Thus the conclusion of $(F_k, m)^+$ is satisfied by v_λ^* for arbitrary $\lambda \in \Lambda$, and so it is satisfied by $(v_\lambda)_{\lambda\in\Lambda}$. We have proved therefore that if an intuitionistic valuation satisfies the premisses of $(F_k, m)^+$ then it also satisfies the conclusion; i.e. $(F_k, m)^+$ preserves validity.

We have now proved that if a sequent is provable in LJ_M then it is intuitionistically valid. Note that the only property of the partial

order used in this proof is the reflexivity; this property however is essential.

THEOREM 2. *A sequent is provable in* LJ_M *if and only if it is intuitionistically valid.*

Proof. In view of what has been proved already, it suffices to show that every valid sequent is provable.

If Ω is an unprovable sequent then there exists an unprovable sequent Ω^* such that

$$\Omega \subseteq \Omega^* \tag{7}$$

and such that for each connective F_k and each truth-value m,

$$\text{if } |F_k a_1 \ldots a_r|_m^- \subseteq \Omega^*, \text{ then } \Pi_i^-(a_1, \ldots, a_r) \subseteq \Omega^* \text{ for some } i \in I^- . \tag{8}$$

This may be seen as follows. If $|F_k a_1 \ldots a_r|_m^- \subseteq \Omega$ then, because Ω is unprovable, there exists $i \in I^-$ such that $\Omega\Pi_i^-(a_1, \ldots, a_r) = \Omega'$ is unprovable; now apply the same argument to Ω' with respect to a different sequent $|F_k a_1 \ldots a_r|_m^- \subseteq \Omega'$; continuing in this way we obtain a sequence $\Omega, \Omega', \Omega'', \ldots$ which must terminate after a finite number of steps in a sequent Ω^* with the desired properties (7) and (8).

If Ω is an unprovable sequent and $\Sigma = |F_k a_1 \ldots a_r|_m^+ \subseteq \Omega$ then there exists an unprovable sequent Ω^Σ such that

$$\text{if } |a|_l^- \subseteq \Omega \quad \text{then} \quad |a|_l^- \subseteq \Omega^\Sigma \quad (a \in \mathfrak{S}, l \in \boldsymbol{M}) , \tag{9}$$

and such that

$$\Pi_i^+(a_1, \ldots, a_r) \subseteq \Omega^\Sigma \quad \text{for some } i \in I^+ . \tag{10}$$

To see this we argue as follows. Let Π be the sequent whose lth place is $\Omega[0] \cap \ldots \cap \Omega[l]$ for each $l \in \boldsymbol{M}$; since Ω is unprovable and $\Pi|F_k a_1 \ldots a_r|_m^+ \subseteq \Omega$, it follows that $\Pi|F_k a_1 \ldots a_r|_m^+$ is unprovable; but then since Π satisfies the restriction (6), we see that $\Pi\Pi_i^+(a_1, \ldots, a_r)$ is unprovable for some $i \in I^+$; the sequent $\Omega^\Sigma = \Pi\Pi_i^+(a_1, \ldots, a_r)$ has the desired properties (9) and (10).

Let Π be an unprovable sequent. We construct a "tree" Λ and a mapping $\lambda \to \Pi_\lambda$ which associates with each node λ an unprovable sequent Π_λ. The construction proceeds by levels: at the 0-th level we place a single node λ_0 with $\Pi_{\lambda_0} = \Pi^*$; if λ is at the kth level and $\Sigma = |F_k a_1 \ldots a_r|_m^+ \subseteq \Pi_\lambda$ then we connect λ to a node $\mu = \mu(\Sigma)$ at the $(k+1)$-th level and set $\Pi_\mu = \Pi_\lambda^{\Sigma *}$. The set Λ is partially ordered in the obvious way.

By (7) and (9) we see that for all $a \in \mathfrak{S}$ and $m \in \boldsymbol{M}$

$$\text{if } |a|_m^- \subseteq \Pi_\lambda \quad \text{then} \quad |a|_m^- \subseteq \Pi_\mu \quad \text{whenever } \lambda \leqslant \mu . \tag{11}$$

No Π_μ is fundamental and so it is possible for each $\alpha \in \mathfrak{A}$ to define $v_\lambda(\alpha)$ as the least m such that $\alpha \notin \Pi_\lambda[m]$. The family $(v_\lambda)_{\lambda\in\Lambda}$ is an intuitionistic valuation in view of (11).

We shall prove that for all $\alpha \in \mathfrak{S}$

$$\text{(12)} \qquad \text{if} \quad \alpha \in \Pi_\lambda[m] \quad \text{then} \quad v_\lambda^*(\alpha) \neq m \quad (\lambda \in \Lambda)\,.$$

This holds for $\alpha \in \mathfrak{A}$ by construction. Suppose (12) holds for $\alpha_1, \ldots, \alpha_r$ and consider $\alpha = F_k\alpha_1 \ldots \alpha_r$. If $\alpha \in \Pi_\lambda[m]$ then either $|\alpha|_m^- \subseteq \Pi_\lambda$ or $|\alpha|_m^+ \subseteq \Pi_\lambda$.

In the first case we have by (11)

$$|F_k\alpha_1 \ldots \alpha_r|_m^- \subseteq \Pi_\mu \quad \text{for all} \quad \mu \geqslant \lambda\,.$$

Hence, by (8), for all $\mu \geqslant \lambda$ we have $\Pi_i^-(\alpha_1, \ldots, \alpha_r) \subseteq \Pi_\mu$ for some $i \in I^-$. Thus by inductive hypothesis we have for each $\mu \geqslant \lambda$

$$\bigvee_{i\in I^-} [v_\mu^*(\alpha_1) \notin R_1^i \wedge \ldots \wedge v_\mu^*(\alpha_r) \notin R_r^i]\,.$$

We deduce by (4) that for each $\mu \geqslant \lambda$

$$f_k\big(v_\mu^*(\alpha_1), \ldots, v_\mu^*(\alpha_r)\big) > m\,.$$

Thus by (2) we have

$$v_\lambda^*(F_k\alpha_1 \ldots \alpha_r) > m\,.$$

In the second case $|F_k\alpha_1 \ldots \alpha_r|_m^+ \subseteq \Pi_\lambda$, so by (10) we have for suitable $\mu \geqslant \lambda$

$$\Pi_i^+(\alpha_1, \ldots, \alpha_r) \subseteq \Pi_\mu \quad \text{for some} \quad i \in I^+\,.$$

Thus by inductive hypothesis

$$\bigvee_{i\in I^+} [v_\mu^*(\alpha_1) \notin R_1^i \wedge \ldots \wedge v_\mu^*(\alpha_r) \notin R_r^i]\,.$$

Hence by (5) we have

$$f_k\big(v_\mu^*(\alpha_1), \ldots, v_\mu^*(\alpha_r)\big) < m\,,$$

and so by (2)

$$v_\lambda^*(F_k\alpha_1 \ldots \alpha_r) < m\,.$$

Thus in either case we have $v_\lambda^*(\alpha) \neq m$, and this completes the proof of (12). If Π were valid then Π_{λ_0} would be valid; hence for some $\alpha \in \mathfrak{S}$ and $m \in \boldsymbol{M}$ we would have

$$v_{\lambda_0}^*(\alpha) = m \quad \text{and} \quad \alpha \in \Pi_{\lambda_0}[m]\,,$$

which contradicts (12). We see therefore that every unprovable sequent is invalid, which was to be shown.

Theorems 1 and 2 solve the problem of constructing sequent calculi for classical and intuitionistic propositional logic. We may consider the same problem for predicate logic. From [3] it follows that for each choice of M-valued truth-functions and quantifiers there exists a calculus of sequents for the corresponding classical M-valued predicate logic. However it remains open whether a similar result holds for intuitionistic M-valued predicate logic. In certain cases the result does hold — e.g. for the quantifiers $\exists X = \sup X$ and $\forall X = \inf X$.

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