Subspaces and altitudes in Noetherian lattice modules*

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§ 0. Introduction. In a recent paper J. A. Johnson [4] introduced the a-adic pseudometric and the a-adic completion of a Noetherian lattice module. The purpose of this paper is to investigate the a-adic topology on a sublattice interval of a Noetherian lattice module and to determine the relation between the altitude of a local Noether lattice and its natural completion. It is shown (Theorem 2.4) that the a-adic topology on a sublattice interval is the subspace (a-adic) topology of the module and that the altitude of a local Noether lattice and its natural completion are the same (Theorem 3.6).

§ 1. Preliminary remarks. Let L be a multiplicative lattice and let M be a complete lattice. Elements of L will be denoted by $a, b, c, \ldots$, except that the least and greatest elements of L will be denoted by 0 and I, respectively. Elements of M will be denoted by $A, B, C, \ldots$, except that the least and greatest elements of M will be denoted by 0_M and $\mathcal{M}$, respectively. When no confusion is possible, 0 will also be used in place of 0_M.

Recall ([4], Definition 2.2) that M is an L-module in case there is a multiplication between elements of L and M, denoted by aA for a in L and A in M, which satisfies: (i) $(ab)A = a(bA)$; (ii) $(\bigvee_\alpha a_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha,\beta} a_\alpha B_\beta$; (iii) $IA = A$; (iv) $0A = 0$; for all a, b, a_α in L and A, B_β in M. For general definitions and properties concerning Noetherian lattice modules which are used in § 2, the reader is referred to [4]. In particular, the a-adic pseudometric is developed in § 3 of [4]. For general definitions and properties concerning local Noether lattices which are used in § 3, the reader is referred to [1] and [2].

Let M be an L-module and let A, B be elements of M such that $A \leqslant B$. Then the set $\{D \in M \mid A \leqslant D \leqslant B\}$ is a sublattice of M and will be denoted by $[A, B]$. We can define a multiplication between elements

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of L and elements of $[A, B]$ as follows: for a in L and C in M, set $a \circ C = aC \vee A$. This multiplication makes $[A, B]$ into an L-module ([4], Remark 2.8). If M is a Noetherian L-module, then $[A, B]$ becomes a Noetherian L-module ([4], Remark 2.9).

§ 2. The a-adic topology on an interval. We will make use of the following definition in the proof of Theorem 2.4.

DEFINITION 2.1. Let M be an L-module, let A, B be elements of M such that $A \leqslant B$, and let a be an element of L. Then, for all C, D in $[A, B]$, let $d(C, D, a, A, B)$ denote the *a-adic distance* ([4], Definition 3.5) between C and D considered as elements of the L-module $[A, B]$. Also, for C in $[A, B]$ and a positive real number ε, set

$$N(C, \varepsilon, a, A, B) = \{D \in [A, B] \mid d(C, D, a, A, B) < \varepsilon\}.$$

We will also need the following two results. The reader is referred to [3] and [4] for their proofs.

PROPOSITION 2.2 ([4], Corollary 3.11). *Let M be an L-module, let A, B be elements of M such that $A \leqslant B$, and let a be an element of L such that the a-adic pseudometric on M is a metric. Then the a-adic pseudometric on the L-module $[A, B]$ is a metric.*

PROPOSITION 2.3 ([3], Lemma 0.4). *Let L be a Noether lattice, let M be a Noetherian L-module, let b be an element of L, let A be an element of M, and let $\langle B_i \rangle$, $i = 1, 2, \ldots$, be a sequence of elements of M satisfying $b^i A \geqslant B_i \geqslant B_{i+1} \geqslant bB_i$ for all positive integers i. Then, there exists a positive integer n such that $B_{n+i} = b^i B_n$ for all nonnegative integers i.*

Let M be an L-module and let a be an element of L such that the a-adic pseudometric on M is a metric. In view of Proposition 2.2, the a-adic topology on $[A, B]$ (i.e. the topology generated by the a-adic metric) is always defined. The following theorem establishes an important relation between the topology on a sublattice interval $[A, B]$ of M, considered as an L-module, and the topology on $[A, B]$ considered as a subspace of M with the a-adic topology.

THEOREM 2.4. *Let L be a Noetherian lattice, let M be a Noetherian L-module, let A, B be elements of M such that $A \leqslant B$, and let a be an element of L such that the a-adic pseudometric on M is a metric. Then, the a-adic topology on $[A, B]$, considered as an L-module, is the same as the topology on $[A, B]$ considered as a subspace of M with the a-adic topology.*

Proof. By well-known properties of metric spaces it is sufficient to prove the following two statements for $C \in [A, B]$:

(i) *given a positive real number ε, there exists a positive real number δ such that*

$$N(A, \delta, a, A, B) \subseteq N(C, \varepsilon, a, 0, \mathcal{M}) \cap [A, B];$$

(ii) *given a positive real number ε, there exists a positive real number δ such that*

$$N(C, \delta, a, 0, \mathcal{M}) \cap [A, B] \subseteq N(C, \varepsilon, a, A, B).$$

First, (i) shall be established. Let ε be a positive real number. Set $\delta = \varepsilon$, and let $D \in N(C, \delta, a, A, B)$. Then, $d(C, D, a, A, B) < \delta$. Let n be the least nonnegative integer k such that $2^{-k} < \delta$. Hence $2^{-n} < \delta$. It follows that $d(C, D, a, A, B) \leqslant 2^{-n} < \delta$. Hence $C \vee a^n \circ B = D \vee a^n \circ B$, which implies $C \vee a^n \mathcal{M} = D \vee a^n \mathcal{M}$. Thus $d(C, D, a, 0, \mathcal{M}) \leqslant 2^{-n} < \varepsilon$, so $D \in N(C, \varepsilon, a, 0, \mathcal{M}) \cap [A, B]$. Since D was arbitrary,

$$N(C, \delta, a, A, B) \subseteq N(C, \varepsilon, a, 0, \mathcal{M}) \cap [A, B].$$

To establish (ii), let ε be a positive real number. Choose m to be the least natural number k such that $2^{-k} < \varepsilon$. Now, consider the sequence $\langle a^i \mathcal{M} \wedge B \rangle$, $i = 1, 2, \ldots$ Since $a^{i+1}\mathcal{M} \geqslant a(a^i \mathcal{M} \wedge B)$ and since $B \geqslant a(a^i \mathcal{M} \wedge B)$, we have $a^i \mathcal{M} \geqslant a^i \mathcal{M} \wedge B \geqslant a^{i+1}\mathcal{M} \wedge B \geqslant a(a^i \mathcal{M} \wedge B)$ for all positive integers i. Hence, the sequence $\langle a^i \mathcal{M} \wedge B \rangle$ satisfies the conditions of Proposition 2.3. Thus, there exists a positive integer n such that

$$(*) \qquad a^{n+i}\mathcal{M} \wedge B = a^i(a^n \mathcal{M} \wedge B)$$

for all nonnegative integers i. Now, set $\delta = 2^{-(n+m)}$ and let $D \in N(C, \delta, a, 0, \mathcal{M}) \cap [A, B]$. Then $A \leqslant D \leqslant B$ and $d(C, D, a, 0, \mathcal{M}) < \delta = 2^{-(n+m)}$. Hence $C \vee a^{n+m}\mathcal{M} = D \vee a^{n+m}\mathcal{M}$, so $B \wedge (A \vee a^{n+m}\mathcal{M}) = B \wedge (D \vee a^{n+m}\mathcal{M})$. Since $B \wedge (C \vee a^{n+m}\mathcal{M}) = C \vee (B \wedge a^{n+m}\mathcal{M}) = C \vee a^m(a^n \mathcal{M} \wedge B)$, and since $B \wedge (D \vee a^{n+m}\mathcal{M}) = D \vee a^m(a^n \mathcal{M} \wedge B)$, by modularity and $(*)$, it follows that $C \vee a^m(a^n \mathcal{M} \wedge B) = D \vee a^m(a^n \mathcal{M} \wedge B)$. Hence, $C \vee a^m B = D \vee a^m B$, and thus $C \vee a^m \circ B = D \vee a^m \circ B$. Consequently, $d(C, D, a, A, B) \leqslant 2^{-m} < \varepsilon$, and so $D \in N(C, \varepsilon, a, A, B)$. Since D was arbitrary,

$$N(C, \delta, a, 0, \mathcal{M}) \cap [A, B] \subseteq N(C, \varepsilon, a, A, B),$$

which establishes (ii), q.e.d.

§ 3. Altitudes in local Noether lattices. Throughout this section, L is a local Noether (Noetherian) lattice with unique proper maximal (prime) element p, and L^* is the p-adic completion of L. Recall ([4], § 8) that L^* is a local Noether lattice with unique proper maximal (prime) element pL^*. In this section we shall use p^* to denote pL^*.

LEMMA 3.1. *Let q be an element of L such that q is p-primary. Then qL^* is p^*-primary.*

Proof. Since q is p-primary, there exists a natural number n such that $p^n \leqslant q$. Consequently, $(p^*)^n = (pL^*)^n = (p^n)L^* \leqslant qL^*$ by [4], Corollary 5.11. It follows that qL^* is p^*-primary, q.e.d.

LEMMA 3.2. *Let q^* be an element of L^* such that q^* is p^*-primary. Then*

(i) *$q^* \cap L$ is p-primary;*

(ii) *$(q^* \cap L)L^* = q^*$.*

Proof. Since q^* is p^*-primary, there exists a natural number n such that $(p^n)L^* = (pL^*)^n = (p^*)^n \leqslant q^*$. This implies $p^n = (p^n)L^* \cap L \leqslant q^* \cap L$ by [4], Proposition 7.2. It follows that $q^* \cap L$ is p-primary.

To show (ii), let $\langle q_i \rangle$, $i = 1, 2, \ldots$, be the completely regular representative of q^*. Since the sequence $\langle p^n \vee p^i \rangle$, $i = 1, 2, \ldots$, is the completely regular representative of $p^n L^*$ ([4], Remark 5.2), and since $(p^n)L^* \leqslant q^*$, we have $p^n \vee p^i \leqslant q_i$ for all positive integers i, by [4], Proposition 5.9. Hence $p^n \leqslant q_i$ for all integers $i \geqslant n$. Since $[p^n, I]$ is finite dimensional ([2], Theorem 1.4) and since $\langle q_i \rangle$, $i = 1, 2, \ldots$, is decreasing ([4], Remark 4.8), there exists a natural number $m \geqslant n$ such that $q_i = q_m$ for all integers $i \geqslant m$. Hence $\bigwedge_i q_i = q_m$. Consequently, $(q^* \cap L)L^* = (\bigwedge_i q_i)L^* = q_m L^* = q^*$, q.e.d.

PROPOSITION 3.3. *There exists a one-to-one correspondence between the p-primary elements q of (L, p) and the p^*-primary elements q^* of (L^*, p^*). The correspondence is such that q and q^* correspond if and only if $qL^* = q^*$ and $q^* \cap L = q$.*

Proof. Let $P(p) = \{q \in L \mid q \text{ is } p\text{-primary}\}$ and let $P(p^*) = \{q^* \in L^* \mid q^* \text{ is } p^*\text{-primary}\}$. Consider the extension mapping from $P(p)$ to L^* (i.e. $q \to qL^*$ of $P(p) \to L^*$). By Lemma 3.1, qL^* is an element of $P(p^*)$ for every q in $P(p)$, and by Lemma 3.2 the extension map is onto $P(p^*)$. Since the extension map is one-to-one ([4], Proposition 5.3) and since $qL^* \cap L = q$ ([4], Proposition 7.2), the proposition follows, q.e.d.

If q is a p-primary element of L, then, for each positive integer n, we let $D(q, n)$ denote the lattice dimension of $[q^n, I]$ (observe that $D(q, n)$ is finite by [2], Theorem 1.4). We will need the following two results. The reader is referred to [2] for their proof.

PROPOSITION 3.4. ([2], Corollary 3.5). *Let q be a p-primary element of L. Then there exists a polynomial $D^*(q, x)$ and a natural number n such that $D^*(q, m) = D(q, m)$ for all integers $m \geqslant n$.*

THEOREM 3.5 ([2], Theorem 3.9). *Let q be a p-primary element of L. Then the degree of the polynomial $D^*(q, x)$ is the altitude of L.*

THEOREM 3.6. *Let (L, p) be a local Noether lattice and let (L^*, p^*) be the p-adic completion of (L, p). Then altitude $L =$ altitude L^*.*

Proof. Clearly p is p-primary and p^* is p^*-primary. Hence, by Proposition 3.4, there exist polynomials $D^*(p, x)$ and $D^*(p^*, x)$, and there exist natural numbers N_1 and N_2 such that $D^*(p, n) = D(p, n)$ for all integers $n \geqslant N_1$, and $D^*(p^*, n) = D(p^*, n)$ for all integers $n \geqslant N_2$. Let N be the greater of N_1 and N_2. Consequently, $D^*(p, n) = D(p, n)$ and $D^*(p^*, n) = D(p^*, n)$ for all integers $n \geqslant N$. We know that each element $q \neq I$ in $[p^n, I]$ is p-primary and that each element $q^* \neq IL^*$ in $[(p^*)^n, IL^*]$ is p^*-primary. Thus, since p^n and $(p^*)^n$ correspond, we have $D(p, n) = D(p^*, n)$, for all positive integers n, by Proposition 3.3. Hence $D^*(p, n) = D^*(p^*, n)$ for all integers $n \geqslant N$. It follows that the polynomials $D^*(p, x)$ and $D^*(p^*, x)$ are equal. Hence their degrees are the same, and consequently the altitudes of L and L^* are equal by Theorem 3.5, q.e.d.

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