On the decision problem for extensions of a decidable theory

by

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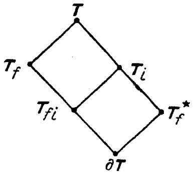
Set theoretically, the most natural partition of a class $\boldsymbol{K}$ of algebraic structures divides them into finite and infinite ones. Syntactically, there arise from this separation—again naturally—no less than five different extensions of the theory $\boldsymbol{T}$ of $\boldsymbol{K}$, namely the theory $\boldsymbol{T}_i$ of infinite models of $\boldsymbol{T}$, the theory $\boldsymbol{T}_f$ of finite models of $\boldsymbol{T}$, the theory $\boldsymbol{T}_{fi}$ of infinite models of $\boldsymbol{T}_f$, the theory $\boldsymbol{T}_f^*$ of models of $\boldsymbol{T}$ that are not models of $\boldsymbol{T}_f$, and, finally, the theory $\partial\boldsymbol{T}$ of those structures that are models of $\boldsymbol{T}_f$ as well as of $\boldsymbol{T}_f^*$. They are arranged in the following self-explanatory diagram:



It will be seen below that $\boldsymbol{T}_f^*$ is the more natural candidate for a dual of $\boldsymbol{T}_f$ than $\boldsymbol{T}_i$.

The question, that we are concerned with in this paper, is how the decidability of $\boldsymbol{T}$ may affect these five extensions. In [7] we showed that a decidable theory $\boldsymbol{T}$ may have an undecidable theory $\boldsymbol{T}_f$. Here we construct, among others, an example of a decidable theory $\boldsymbol{T}$ that has an undecidable extension $\boldsymbol{T}_i$. As a matter of fact, of the 32 possible combinations decidable-undecidable, 7 are ruled out by very simple and general considerations, among the other 25 cases there are 9 basic ones for which we exhibit theories, and from these a trivial construction yields examples for the remaining 16 possibilities.

The diagram above is obviously a special case of a general situation. However, we want to avoid introducing too much machinery in the form of definitions and trivial lemmas, although a more thorough investigation of these generalities might well lead to some interesting results. Thus, in § 1 we shall offer the bare minimum needed for our purpose. § 2 contains the lemmas on the basis of which the seven cases are ruled out, while § 3 deals with those cases. In § 4 we develop some methods that allow for the construction of our examples, and § 5 lists those examples with sketches of the corresponding proofs. Finally, in § 6 we make a few comments and raise some problems.

If one looks for mathematically meaningful theories $\boldsymbol{T}$ that have been proved decidable, one finds that there are very few for which our diagram does not degenerate and none whose decision procedure does not also entail the decidability of the associated theory $\boldsymbol{T}_i$ of infinite models. In the case of decision procedures based on completeness or model completeness the theory $\boldsymbol{T}$ simply coincides with $\boldsymbol{T}_i$, and the known cases of elimination of quantifiers lead to combinations of so-called basic sentences for which it can easily be decided whether or not they have infinite models. The theory of Abelian groups is at any rate rich enough to offer a non-degenerate diagram, but here we find all the pertinent extensions decidable. It is this theory that we extend in various ways in order to obtain our examples.

We shall make much use of Szmielew's decision procedure (cf. [11]) and of notation and terminology introduced in [7].

But, alas, complete as the results of this note may seem, they are very unsatisfactory, for the examples are contrived and definitely not finitely axiomatisable. It is obviously impossible to find finitely axiomatisable examples for our purposes among extensions of theories for which all the extensions in the diagram are decidable. And it seems highly unlikely that our examples could be turned into finitely axiomatisable ones by means of an "enrichment" of the language, for, such a procedure would, no doubt, destroy the decidability properties. Thus, the problem remains open, for which of the 25 cases there exist finitely axiomatisable examples. This seems to be a much deeper problem than the one treated here, and it touches upon the general quest for some intrinsic properties by which finitely axiomatisable theories distinguish themselves. (¹)

§ 1. Bases and co-bases.

We shall assume throughout that we are dealing with a fixed similarity type which determines an elementary language $\boldsymbol{L}$ with equality. By $\boldsymbol{U}$ we denote the set of all those structures of the given type that have a subset of the set N of natural numbers as domain. This restriction is feasible for our purpose and keeps foundational troubles at bay. The familiar maps that associate with a class $K \subseteq \boldsymbol{U}$ its elementary theory and with a set $S \subseteq \boldsymbol{L}$ the class of all its models in $\boldsymbol{U}$ are denoted by τ and μ respectively. The compositions $\mu\tau$ and $\tau\mu$ are closure operations and will both be described by square bracketing. The natural equivalence relations that they induce on their respective domains are those of elementary equivalence for classes of models and for sets of sentences respectively. Closed classes of structures are called *elementary* and closed sets of sentences are *theories*.

If $\boldsymbol{T}$ and $\boldsymbol{R}$ are theories, $A \subset \boldsymbol{L}$ and $a \in \boldsymbol{L}$, we use the notation $\boldsymbol{TR} = [\boldsymbol{T} \cup \boldsymbol{R}]$, $\boldsymbol{T}[A] = [\boldsymbol{T} \cup A]$, $\boldsymbol{T}[a] = [\boldsymbol{T} \cup \{a\}]$, and call $\boldsymbol{T}[A]$ the extension of $\boldsymbol{T}$ by A and $\boldsymbol{T}[a]$ the finite extension of $\boldsymbol{T}$ by a. A is called a *$\boldsymbol{T}$-basis* for $\boldsymbol{T}[A]$, and a proper one if $a \notin \boldsymbol{T}$ for all $a \in A$. If $\boldsymbol{T}$ is the set $\boldsymbol{O}$ of all logically valid sentences of $\boldsymbol{L}$ we omit the qualification $\boldsymbol{T}$.

The concept of a basis has a dual that will prove useful below.

DEFINITION 1. We call the set $B \subseteq \boldsymbol{L}$ a *$\boldsymbol{T}$-co-basis for the theory* $\boldsymbol{R}$, and write $\boldsymbol{R} = \boldsymbol{T}|B|$, if

(i) $\boldsymbol{R} = \bigcap\limits_{b \in B} \boldsymbol{T}[b]$, and

(ii) *whenever* $\boldsymbol{T}[a] \supseteq \bigcap\limits_{b \in B} \boldsymbol{T}[b]$, *then there exists a finite subset B' of B such that* $\boldsymbol{T}[a] \supseteq \bigcap\limits_{b \in B'} \boldsymbol{T}[b]$.

If, moreover, $\sim b \notin \boldsymbol{T}$, for all $b \in B$, then B is a proper $\boldsymbol{T}$-co-basis. Note that, while every extension of $\boldsymbol{T}$ has a $\boldsymbol{T}$-basis, and every set $A \subseteq \boldsymbol{L}$ is a $\boldsymbol{T}$-basis, not every extension of $\boldsymbol{T}$ has a co-basis and not every set B of sentences is a $\boldsymbol{T}$-co-basis, i.e., has property (ii). However, if a theory $\boldsymbol{R}$ can be represented in the form (i) then it does have a $\boldsymbol{T}$-co-basis, in particular the trivial one consiting of all sentences $b \in \boldsymbol{L}$ for which $\boldsymbol{T}[b] \supseteq \boldsymbol{R}$. But, in a sense, such a co-basis contains much redundancy and we shall have occasion to restrict ourselves to certain special types of sets of sentences.

DEFINITION 2. A set $B \subseteq \boldsymbol{L}$ *is $\boldsymbol{T}$-disjoint* if $\sim b \vee \sim b' \in \boldsymbol{T}$ for all $b' \neq b$ of B, and properly so if, moreover, $\sim b \notin \boldsymbol{T}$ for all $b \in B$.

It is easy to see that to every set B there exists a properly $\boldsymbol{T}$-disjoint set B' such that $\bigcap\limits_{b \in B} \boldsymbol{T}[b] = \bigcap\limits_{b \in B'} \boldsymbol{T}[b]$, and such that B' is a $\boldsymbol{T}$-co-basis if and only if B is one. It is convenient to use the notation $\sim B = \{\sim b\colon b \in B\}$, and to call a set A (properly) *$\boldsymbol{T}$-co-disjoint* if $\sim A$ is (properly) $\boldsymbol{T}$-disjoint. It is clear then, that to every set A there exists a properly $\boldsymbol{T}$-co-disjoint set A' which is $\boldsymbol{T}$-equivalent to A, i.e., for which $\boldsymbol{T}[A'] = \boldsymbol{T}[A]$. Note that a properly $\boldsymbol{T}$-co-disjoint $\boldsymbol{T}$-basis for a theory $\boldsymbol{R}$ is an independent $\boldsymbol{T}$-basis in the usual sense. By E we shall always denote the set of all sentences $E(n)$, which state that there are exactly $n+1$ elements. E is properly disjoint and $\bigcap\limits_{n \in N} [E(n)]$ is the theory of all finite structures of

(¹) Some of the results of this paper were obtained in spring 1963 in Berkeley, while the author was working at the University of California on a research project in the foundations of mathematics sponsored by the U.S.N.S.F., Grant G-19673 (cf. [6]). They wre completed while she was attending an N.S.F. seminar in algebra at Bowdoin College in summer 1966. The author is glad of this opportunity to express her sincerest gratitude to Professor A. Tarski, without whose untiring encouragement neither this nor [7] would ever have been written.

our similarity type, but unless L contains only monadic predicates, E is not a co-basis, while $\sim E$ is a properly co-disjoint basis for the theory of infinite structures.

The set-theoretic difference of elementary classes gives rise to the following operations:

DEFINITION 3. $T \ast A = \tau(\mu(T) - \mu(A))$, and
$$T \ast\!\ast A = T \ast (T \ast A).$$

Thus, $T \ast A$ is the theory of all models of T that are not models of A, i.e., the theory of those models of T in which some sentence of A fails, and we may rephrase the definition as follows:
$$T \ast A = \bigcap_{a \in A} T[\sim a].$$
It follows that $\sim s \in T \ast A$ if and only if $(\sim a \to \sim s) \in T$ for all $a \in A$, i.e., if and only if $T[s] \supseteq A$. But the set of all sentences s for which $T[s]$ is an extension of $[A]$ is obviously a T-co-basis, and so we find that
$$T \ast\!\ast A = \bigcap \{T[s]\colon T[s] \supseteq A\} = T|\{s\colon T[s] \supseteq A\}|.$$
Thus, $T \ast\!\ast A$ is the theory of all those T-models that are models of some finite extension of T which is an extension of $[A]$. Such models might well be called *strict A-models of T*, e.g., if T is the theory of Abelian groups, then $T \ast\!\ast \sim E$ is the theory of strictly infinite Abelian groups, and any sentence s such that $T[s] \supset \sim E$ may be called an axiom of infinity for Abelian groups.

The following basic properties of the operations $\ast$ and $\ast\!\ast$ are direct consequences of the definitions:

LEMMA 1. (i) $T \ast A = T \ast T[A]$, $T \ast\!\ast A = T \ast\!\ast T[A]$,

(ii) $T \subseteq T \ast A$, $T[A] \subseteq T \ast\!\ast A$,

(iii) *if* $A \subset A'$, *then* $T \ast A' \subseteq T \ast A$ *and* $T \ast\!\ast A \subseteq T \ast\!\ast A'$,

(iv) $T \ast\!\ast (T \ast A) = T \ast (T \ast\!\ast A) = T \ast A$,

(v) *if* $T \subset T'$, *then* $T \ast A \subseteq T' \ast A$,

(vi) *if* $T \subset T'$ *and* $(T \ast A)T' = T' \ast A$, *then* $(T \ast\!\ast A)T' \subseteq T' \ast\!\ast A$,

(vii) *if* $T \subset T' \subset T \ast\!\ast A$, *then* $T' \ast\!\ast A \subseteq T \ast\!\ast A$,

(viii) $T \ast A = T|B|$ *if and only if* $T \ast\!\ast A = T[\sim B]$.

Of all these statements only the last one may need some comment. First we show

(1) $\quad T \ast\!\ast A = T[A]$ if and only if $\sim A$ is a T-co-basis.

For, let $\sim A$ be a T-co-basis, and assume that $t \in T \ast\!\ast A$, i.e., $T[t] \subseteq T \ast\!\ast A$, then $T[\sim t] = T \ast [t] \supseteq T \ast A = \bigcap_{A} T[\sim a]$, by (i), (iii) and (iv). But then $T[\sim t] \supseteq \bigcap_{A'} T[\sim a]$, for some finite subset A' of A, and hence

$T[t] \subseteq T[A']$, and so, by (ii), $T \ast\!\ast A = T[A]$. Conversely, assume $T \ast\!\ast A = T[A]$, and $T[s] \supseteq \bigcap_{A} T[\sim a] = T \ast A$, then $T[\sim s] \subseteq T \ast\!\ast A = T[A]$, but then $T[\sim s] \subseteq T[A']$ and hence $T[s] \supseteq T \ast A' = \bigcap_{A'} T[\sim a]$ for some finite subset A' of A. Thus $\sim A$ is a T-co-basis. With this we have proved (viii) in case $B = \sim A$, i.e., $T \ast\!\ast A = T[A]$ if and only if $T \ast A = T|\sim A|$. The rest now follows from the fact $T \ast A = T \ast C$ if and only if $T \ast\!\ast A = T \ast\!\ast C$ and from

(2) $\quad$ if $\; T \ast\!\ast A = T[C]$, then $T \ast\!\ast A = T \ast\!\ast C$,

which is a consequence of (i) and (iv). We can now rewrite our Definition 1 as follows:

(3) $\quad R = T|B|$ if and only if $R = T \ast \sim B$ and $T \ast\!\ast \sim B = T[\sim B]$.

If we fix the theory T and let R range over extensions of T, then we obtain an operation * of "dualization" on the lattice $\mathfrak{T}$ of extensions of T by defining $R^* = T \ast R$. The operation R to R^{**} is then a closure operation, and it follows from the above considerations that a theory R is closed if and only if it coincides with the intersection of all finite extensions of T that are extensions of it. We note that we have $(\bigcup_{A} T[a])^* = (T[A])^* = T \ast A = \bigcap_{A} T[\sim a]$, and $(T|B|)^* = T[\sim B] = \bigcup_{B} T[\sim b]$, but only $(\bigcap_{C} T[c])^* \supset T[\sim C]$ unless C is a T-co-basis.

In the following lemma we list some properties of the lattice $\mathfrak{T}$ as enriched by the operations * and **. They are all very easy to verify on the basis of Lemma 1.

LEMMA 2. (i) *If* $R \subset S$, *then* $S^* \subseteq R^*$ *and* $R^{**} \subseteq S^{**}$,

(ii) $(RS)^* = R^* \cap S^*$, $R^*S^* \subseteq (R \cap S)^* = (R^{**} \cap S^{**})^*$,
$(RS)^{**} = (R^{**}S^{**})^{**} \subseteq R^{**}S^{**}$,
$(R \cap S)^{**} \subseteq R^{**} \cap S^{**} = (R^{**} \cap S^{**})^{**}$, $R \subseteq R^{**}$, $R^{***} = R^*$,

(iii) $(T[a])^* = T[\sim a]$, $(T[a])^{**} = T[a]$, $L^* = T$, $T^* = L$,

(iv) *if* $R^* = L$ *then* $R = T$, *and if* $R^{**} = T$ *then* $R = T$,

(v) $R \cap R^* = T$, $(RR^*)^* = T$, $(RR^*)^{**} = L$,
$RR^* = L$, *if and only if* $R = T[a]$ *for some* $a \in L$,
$SR \cap R^* = S \cap R^*$, $RR^* \subseteq R \ast\!\ast R^* \subseteq R \ast R^{**} \subseteq L$,

(vi) *if* R *is complete, then either* $R = T[a]$, *for some* a, *or* $R^{**} = L$.

It is seen from this that, as one would expect, the operation * bears some analogy, but not a complete one, to that of complementation. The closed extensions of T, i.e., those for which $R^{**} = R$, behave in some respects like finite extensions, as will be seen, under further restrictions, in the next section. It is clear that the closed extensions are just those that can be written in the form (i) of Definition 1. Moreover, by Lemma 1 (viii), B is a basis for $\mathbf{R}^*$ if and only if $\sim B$ is a co-basis for $\mathbf{R}^*$. For closed theories the dual can be defined by either $(\mathbf{T}[A])^* = \mathbf{T}|\sim A|$, or by $(\mathbf{T}|B|)^* = \mathbf{T}[\sim B]$, and we have here independence of the choice of basis and co-basis. This, of course, justifies our additional clause (ii) for the definition of a co-basis. Note that, if $\mathbf{R}$ is a finite extension of $\mathbf{T}$, then the inequalities in the conclusion of Lemma 2 (i) become proper and all other inequalities become equalities.

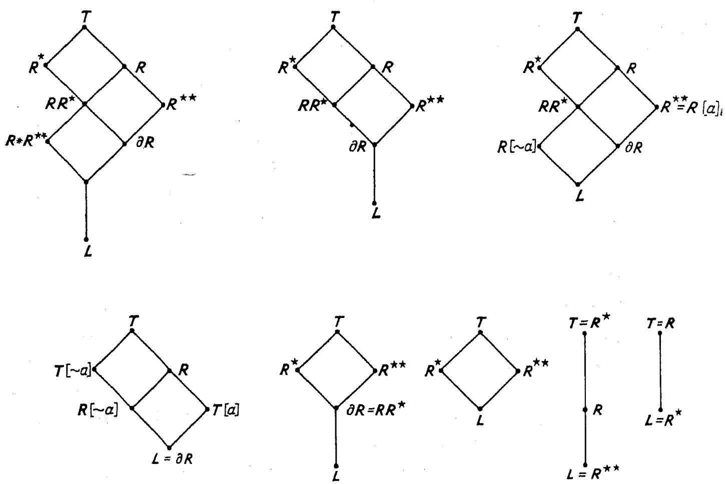
Of particular interest are the extensions of the form $\mathbf{R}\mathbf{R}^*$ and $\mathbf{R}^*\mathbf{R}^{**}$. A model of $\mathbf{R}\mathbf{R}^*$ is, in a sense, a non-standard model for $\mathbf{R}^*$, while a model of $\mathbf{R}^*\mathbf{R}^{**}$ is strictly non-standard, in that it is non-standard for $\mathbf{R}$ as well as for $\mathbf{R}^*$. We shall write $\partial\mathbf{R}$ for $\mathbf{R}^*\mathbf{R}^{**}$. Clearly, if $\mathbf{R}$ has a co-basis, then $\mathbf{R}\mathbf{R}^*$ coincides with $\partial\mathbf{R}$, but the converse also holds, namely:

LEMMA 3. *If* $\mathbf{R}\mathbf{R}^* = \partial\mathbf{R}$, *then* $\mathbf{R} = \mathbf{R}^{**}$.

For, by Lemma 2 (v), we have $\mathbf{R}\mathbf{R}^* \cap \mathbf{R}^{**} = \mathbf{R}$ as well as $\partial\mathbf{R} \cap \mathbf{R}^{**} = \mathbf{R}^{**}$. Note that the lemma can also be phrased as follows: If $(\mathbf{T} \ast A)[A] = (\mathbf{T} \ast A)(\mathbf{T} \ast\ast A)$ then $\mathbf{T}[A] = \mathbf{T} \ast\ast A$. Now if $\mathbf{S}\mathbf{R}^*$ is complete, $\mathbf{S} \subset \partial\mathbf{R}$ and $\mathbf{R}^*$ is not a finite extension of $\mathbf{T}$, then $\partial\mathbf{R} \neq \mathbf{L}$, according to Lemma 2 (v), and thus, since $\partial\mathbf{R} \subseteq \mathbf{S}\mathbf{R}^*$, we must have $\partial\mathbf{R} = \mathbf{S}\mathbf{R}^*$, and hence we obtain the following sufficient criterion for $\mathbf{S}$ to be closed:

COROLLARY 3. *If* $\mathbf{R}^*$ *is not a finite extension of* $\mathbf{T}$, $\mathbf{S} \subset \mathbf{R}^{**}$ *and* $\mathbf{S}\mathbf{R}^*$ *is complete, then* $\mathbf{S} = \mathbf{R}^{**}$.

The following diagrams represent the general case and its possible degeneracies.



Of course, all these concepts are relative to a theory $\mathbf{T}$. On the other hand, any fixed set S of sentences of $\mathbf{L}$ gives rise to a diagram of extensions associated with each theory $\mathbf{T}$, by taking $\mathbf{R} = \mathbf{T}[S]$, and it would be interesting to investigate the behaviour of the operations $\ast S$ and $\ast\ast S$ under change of $\mathbf{T}$. In particular it would be useful to have some sufficient conditions under which $\mathbf{T}' \ast S = (\mathbf{T} \ast S)\mathbf{T}'$ holds, similarly for $\mathbf{T}' \ast\ast S = (\mathbf{T} \ast\ast S)\mathbf{T}'$, in case $\mathbf{T}' \supset \mathbf{T}$. We make here only the trivial observation that, if $\mathbf{T} \ast S = \mathbf{T}|B| = \mathbf{T}[\sim A]$, and $\mathbf{T}' \supset \mathbf{T}$, then $\mathbf{T}' \ast S \supseteq \bigcap_B \mathbf{T}'[b] \supseteq \mathbf{T}'[\sim A] = (\mathbf{T} \ast S)\mathbf{T}'$, and that, if both equalities hold, then $\mathbf{T}' \ast\ast S = \mathbf{T}'|A| \supseteq \mathbf{T}'[\sim B] = (\mathbf{T} \ast\ast S)\mathbf{T}'$, while $\mathbf{T}'|A| = \mathbf{T}'[\sim B]$ of course implies $\mathbf{T}' \ast S = \mathbf{T}'|B|$.

Particularly natural choices for S are, on the one hand, the set C, where C is the set of sentences c for which $[c]$ is a complete theory—we shall simply call these complete sentences—and, on the other hand, the set F, where F, a subset of C, consists of the sentences f for which $\mu[f]$ contains up to isomorphisms just one finite structure. It is the latter case that we shall be concerned with, and we shall work with the set E of sentences stating that there are exactly $n+1$ elements rather than with F. Thus we have $\mathbf{T}_i = \mathbf{T}[\sim E]$, and $\mathbf{T}_f = \mathbf{T} \ast \sim E = \bigcap \mathbf{T}[E(n)]$, write $\partial\mathbf{T}$ for $\partial\mathbf{T}_i = \partial\mathbf{T}_f$, and obtain the diagram mentioned in the introduction. We note that E is a $\mathbf{T}$-co-basis, i.e., that $\mathbf{T}_i = \mathbf{T}_f^*$, if and only if every finite extension of $\mathbf{T}$ that is an extension of $\mathbf{T}_f$ has only finitely many non-isomorphic models. As an illustration we consider the theory $\mathbf{AG}$ of Abelian groups. We use the notation of [7] where (cf. p. 67) it was shown that $\mathbf{AG}_f$ coincides with the theory of periodic Abelian groups (i.e., Abelian groups of bounded torsion), and where a basis D for $\mathbf{AG}_f$ was introduced. The set H of all sentences $H(m)$ expressing periodicity m, $m \geqslant 1$, forms a co-basis for $\mathbf{AG}_f$, and we thus find that the theory $\mathbf{AG}_f^*$, i.e., the theory of strictly infinite Abelian groups, coincides with the theory of non-periodic Abelian groups, whereas $\mathbf{AG}_{fi}$ is the theory of infinite periodic groups and $\partial\mathbf{AG}$ is the theory of non-periodic direct products of finite groups. We collect these results for further use:

$$\mathbf{AG}_i = \mathbf{AG}[\sim E],\quad \mathbf{AG}_f = \mathbf{AG}[D] = \mathbf{AG}|H|,\quad \mathbf{AG}_f^* = \mathbf{AG}[\sim H] = \mathbf{AG}|\sim D|,$$

$$\mathbf{AG}_{fi} = \mathbf{AG}_i[D] = \mathbf{AG}_i|H|,\quad \partial\mathbf{AG} = \mathbf{AG}[D \cup \sim H].$$

We note that we have here a particularly nice co-basis for $\mathbf{T}_f$, which is also a proper T_i-co-basis for T_{fi}. We have chosen it so that it is disjoint, but an equivalent basis would be the set of all laws $r(k)$ of the form $x^k = 1$, and so we see that the theory of finite Abelian groups coincides with the theory of all proper subvarieties of the variety of Abelian groups, or, more precisely, the theory of every proper subvariety of the class of all Abelian groups is an extension of $\mathbf{AG}_f$ and every finitely axiomatisable class of finite Abelian groups is an extension of a finite union of such varieties. If we denote the theory of groups by $\boldsymbol{G}$, it follows that every law s of the form $z^k xy = yx$ is an "axiom of finiteness" for $\boldsymbol{G}$, i.e., is such that $\boldsymbol{G}[s] \supset \boldsymbol{G}_f$. However, Malcev's work shows, that for no k, except for 1, 2 and possibly higher powers of 2, the law $r(k)$, even in conjunction with the law $n(2)$ expressing nilpotence of class 2, can be an axiom of finiteness for $\boldsymbol{G}$. This follows already from [8] where he proves that every theory $\boldsymbol{G}[r(k) \wedge n(2)]$, for $k \geqslant 3$, has a finite essentially undecidable extension. Moreover, he shows in [9] and [10] that the theory of finite models of every such extension of $\boldsymbol{G}$ is hereditarily undecidable. In view of these results the following problems arise:

Problem 1. *Find a law s, such that $\boldsymbol{G}[s]$ is decidable, but has infinitely many non-equivalent non-Abelian models.*

Problem 2. *Find a sentence a such that $\boldsymbol{G}[a]$ is an extension of $\boldsymbol{G}_f$ but has infinitely many non-isomorphic non-Abelian models.*

Problem 2′. *Find some sentences a such that $\boldsymbol{G}[r(3) \wedge n(2) \wedge a] \supset \boldsymbol{G}_f$, and such that this extension has infinitely many non-isomorphic non-Abelian models.*

It seems quite likely that the law $yx^2 = yx^2$ is a sentence as required in problem 1, and that the law $z^4x^2y = yx^2$ will do for a in problem 2 and thus also for s in problem 1. On the other hand, one may ask.

Problem 3. *Are there any finite groups $\mathfrak{G}$ such that all theories $\boldsymbol{G}[a]$, for which $\tau(\mathfrak{G}) \supset \boldsymbol{G}[a] \supset \boldsymbol{G}_f$, have only finitely many non-isomorphic non-Abelian models?*

That these are difficult problems becomes strikingly clear if one observes that, by asking for axioms of finiteness for the theory $\boldsymbol{G}$, one is looking for theorems of $\boldsymbol{G}_f^*$ that are not theorems of $\boldsymbol{G}$, i.e., for sentences that hold in all strictly infinite groups but not in all groups. But these are the groups that satisfy the negation of some theorem of $\boldsymbol{G}_f$ that is not a theorem of $\boldsymbol{G}$. And, as mentioned above, this latter theory, $\boldsymbol{G}_f$, is not axiomatisable, which, loosely speaking, means that it is not easy to find a proof for a sentence that holds in all finite groups, but not in all groups, the most remarkable illustration of this point being the proof of the validity in $\boldsymbol{G}_f$ of the sentence

$$(\forall x)(x \neq 1 \to x^2 \neq 1) \wedge (\exists x)(x \neq 1) \to (\exists x)\big(x \neq 1 \wedge (\forall z)(xzxz^{-1} = zxz^{-1}x)\big)$$

(cf. [4]).

§ 2. Axiomatisability and co-axiomatisability.

From now on we shall assume that a recursive one to one enumeration ν of $\boldsymbol{L}$ is fixed, i.e., a mapping $\nu\colon \boldsymbol{L} \to N$ under which the logical operations are mapped onto resursive operations and with an inverse α, and shall mean by $\boldsymbol{L}$ the corresponding numbered system. A sequence $\langle A(n)\rangle_{n \in N}$ of sentences of $\boldsymbol{L}$ is called *recursive* if $\nu\big(A(n)\big)$ is a recursive function of n. The set $A = \{A(n)\colon n \in N\}$ of values of such a sequence is of course recursively enumerable (r.e.) and is termwise logically equivalent to a recursive sequence with a recursive set of values. But we shall occasionally need a stronger property.

Definition 4. A set $A \subset \boldsymbol{L}$ is $\boldsymbol{T}$-*decidable* if the set $\big\{b \in \boldsymbol{L}\colon (\exists a \in A)\big((b \leftrightarrow a) \in \boldsymbol{T}\big)\big\}$ is recursive, and a sequence $\langle A(n)\rangle_{n \in N}$ is $\boldsymbol{T}$-decidable if it is recursive and its set of values A is $\boldsymbol{T}$-decidable.

A sequence $\langle A(n)\rangle$ with range A is called an *axiomsystem* for $\boldsymbol{R}$ relative to $\boldsymbol{T}$, or a $\boldsymbol{T}$-*axiomsystem* for $\boldsymbol{R}$, if it is a recursive sequence and $\boldsymbol{R} = \boldsymbol{T}[A]$. $\boldsymbol{R}$ is a recursive extension of $\boldsymbol{T}$, or, is $\boldsymbol{T}$-axiomatisable, if it has a $\boldsymbol{T}$-axiomsystem; it is finitely $\boldsymbol{T}$-axiomatisable if it is an extension of $\boldsymbol{T}$ by a single sentence. (Note that, contrary to some usage, we do apply the term "theorem" to valid sentences of $\boldsymbol{T}$ whether $\boldsymbol{T}$ is axiomatisable or not.) If $\boldsymbol{R} \supset \boldsymbol{T}' \supset \boldsymbol{T}$ and $\boldsymbol{R}$ is $\boldsymbol{T}'$-axiomatisable and $\boldsymbol{T}'$ is $\boldsymbol{T}$-axiomatisable, then $\boldsymbol{R}$ is $\boldsymbol{T}$-axiomatisable. Moreover, if $\boldsymbol{R}$ is $\boldsymbol{T}$-axiomatisable, then it is $\boldsymbol{T}'$-axiomatisable for all $\boldsymbol{T}'$ such that $\boldsymbol{R} \supset \boldsymbol{T}' \supset \boldsymbol{T}$, and thus in particular a $\boldsymbol{O}$-axiomatisable theory is axiomatisable relative to all its subtheories. We call a $\boldsymbol{O}$-axiomatisable theory simply axiomatisable. Clearly a theory $\boldsymbol{T}$ is axiomatisable if and only if it is an r.e. subset of $\boldsymbol{L}$.

Definition 5. A sequence $\langle B(n)\rangle$ is a *co-axiomsystem for* $\boldsymbol{R}$ *relative to* $\boldsymbol{T}$ if it is recursive and

(i) $\boldsymbol{R} = \bigcap_{n \in N} \boldsymbol{T}[B(n)]$, *and the set*

(ii) $\{m\colon (\exists n)(\boldsymbol{T}[\sim\alpha(m) \wedge B(n)] \neq \boldsymbol{L}\}$ *is* r.e.

If a theory $\boldsymbol{R}$ has a $\boldsymbol{T}$-co-axiomsystem we call it $\boldsymbol{T}$-*co-axiomatisable*. In this case $\boldsymbol{R}$ is a $\divideontimes$-closed extension of $\boldsymbol{T}$, i.e., $\boldsymbol{T} \divideontimes \boldsymbol{R} = \boldsymbol{R}$. If $\boldsymbol{R} \supset \boldsymbol{T}' \supset \boldsymbol{T}$ and $\boldsymbol{R}$ is $\boldsymbol{T}'$-co-axiomatisable, and $\boldsymbol{T}'$ is $\boldsymbol{T}$-co-axiomatisable, then $\boldsymbol{R}$ is $\boldsymbol{T}$-co-axiomatisable. Moreover, if $\boldsymbol{R}$ is $\boldsymbol{T}$-co-axiomatisable, then it is $\boldsymbol{T}'$-co-axiomatisable for all $\boldsymbol{T}'$ such that $\boldsymbol{R} \supset \boldsymbol{T}' \supset \boldsymbol{T}$, and thus a $\boldsymbol{O}$-co-axiomatisable theory is co-axiomatisable relative to all its subtheories. But, whereas a theory is always axiomatisable in a trivial way relative to itself, it is just the corresponding property that is significant in the dual case; thus we call a theory co-axiomatisable if it is co-axiomatisable relative to itself. In this case, of course, condition (i) becomes trivial, while for $\boldsymbol{T} = \boldsymbol{O}$ it is very strong. Clearly, a theory $\boldsymbol{R}$ is co-axiomatisable if and only if it is co-r.e., i.e., is the complement of an r.e. set, or, the set of non-theorems of $\boldsymbol{R}$ is r.e. In this case $\alpha(\boldsymbol{L} - \boldsymbol{R})$ is trivially a co-axiomsystem for $\boldsymbol{R}$. We call $\boldsymbol{R}$ a *recursive co-extension* of $\boldsymbol{T}$ if $\boldsymbol{R} = \boldsymbol{T}|B|$, for some recursive sequence B. But note that, while $\boldsymbol{R}$ is $\boldsymbol{T}$-axiomatisable if and only if it is a recursive extension of $\boldsymbol{T}$, the existence of a recursive $\boldsymbol{T}$-co-basis is neither necessary nor sufficient for $\boldsymbol{T}$-co-axiomatisability.

A *decidable* theory is a theory that is both axiomatisable and co-axiomatisable.

To give some justification for our definition of a co-axiomsystem we note that a theory is usually said to have a disproof-procedure (cf. [1]) if the set of its non-theorems is r.e. More precisely, a disproof procedure for a theory $\boldsymbol{R}$ can be described as follows: first a recursive sequence $\langle B'(n)\rangle$ of sentences, each of which is consistent with $\boldsymbol{R}$ is given and then an algorithm is exhibited which yields for exactly the non-theorems a of $\boldsymbol{R}$ a consistency proof of $\boldsymbol{R}[\sim a\wedge B'(n)]$ for some n. In this sense the set $\sim B'$ forms a basis for the set of non-theorems of $\boldsymbol{R}$. Now such a sequence $\langle B'(n)\rangle$ must clearly have property (i) for $\boldsymbol{T}=\boldsymbol{R}$ while the existence of the algoritm is expressed by (ii). Thus $\langle B'(n)\rangle$ is a co-axiomsystem for R. Conversely, every $\boldsymbol{T}$-co-axiomsystem does yield such a procedure, since, according to (ii), the set $\{n\colon \boldsymbol{T}[B(n)]\neq \boldsymbol{L}\}$ is r.e. and, by (i), $\boldsymbol{T}[B(n)]=\boldsymbol{R}[B(n)]$, and thus a sequence $\langle B'(n)\rangle$ as above can be obtained from the sequence $\langle B(n)\rangle$. Clearly, if $\boldsymbol{R}$ is a recursive co-extension by B of a co-axiomatisable theory $\boldsymbol{T}$, then B is a $\boldsymbol{T}$-co-axiomsystem for $\boldsymbol{R}$, and the disproof procedure for $\boldsymbol{T}$ induces one for $\boldsymbol{R}$. But the usual way of obtaining a sequence B', in the non-trivial cases where $\boldsymbol{R}$ has infinitely many non-equivalent complete extensions, and provided that $\boldsymbol{R}$ is axiomatisable, is as follows: first one finds a recursive sequence $\langle \boldsymbol{R}^k\rangle$ of axiomatisable complete extensions of $\boldsymbol{R}$, which is representative of the class $\mu(\boldsymbol{R})$ of models of $\boldsymbol{R}$ in the sense that the class $\mu(\boldsymbol{R})$ is the elementary closure of the set of "basic" models $\mu(\boldsymbol{R}^k)$, i.e., is such that $\boldsymbol{R}=\bigcap_{k\in N}\boldsymbol{R}^k$. Now, the condition of recursiveness of the sequence means that for each theory $\boldsymbol{R}^k$ there is an axiomsystem $\langle B(k,n)\rangle_{n\in N}$ such that $\nu\big(B(k,n)\big)$ is a recursive function of k and n. But then the set $\{\bigwedge_{j\leqslant k} B(j,n)\colon n,k\in N\}$, where $\bigwedge_{j\leqslant k}$ stands for the obvious k-fold conjunction, can be arranged in a simple recursive sequence B' which will have the desired properties, since we have

($\mathrm{i_c}$) $\boldsymbol{R}=\bigcap_{B'}\boldsymbol{R}[b]$,

($\mathrm{i'_c}$) $\boldsymbol{R}[b]\neq\boldsymbol{L}$ *for all* $b\in B'$, *and*

($\mathrm{ii_c}$) $a\notin\boldsymbol{R}$ *if and only if* $(\exists b\in B')\big((b\to\sim a)\in\boldsymbol{R}\big)$.

Note that in this application ($\mathrm{i'_c}$) is of course indispensable, and that by ($\mathrm{ii_c}$) the disproof procedure is reduced to the proof procedure for $\boldsymbol{R}$. A typical illustration for this method, occasionally quoted as Ershof's theorem (cf. [3]), is to be found in Szmielew's decision procedure [11] (cf. [7] for a summary).

All this is, of course, based on the general and trivial principle according to which the intersection $\boldsymbol{T}=\bigcap \boldsymbol{T}_k$ of a recursive sequence of uniformly co-axiomatisable theories has a disproof procedure. It is clear that the co-axiomatisability of $\boldsymbol{T}$ does, however, not follow from the mere existence of individual disproof procedures for the theories $\boldsymbol{T}_k$. For example, let $\boldsymbol{T}=\boldsymbol{AG}_f[A]$, where A is the sequence of sentences $\sim\tau(\mathfrak{C}_{g(n)})$, where g is some recursive function with non-recursive range and $\mathfrak{C}_m$ is a cyclic group of order m. Then we have $\boldsymbol{T}=\bigcap\boldsymbol{T}[H(n)]$ and each theory $\boldsymbol{T}[H(n)]$ is decidable and hence co-axiomatisable, because $\boldsymbol{T}[H(n)]=\boldsymbol{AG}_f[H(n)\wedge\sim\tau(\mathfrak{C}_n)]$ or $=\boldsymbol{AG}_f[H(n)]$ according as n belongs to the range of g or not, and these theories are finite extensions of a decidable theory. Yet the set

$$\{m\colon \boldsymbol{T}[\tau(\mathfrak{C}_m)]\neq\boldsymbol{L}\}=\{m\colon (\exists n)(\boldsymbol{T}[\tau(\mathfrak{C}_m)\wedge H(n)]\neq\boldsymbol{L})\}$$

is the complement of the range of g and thus is not r.e., i.e., $\boldsymbol{T}$ is not co-axiomatisable.

We state now some useful lemmas. Let us call a sentence $c\in\boldsymbol{L}$ complete if the theory $[c]$ is complete.

LEMMA 4. *Let C be a recursive sequence of complete sentences, and $T_C=\bigcap_C \boldsymbol{T}[c]$, the theory of C-models of $\boldsymbol{T}$. Then:*

(i) *If $\boldsymbol{T}$ is finitely axiomatisable and all c's in C are consistent, then $\boldsymbol{T}_C$ is co-axiomatisable.*

(ii) *If $\boldsymbol{T}$ is co-axiomatisable then so is $\boldsymbol{T}_C$.*

For the proof note that in case $\boldsymbol{T}=[a]$, the set $\{c\in C\colon (c\to a)\in O\}$ serves as co-axiomsystem, while in case (ii) the set $\{c\in C\colon \boldsymbol{T}[c]\neq\boldsymbol{L}\}$ does it.

COROLLARY 4. *The theory $\boldsymbol{T}_f$ of finite models of a finitely axiomatisable or of a co-axiomatisable theory $\boldsymbol{T}$ is co-axiomatisable.*

Remark. If the set C_0 of all complete and consistent sentences of $\boldsymbol{L}$ were known to be recursively enumerable, then it could be used for C in the above lemma and it would follow that a finitely axiomatisable theory with only a countable number of inequivalent complete extensions is decidable, since it coincides with its theory $\boldsymbol{T}_{C_0}$. To our knowledge the question whether C_0 is r.e. or not has not yet been answered (for related problems and results, cf. [5]).

LEMMA 5. *If $\boldsymbol{T}$ is co-axiomatisable, then the $\boldsymbol{T}$-dual of any recursive extension of $\boldsymbol{T}$ is co-axiomatisable, and if $\boldsymbol{T}$ is axiomatisable, then the $\boldsymbol{T}$-dual of any recursive co-extension of $\boldsymbol{T}$ is axiomatisable.*

This is obvious, but note that mere co-axiomatisability of an extension $\boldsymbol{R}$ of an axiomatisable theory $\boldsymbol{T}$ does not entail axiomatisability of $\boldsymbol{R}^*$, i.e., the condition that some $\boldsymbol{T}$-co-basis of $\boldsymbol{R}$ be a co-axiomsystem is indispensable.

LEMMA 6. (i) *If* R *and* R' *are* T*-axiomatisable, then so are* $R \cap R'$ *and* RR'.

(ii) *If* R *and* R' *are* T*-co-axiomatisable, then so are* $R \cap R'$ *and all finite extensions of* R.

(iii) *If* T *and* T' *are axiomatisable and* $T \cap T'$ *and* TT' *are co-axiomatisable, then all four theories are decidable.*

For (i) note that

$$T[\{A(n)\}] \cap T[\{A'(n)\}] = T[\{\bigwedge_{i\leqslant n} A(i) \vee \bigwedge_{i\leqslant n} A'(i)\}],$$

and

$$T[A]T[A'] = T[A \cup A'].$$

(ii) follows from the fact that

$$\bigcup T[B(n)] \cap \bigcap T[B'(n)] = \bigcap T[B(n) \vee B'(n)]$$

and

$$(\bigcap T[B(n)])[a] = \bigcap T[B(n) \wedge a].$$

For (iii) it suffices to show that T is co-axiomatisable under the assumptions, but this follows from the fact that $a \notin T$ if and only if either $a \notin TT'$ or there exists a sentence $d \in T$ such that $(d \to a) \in T'$ but $(d \to a) \notin T \cap T'$.

COROLLARY 6. *If a theory* T *is finitely axiomatisable and undecidable then so is any theory* T' *for which* $T' \subseteq T_f^*$, i.e. T_f^* is hereditarily undecidadle.

For the proof of the corollary we observe that under the assumptions T_f is co-axiomatisable by Corollary 4. Hence T' cannot be co-axiomatisable, since otherwise $T = T_f \cap T'T$ would be decidable according to Lemma 6 (ii).

In particular then, a finitely axiomatisable theory T with a decidable theory T_i of infinite models is itself decidable. That the analogous situation for T_i replaced by T_f does not prevail was shown in [7], Theorem 3, by an example of a finitely axiomatisable undecidable theory for which T_f is decidable.

The condition of finite axiomatisablity in the corollary is indispensable as the following example shows. Let g again be a recursive function with non-recursive range $R(g)$. Then the theory of cyclic groups of prime order p_m with $m \in R(g)$ is co-axiomatisable, and the theory of cyclic groups of order p_m with $m \notin R(g)$ is axiomatisable. Neither theory is decidable, but both have the same complete and axiomatisable, hence decidable, theory of infinite models, namely the theory of the additive group of the rationals.

§ 3. Seven possibilities ruled out. Lemma 6 has, of course, many corollaries of theories related as in the diagrams of § 1. We are, however, only interested here in the properties of decidability and undecidability rather than the finer distinstions between axiomatisability, co-axiomatisability and their negations and combinations, and even then only in the special case where $R = T_i$. Since $R = RR^* \cap R^{**}$, (see Lemma 2), it follows from Lemma 6 that the decidability of RR^* and R^{**} entails that of R. Moreover, part (iii) of Lemma 6 shows that if T and R^* are decidable and R is undecidable but axiomatisable, then RR^* cannot be decidable. And finally, if R, R^*, R^{**} and ∂R are all decidable then so is RR^*, since by 6 (i) it is axiomatisable and hence 6 (iii) is applicable to the pair R^{**}, RR^*. Thus we have

THEOREM 1. (i) *If the theory* T_f^* *of strictly infinite models of a theory* T *and the theory* T_{fi} *of infinite models of the theory of finite models of* T *are both decidable, then so is the theory* T_i *of infinite models of* T.

(ii) *If the theory* T_f *of finite models of a decidable theory* T *is decidable but the theory* T_i *of ifninite models of that theory is undecidable, then the theory* T_{fi} *of infinite models of* T_f *is undecidable as well.*

(iii) *If* T_f, T_i, T_f^* *and* $\partial T = T_f T_f^*$ *are all decidable, then so is* T_{fi}.

If we arrange the extensions considered in the sequence $(T_i, T_f, T_{fi}, T_f^*, \partial T)$ and write quintuples with entries 'u' and 'd' as abbreviations for the statements that the kth extension in this list is undecidable or decidable respectively, then we can paraphrase the above theorem as follows: If T is decidable then the following cases cannot occur: $(ddudd)$, $(udddd)$, $(udddu)$, $(uddud)$, $(udduu)$, $(uuddd)$, and $(uuddu)$.

If we replace u by 0 and d by 1 and use the usual binary number system, then these cases are represented respectively by the numbers 27, 15, 14, 13, 12, 7 and 6. We shall use these numbers to refer to them. Part (iii) of Theorem 1 expresses the impossibility of case 27, while cases 15, 14, 13 and 12 are ruled out by part (ii) and 7 and 6, as well as 14 and 15, by part (i).

This is a very trivial and general theorem. For, if we simply consider the lattice diagram underlying the diagram of § 1, disregarding the relation of duality by means of the operation *, and label the vertices with 'u' or 'd' in accordance with our cases we note that on the basis of § 2, all but 12 and 13 are already ruled out, whereas in these two cases the only property that is needed for their exclusion, in addition to the purely lattice theoretic relation between the extensions, is the axiomatisability of T_i. It is therefore rather surprising that actually these 7 cases should be the only ones that cannot occur for the special case of $R = T_i$, as we shall show in § 5 by means of examples with infinitely axiomatisable theories T. One cannot help feeling that finite axiomatisability of T should add some deeper ties to the diagram.

§ 4. Some decidable extensions of a decidable theory. We shall construct our examples for the remaining 25 possibilities by extending the decidable theory AG of Abelian groups. To this end we need some criteria under which an infinite extension of a decidable theory is again decidable. We fix a decidable theory $\boldsymbol{T}$ and with it the structure $\mathfrak{T} = \langle \mathcal{C}, \cap, [\ ], {}^*\rangle$ of the set $\mathcal{C}$ of extensions of $\boldsymbol{T}$ under the operations of intersection, join and dualisation. We write $\mathcal{C}_d$ for the set of decidable extensions of $\boldsymbol{T}$. While the set $\mathcal{C}_0$ of finite extensions of $\boldsymbol{T}$ is a subset of $\mathcal{C}_d$ and forms a substructure of $\mathfrak{T}$ that is a Boolean algebra, and the set of axiomatisable extensions forms a sublattice of $\mathfrak{T}$, nothing algebraically nice can be said—as far as we can see—about $\mathfrak{T}_d$ itself. Those extensions that are closed under the operation ** of taking the double $\boldsymbol{T}$-dual form, according to Lemma 2, a Boolean algebra $\mathfrak{T}^{**}$ under the operations $\cap$, $[\ ,\ ]^{**}$, and *. Now, since both axiomatisability and $\boldsymbol{T}$-co-axiomatisability are preserved under intersection by Lemma 6 (i) and (ii), the subset $\mathcal{C}_1$ consisting of those theories that are both axiomatisable and $\boldsymbol{T}$-co-axiomatisable is closed under intersection, but not in general under * nor under the new join $[\ ,\ ]^{**}$. On the other hand, the set $\mathcal{C}_{sd}$ of all axiomatisable theories that have an r.e. $\boldsymbol{T}$-co-basis is closed under * but not under the other two operations, for the union of two r.e. $\boldsymbol{T}$-co-bases is a $\boldsymbol{T}$-co-axiomsystem but not in general again a $\boldsymbol{T}$-co-basis, so that this strong form of $\boldsymbol{T}$-co-axiomatisability need not be preserved by intersection, and the union of two $\boldsymbol{T}$-axiomsystems is as a rule weaker than the double dual of the join, so that axiomatisability need not be preserved by the operation $[\ ,\ ]^{**}$, although the property of having an r.e. $\boldsymbol{T}$-co-basis is preserved, since the set $\{b \wedge b' \colon b \in B, b' \in B'\}$ is a cobasis for $[\boldsymbol{T}|B|, \boldsymbol{T}|B'|]^{**}$. We shall call these latter theories, i.e., the theories $\boldsymbol{R}$ for which there are r.e. sets A and B such that $\boldsymbol{R} = \boldsymbol{T}[A] = \boldsymbol{T}|B|$, strongly decidable extensions of $\boldsymbol{T}$. This seems the natural generalization of the concept of a finite extension from the point of view of decidability.

We have the following inclusions:

$$\mathcal{C} \supset \mathcal{C}_d \supset (\mathcal{C}_d \cap \mathcal{C}^{**}) \supset \mathcal{C}_1 \supset \mathcal{C}_{sd} \supset T_0\,,$$

which—as might be expected, and is demonstrated by examples in § 5—are all proper. Moreover, we note that, whenever B is a recursive sequence, $\boldsymbol{T} \ast B \in \mathcal{C}_d$, $\boldsymbol{T}' \in \mathcal{C}_d$ and $\boldsymbol{T}' \ast B = (\boldsymbol{T} \ast B)\boldsymbol{T}'$ then $\boldsymbol{T}' \ast B \in \mathcal{C}_d$. This is of particular interest in case $B = \sim E$ and we see how useful it would be to have some conditions for $(\boldsymbol{T}[A])_f = \boldsymbol{T}_f[A]$.

We state now a very weak and very general principle that leads occasionally to a decision procedure.

LEMMA 7. *Assume that*

(i) $\boldsymbol{T}_1 \subset \boldsymbol{T}_2 \subset \ldots \subset \boldsymbol{T}_n \subset \ldots$ *is a recursive chain of axiomatisable theories such that its union* $\boldsymbol{T} = \bigcup \boldsymbol{T}_n$ *is decidable and such that* $\boldsymbol{R} \subset \boldsymbol{T}$, *and*

(ii) $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_n, \ldots$ *is a recursive sequence of uniformly decidable theories such that* $\boldsymbol{R} \cap \boldsymbol{T}_n = \boldsymbol{T} \cap \boldsymbol{S}_n$, *for all* n, *then* $\boldsymbol{R}$ *is decidable.*

By the assumptions we require the existence of recursive functions h and f such that for each n the set $\{a(h(n, m)) \colon m \in N\}$ is an axiom system for $\boldsymbol{T}_n$ and that $f(n, m) = 0$ if and only if $a(m)$ is a theorem of $\boldsymbol{S}_n$. (It is by no means sufficient to have merely a recursive sequence of individually decidable theories $\boldsymbol{S}_n$, as is easily shown by examples. For instance, let $\boldsymbol{T}_0 = \boldsymbol{AG}$, $\boldsymbol{T}_{n+1} = \boldsymbol{T}_n[\sim E(n)]$, $\boldsymbol{R} = \bigcup \boldsymbol{R}_n$, where $\boldsymbol{R}_0 = \boldsymbol{AG}$, $\boldsymbol{R}_{n+1} = \boldsymbol{R}_n[\sim E(g(n))]$, and g is a recursive function with non-recursive range Then (i) is satisfied, but, although $\langle \boldsymbol{R}_n \rangle$ is a sequence of uniformly decidable theories, the sequence $\langle \boldsymbol{S}_n \rangle$, with $\boldsymbol{S}_n = \boldsymbol{R} \cap \boldsymbol{T}_n$, is only a uniformly axiomatisable sequence of decidable theories, and $\boldsymbol{R} = \bigcup \boldsymbol{R}_n$ is not decidable. The function f, such that

$$a(f(n, m)) = \begin{cases} \sim E(g(m)) & \text{if} \quad g(m) < n\,, \\ x = x & \text{otherwise}\,, \end{cases}$$

uniformly prescribes the axiom systems relative to $\boldsymbol{AG}$ for the theories $\boldsymbol{S}_n$, which are finite extensions of $\boldsymbol{AG}$, and thus decidable, but the sequence $\langle a(f(n, m)) \rangle$ is not decidable in the sense of Definition 4.)

For the proof of the lemma simply note that $a \in \boldsymbol{R}$ if and only if, for some n, $a \in \boldsymbol{T}_n$ and $a \in \boldsymbol{S}_n$, while $a \notin \boldsymbol{R}$ if and only if either $a \notin \boldsymbol{T}$ or, for some n, $a \in \boldsymbol{T}_n$ but $a \notin \boldsymbol{S}_n$.

An illustration of the use of Lemma 7 is given by the following situation. Let $\boldsymbol{M}$ be an r.e. sequence of finite structures, i.e., a sequence of structures $\mathfrak{M}(n)$ such that the sequence of sentences $D(n)$ characterising them—that is, their diagrams—forms a recursive sequence. Assume that the theory $(\tau(\boldsymbol{M}))_i$ is complete and axiomatisable by some sequence A of sentences. Every sentence $A(n)$ is then, relative to $\tau(\boldsymbol{M})$, equivalent to a sentence $\sim F(n)$ where $F(n)$ is the disjunction of the diagrams of exactly those finite structures of $\boldsymbol{M}$ for which $A(n)$ fails. There are only finitely many such models for each n, and if there are none we set for $F(n)$ any fixed contradiction. If this sequence F is also recursive, then the sequence $\langle A(n) \vee F(n) \rangle$ is an axiom system for the theory $\tau(\boldsymbol{M})$, and so $\tau(\boldsymbol{M})$ is decidable. In particular, using our terminology, we can say that if $\boldsymbol{T}$ is a co-axiomatisable theory of finite models with a complete theory $\boldsymbol{T}_i$ that is axiomatisable by a sequence that is decidable relative to $\boldsymbol{T}$, then $\boldsymbol{T}$ is decidable.

We shall now state some useful consequences of Lemma 7 that are most appropriately phrased in terms of recursive sequences of sentences relative to $\boldsymbol{T}$ rather than in terms of extensions. From now on all sequences are assumed to be recursive. We introduce a partial ordering on the set of all such sequences by

DEFINITION 6. *The sequence* $A = \langle A(n) \rangle$ *is weaker than the sequence* $B = \langle B(n) \rangle$, *relative to* $\boldsymbol{T}$, *if* $(B(n) \rightarrow A(n)) \in \boldsymbol{T}$, *for all* n.

We represent this relation by the symbol $\leqslant_T$ and omit the subscript T when there is no danger of ambiguity. Thus $A \leqslant B$ means that there is a recursive sequence D such that, for each n, $A(n)$ is equivalent in $\boldsymbol{T}$ to the disjunction $D(n) \vee B(n)$, and by the same token, B is stronger than A, $B \geqslant A$, if there is a recursive sequence C such that, for all n, $B(n)$ is equivalent in $\boldsymbol{T}$ to the conjunction $C(n) \wedge A(n)$. We shall also use the phrases: 'A is a weakening of B by the sequence 'D' and 'B is a strengthening of A by the sequence C'. We shall say that an extension $\boldsymbol{R}$ of $\boldsymbol{T}$ is recursively weaker than the extension $\boldsymbol{R}'$ if there is a $\boldsymbol{T}$-axiomsystem for $\boldsymbol{R}$ that is weaker than some $\boldsymbol{T}$-axiomsystem for $\boldsymbol{R}'$, and shall write $\boldsymbol{R} \leqslant_T \boldsymbol{R}'$, again usually omitting the subscript $\boldsymbol{T}$. Occasionally we shall indulge in some abuse of language, by using phrases such as '$\boldsymbol{R}'$ is a weakening of $\boldsymbol{R}$'. It is clear that the partial ordering $\leqslant$ on the set of recursive extensions of $\boldsymbol{T}$ is strictly stronger than the relation of inclusion. We have a lattice of sequences under the operations $\vee$ and $\wedge$ defined by $D \vee B = \langle D(n) \vee B(n) \rangle$ and $D \wedge B = \langle D(n) \wedge B(n) \rangle$, and with this also a new lattice of extensions of $\boldsymbol{T}$. But, while we do have $\boldsymbol{T}[D \wedge B] = \boldsymbol{T}[D]\boldsymbol{T}[B]$ we have, of course, only $\boldsymbol{T}[D \vee B] \subseteq \boldsymbol{T}[D] \cap \boldsymbol{T}[B]$. Finally, we recall Definition 2 and observe that the set of $\boldsymbol{T}$-disjoint sequences is closed under strengthening, and the set of $\boldsymbol{T}$-co-disjoint ones under weakening. Using the notations B^n for the sequence $\{B(i)\colon i \geqslant n\}$ and B_n for $\{B(i)\colon i < n\}$, we can now state:

LEMMA 8. *Let $\boldsymbol{T}$ be an axiomatisable theory, then*

(i) *if D is a recursive sequence, such that $\langle \boldsymbol{T}[D^n] \rangle$ is a sequence of uniformly decidable theories, then every weakening of $\boldsymbol{T}[D]$ by a $\boldsymbol{T}$-disjoint recursive sequence is decidable, and*

(ii) *if C is a $\boldsymbol{T}$-co-disjoint recursive sequence, such that $\langle \boldsymbol{T}[C^n] \rangle$ is a sequence of uniformly decidable theories, then every recursive weakening of $\boldsymbol{T}[C]$ is decidable.*

To prove (i), let B be a $\boldsymbol{T}$-disjoint sequence. Then $\boldsymbol{T}[D \vee B] = (\bigcap_{n \in N} \boldsymbol{T}[D^{n+1}][B(n)][D_n]) \cap \boldsymbol{T}[D]$; for certainly $\boldsymbol{T}[D \vee B]$ is a subtheory of each of the theories mentioned on the right side, and thus the inclusion $\subseteq$ holds. Moreover, if $s \notin \boldsymbol{T}[D \vee B]$, then, $\sim s$ has a $\boldsymbol{T}$-model in which either all sentences of D are satisfied, or else—since B is $\boldsymbol{T}$-disjoint—all but one, say $D(n)$, in which case then $B(n)$ holds. Thus s is a non-theorem of one of the theories over which the intersection ranges, and the inclusion $\supseteq$ holds as well. By our assumption about D, the sequence of these theories is uniformly decidable, since they are finite extensions of the theories $\boldsymbol{T}[D^n]$. Hence, the representation of $\boldsymbol{T}[D \vee B]$ as an intersection yields a disproof procedure, and since $\boldsymbol{T}[D \vee B]$ is clearly axiomatisable it is decidable. To prove (ii) we use Lemma 7. Let A be any recursive sequence and set $\boldsymbol{R} = \boldsymbol{T}[A \vee C]$, $\boldsymbol{T}_n = \boldsymbol{T}[C_n]$, and the theories $\boldsymbol{S}_n = \boldsymbol{T}[C^{n+1} \cup (A \vee C)_n] \cap \boldsymbol{T}_n$. Clearly $\boldsymbol{R} = \bigcup \boldsymbol{T}_n$, and, by the assumptions about C, $\boldsymbol{S}_n$ are uniformly decidable. If we can show that $\boldsymbol{R} \cap \boldsymbol{T}_n = \boldsymbol{S}_n$, then all the conditions of Lemma 7 are fulfilled and our result follows. Now the inclusion $\subseteq$ is obvious, since $\boldsymbol{R} \subseteq \boldsymbol{T}[C^{n+1} \cup (A \vee C)_n]$. To establish the converse inclusion, assume that $s \notin \boldsymbol{R} \cap \boldsymbol{T}_n$. Now, if $s \notin \boldsymbol{T}_n$, then, clearly, $s \notin \boldsymbol{S}_n$, and so we may assume that $s \in \boldsymbol{T}_n$ but $s \notin \boldsymbol{R}$. But then we have $\left(\sim s \rightarrow \bigvee_{i \leqslant n} \sim C(i)\right) \in \boldsymbol{T}$, whence, because C is $\boldsymbol{T}$-co-disjoint, the theory $\boldsymbol{T}[\sim s]$ is an extension of $\boldsymbol{T}[C^{n+1}]$. Thus we find that $\boldsymbol{L} \neq \boldsymbol{T}[A \vee C][\sim s] = \boldsymbol{T}[(A \vee C) \cup C^{n+1}][\sim s] = \boldsymbol{S}_n[\sim s]$, and hence $s \notin \boldsymbol{S}_n$.

If we set $A = \sim C$ in Lemma 8 (ii) (or $D = C$, $B = \sim C$ in (i)), note that, for any $\boldsymbol{T}$-co-disjoint sequence C, we have $\boldsymbol{T}[C^n] = \boldsymbol{T}[C] \cap \boldsymbol{T}[\bigvee_{i \leqslant n} \sim C(i)]$, and using Lemma 6 we obtain

COROLLARY 8. *If $\boldsymbol{T}$ is axiomatisable and C is a $\boldsymbol{T}$-co-disjoint recursive sequence such that $\boldsymbol{T}[C]$ is decidable, then $\boldsymbol{T}$ is decidable if and only if $\langle \boldsymbol{T}[C^n] \rangle$ is a sequence of uniformly decidable theories.*

From this together with Lemma 8 (ii) it follows immediately that, for a decidable theory $\boldsymbol{T}$, the set of decidable extensions by $\boldsymbol{T}$-co-disjoint sequences is closed under weakening. Because of its usefulness we state this result as a lemma:

LEMMA 9. *If $\boldsymbol{T}$ is a decidable theory and C is a $\boldsymbol{T}$-co-disjoint sequence such that $\boldsymbol{T}[C]$ is decidable, then every recursive weakening $\boldsymbol{T}[A \vee C]$ of $\boldsymbol{T}[C]$ is decidable as well.*

COROLLARY 9. *If $\boldsymbol{T}$ and C satisfy the hypothesis of Lemma 9 then $\boldsymbol{T}[C']$ is decidable, for every recursive subset C' of C.*

To establish the corollary let $C' = \{C(n)\colon n \in M\}$, where M is a recursive set of natural numbers. The sequence $\langle A(n) \rangle$, where $A(n) = C(n)$ or $\sim C(n)$ according as $n \in M$ or not, is then recursive and so $\boldsymbol{T}[C'] = \boldsymbol{T}[A \vee C]$ is a recursive weakening of $\boldsymbol{T}[C]$ and hence decidable. It follows from this in particular that if $\boldsymbol{T}$ and $\boldsymbol{T}_i$ are both decidable, then so is any extension of $\boldsymbol{T}$ obtained by excluding a decidable set of finite models, i.e., a set of finite models for which the set of diagrams is decidable. This fact can be used for constructing decidable theories with decidable theories of infinite models, but undecidable theories of finite models, for, a decidable set of finite structures need by no means have a decidable theory.

We close this section with the rather vague observation that the relation $<$ inherits some of the special properties that the relation of being a finite extension has, but mere $\subset$ does not have. Lemmas 8 and 9 may be viewed in this light, and so can the following variant of Lemma 1 (v) and (vii).

LEMMA 10 (i). *If* $T \subset T' < R$ *such that* T' *is weakening of* R *by an increasing sequence* C *(i.e.,* $(C(n) \to C(n+m)) \in T$ *for all* n *and* m*), and* $R \ast A = (T \ast A)R$*, then* $T' \ast A = (T \ast A)T'$.

(ii) *If* $T \subset T' < T \ast\ast A = T[\sim H]$*, with* H *disjoint, then* $T' \ast\ast A = T \ast\ast A$.

For (i) we only need to show that $T' \ast A \subseteq (T \ast A)T'$. Assume then that $s \notin (T \ast A)T'$. If also $s \notin R \ast A$, then certainly $s \notin T' \ast A$ since $T' \ast A \subseteq R \ast A$. If, however, $s \in R \ast A$, then, by assumption, $s \in (T \ast A)R$, and if we have $R = T[B]$, we find that $s \in (T \ast A)[B_n] = \bigcap_A T[B_n, \sim a]$, for some $n \in N$. But, by the assumption on C, the conjunction over the set $(B \vee C)_k$ is logically equivalent to the disjunction over the sentences $C(h) \wedge \bigwedge_{i<h} B(h)$, $0 \leqslant h \leqslant k$, and so $s \notin T' \ast A$ follows from the sentence $s \notin (T \ast A)$ $[(B \vee C)_n] = \bigcap_A T[(B \vee C)_n, a]$.

For (ii) assume that $T' \ast\ast A \subset T \ast\ast A$ and $T' = T[\sim H \vee B]$. Then there exists a sentence s such that $L \neq T'[s \wedge H(n)] \supseteq T'[A] \supseteq T[A]$ for some n. Since H is disjoint, we have $T'[s \wedge H(n)] = T[s \wedge H(n) \wedge B(n)]$ and so $L \neq T[s \wedge H(n) \wedge B(n)] \supseteq T[A]$, contradicting the hypothesis that $\sim H(n) \in T \ast\ast A$. By Lemma 1 (vii) this proves (i).

Though uniteresting as well as trivial, this lemma will prove labour saving later.

§ 5. 25 examples. We are now ready for the proof of the counterpart to Theorem 1 which we shall phrase rather loosely so as to avoid an abundance of clumsy verbiage.

THEOREM 2. *Theorem* 1 *is best possible.*

By this we mean that, as long as no further restrictions are imposed on T, all those combinations decidable-undecidable for the quintuple of extensions $(T_i, T_f, T_{fi}, T_f^*, \partial T)$ that are not ruled out by Theorem 1 actually do occur. The proof of this statement obviously requires the construction of 25 theories to exemplify the 25 possibilities. Fortunately, we are able to reduce the labour to the construction of nine basic examples.

We recall the coding introduced in § 3 and shall mean by the phrase "T has property (n)" that the quintuple of extensions associated with T has the decidability properties as indicated by the binary representation of n where 0 stands for undecidable and 1 for decidable. For instance, saying that T has (26) we mean to say that T, T_i, T_f and T_f^* are decidable, but T_{fi} and ∂T are undecidable. A theory that has property (31) will be called *fully decidable*. Now, the set of ordered five-tuples is obviously a semilattice under the operation of componentwise ordinary multiplication. Thus, the set of numbers < 32 is a semilattice under the operation

induced on them via their binary representation. We denote this operation by $\circ$, so that, e.g., $5 \circ 9 = 1$. On the other hand, we introduce a composition for theories. Let R and S be theories whose languages $L(T)$ and $L(S)$ have no predicate symbols and no term symbols in common. We form the language $L' = L(T, S, P, a)$ which contains in addition to the symbols occurring in T and S the unary predicate symbol P and the individual constant a, and no other symbols. In this new language we define theories T' and S' as follows. The axioms of T' are the relativisations of all theorems of T to P, as defined in [12] p. 24, the sentences $\sim P(a)$, $(\forall x)(P(x) \vee x = a)$, and all the closures of all formulas $\sim \Phi(\xi_1, \ldots, \xi_n)$ or $\sigma(\xi_1, \ldots, \xi_n) = a$, for all $n \geqslant 0$, where Φ and σ are respectively an n-ary predicate symbol or n-ary term symbol belonging to $L(S)$ and the ξ_i are any terms or Φ and σ are such symbols belonging to $L(T)$ and at least one of the ξ_i is a. Analogously we define S' with S and T interchanged as well as P replaced by $\sim P$. Now it is clear that T' and S' can also be defined in case their languages do have some symbols in common, namely by first renaming them. It is also clear that all this can be done in some unique canonical way, too tedious to set down for our purpose. Therefore we can define

DEFINITION 7. *Given two theories* T *and* S*, let* T' *and* S' *be as described above. Then* $T \circ S = T' \cap S'$.

The models of $T \circ S$ are then all such that upon deletion of the one element that is the value of a they become models of either T or S. It is easy to see that every sentence A of L' is equivalent in $T \circ S$ to a disjunction of the form $(\sim P(a) \wedge B^P) \vee (P(a) \wedge C^{\sim P})$ for some sentences $B \in L(T)$ and $C \in L(S)$ and such that A is a theorem of $T \circ S$ if and only if B is a theorem of T and C is a theorem of S. (We are using the familar notation D^Q for the relativisation of the sentence D to the unary predicate Q.) In particular, given any sentence $B \in L(T)$, the sentence $P(a) \vee B^P$ is a theorem of $T \circ S$ if and only if $B \in T$, and similarly for any $C \in L(S)$. It follows that $T \circ S$ is decidable if and only if both T and S are decidable. Moreover, it is easy to see that $(T \circ S)_i = T_i \circ S_i$, $(T \circ S)_f = T_f \circ S_f$, $(T \circ S)_{fi} = T_{fi} \circ S_{fi}$, $(T \circ S)_f^* = T_f^* \circ S_f^*$ and $\partial(T \circ S) = \partial T \circ \partial S$. It follows that this construction serves our purpose.

LEMMA 11. *If the theories* T *and* S *have the properties* (n) *and* (m) *respectively, then the theory* $T \circ S$ *has the property* $(n \circ m)$.

Obviously there is no harm done, then, by using the same symbol $\circ$ for both operations. Considering the semilattice of natural numbers < 32 under the operation $\circ$ it is a trivial matter to verify that the set of numbers other than 27, 15, 14, 13, 12, 7 or 6 forms a sub-semilattice generated by the numbers 31, 30, 29, 25, 23, 19, 11 and 5. This set of generators is minimal. We list the numbers in descending order because $n \circ m \leqslant \min(n, m)$ and thus it is in descending order that one finds minimal sets of generators. It follows that we are left with the task of finding for each of these nine numbers a theory with the corresponding property.

We shall now exhibit our examples in a sequence of nine propositions, for each of which we give at least a sketch of a proof.

PROPOSITION 1. *The theory of Abelian groups is fully decidable.*

Proof. The theory $\boldsymbol{AG}$ is thus an example of a theory that has property (31). Moreover, it is non-degenerate in the sense that all the pertinent extensions are distinct. This is why it is rich enough to offer among its extensions examples for the other eight possibilities. That $\boldsymbol{AG}$ has indeed property (31) is seen easily enough from an analysis of Szmielew's decision-procedure [11]. In particular it was shown in [3] as well as in [7] that $\boldsymbol{AG}_f$ is decidable. But, since we shall make much use of the peculiarities of this procedure in the sequel, we shall list here some of the structural features of the Lindenbaum algebra associated with $\boldsymbol{AG}$. For detail and notation we refer the reader to [11] and [7]. Every sentence of $\boldsymbol{L}(\boldsymbol{AG})$ is equivalent in $\boldsymbol{AG}$ to a Boolean combination of sentences belonging to the basic set B. B splits into two sets H and Q. Every sentence of H is of the form $H(m)$, where m is a positive integer, and an Abelian group satisfies $H(m)$ if it is of exponent m. Q consists of the sentences $Q^{(i)}(q,k,n)$, where i is 1, 2 or 3, q ranges over all primes, k over the positive integers and n over the natural numbers. An Abelian group is a model for $Q^{(1)}(q,k,n)$ if its q^k-rank is n, i.e., if its subgroup consisting of all those q^{k-1}th powers that are of order q is of rank n. $Q^{(2)}(q,k,n)$ is valid in $\mathfrak{A}$ if the factor group of the group of all q^{k-1}th powers of elements of $\mathfrak{A}$ over the group of all q^kth powers is of rank n. Finally, $Q^{(3)}(q,k,n)$ is valid in $\mathfrak{A}$ if $\mathfrak{A}$ is of pure q^k-rank n, i.e., if the subgroup consisting of all elements that are of q-height $k-1$ and of order q has rank n. We denote the set of all primes by Λ, and if $P\subset\Lambda$ we write $\bar{P}$ for $\Lambda-P$. If $P=\{q\}$, we shall omit the brackets. Moreover, we denote by $Q_P^{(i)}$ the set of all sentences $Q^{(i)}(q,k,n)$ with $q\in P$, and write Q_P for the set $Q_P^{(1)}\cup Q_P^{(2)}\cup Q_P^{(3)}$. Finally, if $S\subseteq B$, we shall denote the set of Boolean combinations of elements of S by $\|S\|$. The Lindenbaum algebra of $\boldsymbol{AG}$ is isomorphic to the Boolean algebra generated by the set B and subject to the defining relations as given, e.g., in Lemma 1 of [7]. We repeat those relations in a form that will prove convenient:

(1) (i) *The sets H and the sets $\{Q^{(i)}(q,k,n)\colon n\in N\}$ are each disjoint.*

(ii) $Q^{(i)}(q,k,n)=\bigvee\limits_{r\leqslant n}\big(Q^{(i)}(q,k+1,r)\wedge Q^{(3)}(q,k,n-r)\big)$ *for* $i=1$ *or* 2 *and all* q,k,n.

(iii) $\boldsymbol{AG}[H(m)]\cap\|Q\|=\boldsymbol{AG}[Q(m)]\cap\|Q\|$, *where*

$Q(m)=\{Q^{(i)}(q,k,0)\in Q\colon (q^k,m)<q^k\}\cup\{\sim Q^{(3)}(q,k,0)\in Q\colon (q^{k+1},m)=q^k\}$ *and* (n,m) *denotes the greatest common divisor of* n *and* m.

Once it is proved that this yields a complete set of relations (cf. [11] and [7]), it is easy to see that

(2) (i) $\boldsymbol{AG}[S]\cap\|Q_P\|=\boldsymbol{AG}\cap\|Q_P\|$ *whenever* $S\subset\|Q_{\bar{P}}\|$ *and* $\boldsymbol{AG}[S]\neq\boldsymbol{L}$,

(ii) $\boldsymbol{AG}[\sim H]\cap\|Q\|=\boldsymbol{AG}\cap\|Q\|$.

From (1) (i) and (ii) it is seen that the sets $\boldsymbol{AG}\cap\|Q_q\|$ form a sequence of uniformly recursive subsets of B and that thus—because of the independence property (2) (i)—also $\boldsymbol{AG}\cap\|Q\|$ is a recursive subset of B. (More precisely, it is easy, as is done in [7], to exhibit for each filter $\boldsymbol{AG}\cap\|Q_q\|\subset\|Q_q\|$ what we might call a full recursive set of generating reduced disjunctions of elements of $Q_q\cup\sim Q_q$, so that the union of these sets is recursive $\subset\|Q\|$. We call S a full generating set for a filter $\mathfrak{F}\subset\|Q\|$, if, whenever a disjunction of elements of $Q\cup\sim Q$ belongs to $\mathfrak{F}$, then some sub-disjunction of it belongs to S, and reduced, if no sub-disjunction of a disjunction that belongs to S belongs to $\mathfrak{F}$. That these or similar concepts are indispensable, is seen by examples. For instance, if we add to $\boldsymbol{AG}$ the axioms $Q^{(1)}(q_{g(n)},1,n+1)$, where g is a recursive one-one function with non-recursive range, and q_n denotes the nth prime, then the resulting set is again generating and recursive but neither full nor reduced, while if we reduce it, the individual sets remain recursive, but no longer uniformly so, for, $\sim Q^{(1)}(q_n,1,0)$ belongs if and only if n is in the range of g, and indeed the resulting theory is not decidable.) From this in turn together with (2) (ii) it follows that

(3) $\boldsymbol{AG}[\sim H]$ *is decidable,*

since all basic sentences belong either to H or to Q.

Now, according to (1), the complete sentences are exactly those of the form $H(m)\wedge\bigwedge\limits_{q^k|m}Q^{(3)}(q,k,n_{q^k})=H(m)\wedge U(m,r)$, subject to the condition that $n_{q^k}\neq 0$, whenever $(q^{k+1},m)=q^k$. These sentences are, of course, the atoms of the Boolean algebra $\|B\|/(\sim\boldsymbol{AG}\cap\|B\|)$. On the other hand, they clearly all describe finite Abelian groups, and so we have here the particularly nice situation in which $\boldsymbol{T}_f$ is fully determined by the Lindenbaum algebra of $\boldsymbol{T}$. So far then $\boldsymbol{AG}_c=\boldsymbol{AG}_f\supseteq\bigcap\boldsymbol{AG}[H(m)]$. But it is easily seen from (1) and (2), that any $\boldsymbol{AG}$-consistent sentence of the form $H(m)\wedge s$ can be "completed" to such an atom, and that hence $\boldsymbol{AG}[H(m)]\supset\boldsymbol{AG}_f$, whence we have equality in the above. To show that H is actually a co-basis for $\boldsymbol{AG}_f$, we introduce the set D of sentences

$$D(q,k,n)=\big(\sim Q^{(1)}(q,k,n)\vee Q^{(2)}(q,k,n)\big)\wedge\big(\sim Q^{(2)}(q,k,n)\vee Q^{(1)}(q,k,n)\big)$$

and observe that (1) implies $\boldsymbol{AG}[H(m)]\supset\boldsymbol{AG}[D]$, for all m. On the other hand, it follows from (2) that whenever $\boldsymbol{AG}[s][\sim H]\neq\boldsymbol{L}$, then there exist primes q such that $AG[s\wedge \sim D(q,k,n)] \neq L$, for any n and k. Thus, if $AG[s] \supseteq \bigcap AG[H(m)]$, then s must imply, in AG, a finite disjunction of sentences of H and thus H is indeed a co-basis. In fact we have

(4) $AG_f = AG_c = AG|H| = AG[D]$, *and is decidable.*

It was shown in [7] that the last equality holds, but we shall give here a proof that illustrates the methods introduced in § 4. Since $AG_f = AG|H|$, we have $AG_f^* = AG[\sim H]$, and, since every finite subset of the set $\sim Q$, as well as every finite subset of the set $Q_0 = \{Q^{(i)}(q,1,0)\}$ is consistent with some sentence $H(m)$, with $m\neq 1$, we find that both the theory of the direct product of all finite cyclic groups and the theory of the additive group $\mathfrak{Q}$ of the rationals are extensions of ∂AG, i.e.,

(5) (i) $AG[\sim B] = AG[\sim Q] = \tau\big(\prod_{1<m\in N} \mathfrak{C}_m\big) \supset \partial AG$, *and*

(ii) $AG[Q_0 \cup \sim H] = AG[Q_0][\sim H(1)] = \tau(\emptyset) \supset \partial AG$.

Now from this, $AG_f = AG|H|$ and (2) it follows that

$$AG_f = AG[\sim Q] \cap AG_f = AG\big[(AG[\sim Q]\cap \|Q\|)\cap AG_f\big]$$
$$= AG\big[\bigcup_{q\in A}(AG[\sim Q_q]\cap AG_f)\big].$$

If we set $Q_{q,m} = \{Q^{(i)}(p,k,n)\in Q:\ p=q, \text{ and } k-1,\ n\leqslant m\}$, and similarly for $D_{q,m}$, then clearly $AG[\sim Q_q] = \bigcup_{m\in N} AG[\sim Q_{q,m}]$, and it is easily seen that $AG_f\cap AG[\sim Q_{q,m}] = AG[D_{q,m}]$ and therefore $AG_f \cap AG[\sim Q_q] = AG[D_q]$. We note that, since the theory $AG[\sim Q]$, as well as all the theories $AG[\sim Q_q]$ are obviously decidable, we could now invoke Lemma 7 to conclude the decidability of AG_f. However, since the co-axiomatisability of AG_f is ensured even by the mere finite axiomatisability of AG, the significant outcome of this argument is that D is an axiomsystem for AG_f relative to AG. Thus we have established (4) and obtain, using (3),

(6) $AG_f^* = AG|\sim D| = AG[\sim H]$, *and is decidable.*

It is at this point that, using Lemma 5, we automatically get

(7) AG *is decidable.*

We observe that this seems really the natural chain of reasoning. The decidability of AG_f can either be obtained as above, i.e., by finding an axiomsystem for it, or else by model theoretic considerations as in [3] where Feferman and Vaught's method of building up from cyclic models by direct product formation is used. The decidability of $AG[\sim H]$, on the other hand, follows, as we have seen, easily from an analysis of the relations between the basic sentences, which shows, cf. 2 (ii), that for this theory only the basic set Q need be considered. That the two results together yield the decidability of AG then hinges on the fact that H is a co-basis of AG_f, so that the two theories which have separately been

proved decidable are duals of each other. It is clear then, what role the division of B into the sets Q and H plays.

Now the decidability of $\partial AG = AG[\sim H \cup D] = AG_f^*[D] \cap AG_f^*[\sim Q]$ follows easily by two applications of Lemma 7 and using the decidability of AG_f^*, of each theory $AG_f^*[\sim Q_q]$ and of $AG_f^*[\sim Q]$. We also note that ∂AG can be represented as the intersection of the decidable extensions $AG_f[\sim Q_q]$ of AG_f, although it can, of course, not be the intersection of finite extensions of AG_f. At any rate we have

(8) ∂AG *is decidable.*

In order to prove the decidability of AG_{fi} and of AG_i we observe first that $AG_i = AG[\sim E] = AG[\sim U]$ if we denote by U the set of atoms $H(m)\wedge U(m,r)$, where r is a sequence of natural numbers subject to conditions induced by m as described above. Again, for AG, the theory of infinite models is completely characterised by the algebraic structure of the Lindenbaum algebra. Clearly,

$$AG_{fi} = \big(\bigcap AG[H(m)]\big)[\sim U] \subseteq \bigcap \big(AG[H(m)][\sim U]\big) = \bigcap AG_i[H(m)],$$

where actually $AG[H(m)][\sim U] = AG[H(m)][\sim U_m]$, if we write U_m for the set of sentences $U(m,r)$, with fixed m. But we have here the special case, where equality holds, i.e., the theory AG_{fi} coincides with the theory of infinite periodic groups. This follows from the fact, easily verified on the basis of (1) and (2), that every sentence $s\in\|B\|$ either implies a finite disjunction of sentences of $H\cup\sim D$ or else is consistent with $AG[H(mq)][\{\sim Q^{(1)}(q,1,n):\ n\in N\}]$, for some m and q. Roughly speaking, this means that there are enough periodic infinite groups to yield a basis for the set of models of AG_{fi}. More precisely, it means that H is a cobasis for AG_{fi} relative to AG_i. We remark that in general, if $R^* = T|H|$, then $\bigcap_{h\in H} R[h] = R \ast R^{**} \supseteq R \ast\!\ast R^*$, and thus $RR^* = R|H| \cap \partial R \subseteq R|H|$, as can easily be seen from Lemma 2. Thus, equality holds only if $\partial R \supseteq R|H|$, as is the case here. That each theory $AG_i[H(m)]$ is decidable can now easily be shown. For instance it follows from (1) that

$$AG_i[H(m)] = \bigcap_{q|m} \big(AG[H(m)][\{\sim Q^{(1)}(q,1,n):\ n\in N\}]\big)$$

and is thus even a finite intersection of obviously decidable theories. From the decidability of AG_{fi}, (4) and (7) now follows the decidability of AG_i by Lemma 5. Thus

(9) $AG_{fi} = AG_i|H|$, *and both AG_{fi} and AG_i are decidable.*

That AG is thus fully decidable is fairly easy to see from model theoretic considerations, once one has the decidability of AG. We have gone here to the trouble of trying to analyze how these results hang together with the structural properties of the algebra $\|B\|/(\sim AG \cap \|B\|)$, because we are interested in the question, what properties of a decision procedure for a theory $\boldsymbol{T}$ are sufficient to ensure the full decidability of $\boldsymbol{T}$? So far, we are not able to phrase a concise and meaningful general principle. At least, we shall see in the discussion of the next examples in what ways things may break down so that full decidability fails to hold. However, it must be pointed out that our examples are all very contrived and that none of them are finitely axiomatisable. Thus, the question remains open, how many of the regularities of the theory of Abelian groups are connected with its being finitely axiomatisable. Another striking feature of $\boldsymbol{AG}$ is the fact that the set of laws other than the one defining $\boldsymbol{AG}$ forms a co-basis for $\boldsymbol{AG}_f$. One may well ask whether the result that this set of laws also forms a co-basis for $\boldsymbol{AG}_{fi}$ relative to $\boldsymbol{AG}_i$ necessarily follows from this together with the fact that $\boldsymbol{AG}$ itself is an equational theory. That the latter condition would be indispensable can easily be seen by, e.g., adding to $\boldsymbol{AG}$ the axioms stating that every group of even exponent is cyclic. Furthermore, we remark that when all $\boldsymbol{T}$-complete sentences, i.e., all sentences such that $\boldsymbol{T}[s]$ is complete, define finite $\boldsymbol{T}$-models—as is the case for $\boldsymbol{AG}$—and once a basic set of sentences, e.g., $\boldsymbol{L}(\boldsymbol{T})$ itself, has been exhibited, then, while the decision problem for $\boldsymbol{T}$ becomes the so-called "word-problem" for the algebra $\mathfrak{B} = \|B\|/(\sim\boldsymbol{T} \cap \|B\|)$ relative to the set of generators, the decision-problems for the various extensions considered here all become "word-problems" relative to B for various natural quotient-algebras of $\mathfrak{B}$. For, if we denote the set of atoms by A, the set of elements that generate atomic or atomless ideals by H and K, respectively, and express the formation of filters by square-bracketing, then we obtain corresponding to our extensions the filters $[\sim A]$, $\bigcap\limits_A [a] = \bigcap\limits_H [h] = [\sim K]$, $[\sim A \cup \sim K]$, $\bigcap\limits_K [k] = [\sim H]$ and $[\sim H \cup \sim K]$. The ideals that give the above-mentioned algebras are then, of course, obtained by dualisation, i.e., in our notation by adding the prefix $\sim$. We note that H is the "largest" cobasis for $\boldsymbol{T}_f$ and that any set of generators for the ideal generated by H is a co-basis for $\boldsymbol{T}_f$. The ideal generated by H corresponds to $\boldsymbol{T}_f^*$ and is the maximal atomic ideal while the ideal generated by $\boldsymbol{K}$ corresponds to $\boldsymbol{T}_f$ and is the maximal atomless ideal.

Finally, we close this discussion with the remark that there are—as is to be expected—many known cases of fully decidable theories $\boldsymbol{T}$ with non-trivial theories $\boldsymbol{T}_f$ and $\boldsymbol{T}_i$, for which $\boldsymbol{T}_f$ does not coincide with the theory $\boldsymbol{T}_c$ of finitely axiomatisable models. One such example is the theory $\boldsymbol{BA}$ of Boolean algebras, which actually coincides with $\boldsymbol{BA}_c$. A set of basic sentences consisting of a single sentence a and a disjoint sequence A can be chosen so that the atoms are of the form $a \wedge A(n)$, $\sim a \wedge A(n)$, and that $\boldsymbol{BA}_i = \boldsymbol{BA}[\{\sim a \vee \sim A(n)\}]$. $\boldsymbol{BA}_f = \bigcap \boldsymbol{BA}[a \wedge A(n)] = \boldsymbol{BA}[a]$, $\boldsymbol{BA}_{fi} = \boldsymbol{BA}[a][\sim A]$, $\boldsymbol{BA}_f^* = \bigcap \boldsymbol{BA}[\sim a \wedge A(n)] = \boldsymbol{BA}[\sim a]$, and $\partial\boldsymbol{BA} \doteq \boldsymbol{L}$, while $\boldsymbol{BA}_{fi}$ is complete. So, here we have the particularly simple situation where a single sentence separates the atoms that determine the finite models from those that determine the strictly infinite ones.

One consequence of the above discussion of the set B of basic sentences for $\boldsymbol{AG}$ will prove particularly useful. We shall denote the nth prime in the natural order by p_n, and write P_n for the set of all primes p_r with $r > n$.

LEMMA 12. *There exists a recursive function f such that, for all sentences $s \in \boldsymbol{L}(\boldsymbol{AG})$*

(i) *if $\boldsymbol{AG}[s \wedge \bigwedge\limits_{m \leqslant f(\nu(s))} \sim H(m)] \neq \boldsymbol{L}$, then $\boldsymbol{AG}_f^*[s] \cap \|Q_{P_{f(\nu(s))}}\| = \boldsymbol{AG} \cap \|Q_{P_{f(\nu(s))}}\|$ and s has, in particular, a strictly infinite $\boldsymbol{AG}$-model, in which the sentences $Q^{(i)}(p, k, 0)$ hold for all $p \in P_{f(\nu(s))}$ with the possible exception of one such prime, that can be chosen arbitrarily;*

(ii) *if $\boldsymbol{AG}_f[s] \neq \boldsymbol{L}$, then $\boldsymbol{AG}_f[s] \cap \|Q_{P_{f(\nu(s))}}\| = \boldsymbol{AG}_f \cap \|Q_{P_{f(\nu(s))}}\|$ and s has in particular a finite $\boldsymbol{AG}$-model, in which the sentences $Q^{(i)}(p, k, 0)$ hold for all $p \in P_{f(\nu(s))}$ with the possible exception of one such prime, that can be chosen arbitrarily, and*

(iii) *if $\boldsymbol{AG}_f[s \wedge \bigwedge\limits_{m \leqslant f(\nu(s))} \sim H(m)] \neq \boldsymbol{L}$, then $\boldsymbol{AG}_{fi}[s] \cap \|Q_{P_{f(\nu(s))}}\| = \boldsymbol{AG}_{fi} \cap \|Q_{P_{f(\nu(s))}}\|$ and s has in particular an infinite periodic $\boldsymbol{AG}$-model, in which the sentences $Q^{(i)}(p, k, 0)$ hold for all $p \in P_{f(\nu(s))}$ with the possible exception of one such prime that can be chosen arbitrarily.*

This function was used in [7] and it is clear what it is. From now on f shall always stand for this particular function. Moreover, we let g be a fixed recursive one-one function with non-recursive range $R(g) \subset N - \{0\}$. Finally, we assume that a recursive ordering of the set D of all sentences $D(q, k, n)$ has been fixed so that $D = \{D(n)\colon\ n \geqslant 1\}$. Now we are ready for the eight counter-examples.

PROPOSITION 2. *$\boldsymbol{T} = \boldsymbol{AG}[A]$ has property* (30), *if*

$$A(n) = \Big(\bigwedge_{k \leqslant n} \big(D(k) \wedge \sim H(k)\big) \to Q^{(1)}(p_{g(n)}, 1, 0)\Big), \qquad n \geqslant 1 .$$

Proof. We start with the observation that, whenever $\boldsymbol{T} \supset \boldsymbol{AG}$, then $\boldsymbol{T}_f \supseteq \boldsymbol{T}[H(m)] \supseteq \boldsymbol{T}[D]$ and $\boldsymbol{T}_f^* \subseteq \bigcap \boldsymbol{T}[D(m)]$, and that $\boldsymbol{T}_f = \boldsymbol{T}[D]$ if and only if $\boldsymbol{T}_f^* = \boldsymbol{T}|\sim D|$, while $\boldsymbol{T}_f^* = \boldsymbol{T}[\sim H]$ if and only if $\boldsymbol{T}_f = \boldsymbol{T}|H|$. This follows by Lemma 1 from the properties of $\boldsymbol{AG}$ exhibited in connection with Proposition 1. For the theory $\boldsymbol{T}$ under consideration here the equalities hold. To show that $\boldsymbol{T}_f \subseteq \boldsymbol{T}[D]$ we note that the latter theory coincides with $\boldsymbol{AG}_f$ extended by the set of sentences $\bigwedge\limits_{k \leqslant n} \big(\sim H(k) \to Q^{(1)}(p_{g(n)}, 1, 0)\big)$. Now assume that $\boldsymbol{T}[D][s] \neq \boldsymbol{L}$. If also $\boldsymbol{T}[D \cup \sim H][s] \neq \boldsymbol{L}$, then certainly

$AG_f[s]$ has a model, in which all sentences $Q^{(1)}(p_{g(n)}, 1, 0)$ hold. But then, since the range of g, being non-recursive, cannot be co-finite, s has, according to Lemma 12 (ii), a finite such model, and thus a finite T-model. On the other hand, if $T[D \cup \sim H][s] = L$, then $\bigvee_{k\leqslant m} H(k) \in T[D][s]$ for some m, but then $T[D][s] = AG[s \wedge \bigwedge_{k<m} A(k) \wedge \bigvee_{k\leqslant m} H(k)]$, since the remaining axioms of $T[D][s]$ are automatically theorems of this latter theory. Now this theory, being a finite extension of AG_f, certainly has finite models. Having thus proved $T_f = T[D]$, from which follows $T_f^* = T|\sim D| \supseteq T[\sim H]$, it remains to show that $T|\sim D| \subseteq T[\sim H]$, which is done in an analogous way. Assume that $T[\sim H][s] \neq L$. If also $T[\sim H \cup D][s] \neq L$, then $AG_f^*[s]$ has a model in which all conclusions of the axioms A hold, and, as above, s has a strictly infinite AG-model, i.e., a model in which some sentence of D fails, which is also a model for T. On the other hand, if $T[\sim H \cup D][s] = L$, there is nothing to prove here, because then some sentence of $\sim D$ is a theorem of $T[\sim H][s]$, and so certainly $T|\sim D|[s] \neq L$. We note that we have here examples where the operations of intersection via co-basis and of extension commute. Since it is clear that the theories over which we take the intersections are finite extensions of AG, almost all axioms of T being vacuously theorems, we immediately obtain the decidability of T_f, T_f^* and hence also of T, or more explicitly, we have

$$T_f = T[D] = T|H| = \bigcap_{m\in N} AG[H(m) \wedge \bigwedge_{n<m} A(n)]\,,$$

and

$$T_f^* = T[\sim H] = T|\sim D| = \bigcap_{m\in N} AG[\sim D(m) \wedge \bigwedge_{n<m} A(n)]\,.$$

Similarly it follows that

$$T_{fi} = \bigcap_{m\in N} AG_i[H(m) \wedge \bigwedge_{n<m} A(n)]\,,$$

and that therefore both T_{fi} and T_i are decidable. However, ∂T is undecidable, since $Q^{(1)}(p_n, 1, 0) \in T$ if and only if $n \in R(g)$. Thus, T is almost as well behaved as AG, i.e., T_f and T_f^* are not only decidable but have absolute co-bases that are coaxiomsystems, and $T_{fi} = T_i \ast T_f^*$. However, ∂T is badly undecidable, in that it has no finite decidable and consistent extensions. On the other hand, of course, $\tau(\mathfrak{Q})$ is still an extension of ∂T.

Proposition 3. *$AG[A]$ has property* (29) *if A is the set of sentences*

$$A(n, k) = \big(Q^{(1)}(p_{g(n)}, 1, n) \to \sim Q^{(1)}(2, 1, k)\big)\,,$$

for all $k \geqslant 0$, $n \geqslant 1$.

Proof. Let $T_1 = AG[\{\sim Q^{(1)}(2, 1, k)\colon k \geqslant 0\}]$ and

$$T_2 = AG[\{\sim Q^{(1)}(p_{g(n)}, 1, n)\colon n \geqslant 1\}].$$

Then it is easily seen from the discussion of Proposition 1 that both these theories are decidable. (Observe, however, that the same could not be said for sequences like $\langle Q^{(1)}(p_{g(n)}, 1, n)\rangle$ or $\langle \sim Q^{(i)}(p_{g(n)}, n, 0)\rangle$). Clearly $T = T_1 \cap T_2$ and is thus likewise decidable. Moreover, $T_f = AG_f T_2$ and $T_{fi} = AG_{fi} T_2$ and so these theories as well as T_i are decidable too. We have here a case where the inclusions $T_f \supset AG_f T$ and $T_f \supset \bigcap T[H(m)]$ are proper, and as a matter of fact T_f, though decidable, does not have a recursive cobasis, i.e., T_f^* is not axiomatisable. For, we have now in addition to the old "axioms of infinity" $\sim D$ the new ones of the form $Q^{(1)}(p_{g(n)}, 1, n)$. Now, since we have chosen $n \geqslant 1$, we see that if m is either not divisible by any prime of the form $p_{g(n)}$ or odd, then $\sim H(m)$ is a consequence in T of every "axiom of infinity". On the other hand, for every n, the group $\mathfrak{C}_2^\omega \times \mathfrak{C}_{p_{g(n)}}$ is a model for $T_f^*[H(2p_{g(n)})]$ and so we find that in particular $\sim H(2p_m)$ is a theorem of T_f^* if and only if $m \notin R(g)$, and thus T_f^* is not axiomatisable. Nevertheless ∂T is decidable, for it coincides with the theory $(\partial AG)T_2$, which is again easily seen to be decidable. Since $T_f^* \subset T[\sim H]$—for, it is easily seen that $\bigcap T[\sim D(m)] = T[\sim H]$—it remains to show that all sentences of $\sim H$ are indeed theorems of ∂T, and it suffices to show this for sentences of the form $H(2m)$ with m divisible by some primes of the form $p_{g(n)}$. Let $p_{g(n_1)}, \ldots$ $\ldots, p_{g(n_r)}$ be all such prime divisors of m. Then the sentence $H(2m) \to$ $\to \bigvee_{i\leqslant r} Q^{(1)}(p_{g(n_i)}, 1, n_i)$ is valid in all strictly infinite models of T, but then, the negation of the conclusion being a theorem of T_f, we find that $\sim H(2m)$ is indeed a theorem of ∂T. It follows that T has the desired properties.

Proposition 4. *$AG[A \cup B]$ has property* (26), *if*

$$A(n) = \big(\sim Q^{(1)}(p_n, 1, 0) \to Q^{(1)}(p_n, 1, 1) \wedge Q^{(2)}(p_n, 1, 1) \wedge Q^{(3)}(p_n, 1, 1)\big)$$

and

$$B(n) = \big(\sim Q^{(1)}(p_n, 1, 0) \to \bigwedge_{j\leqslant n} Q^{(1)}(p_{g(j)}, 1, 0)\big)\,.$$

Proof. Let us denote $AG[A]$ by R. R is then the theory of those Abelian groups that are direct products of torsion free groups and finite groups of squarefree order. If we denote by M the set of squarefree positive integers, we see that $\{H(m)\colon m \in M\}$ is a proper cobasis for R_f; and since every extension $R[H(m)]$ is complete it follows that R_i and R_f^* coincide. Altogether we have

$$R_f = R|H| = AG_f[A] = AG[A \cup D]\,,$$

$$R_f^* = R_i = AG_i[A] = AG[A \cup \sim H] = R|\sim D|$$
$$= \bigcap_{A} R[Q^{(1)}(p, 1, 0) \wedge \sim Q^{(2)}(p, 1, 0)]\,,$$

$$\partial R = R_{fi} = \partial AG[A] = AG[A \cup D \cup \sim H]\,.$$

It is not difficult to see that all these theories, including $\boldsymbol{R}$, are decidable, for $\boldsymbol{R}$ and $\partial\boldsymbol{R}$, one gives an argument showing that if a sentence is a theorem then it is a consequence in $\boldsymbol{AG}$, respectively $\partial\boldsymbol{AG}$, of the first $f\big(\nu(s)\big)$ axioms, and then the rest follows. Now we adjoin the axioms B to $\boldsymbol{R}$ and find first of all that $\boldsymbol{T}_f = \boldsymbol{R}|H'| = \boldsymbol{R}_f[S \cup \sim H'']$, where $H' = \{H(m)\colon m \in M'\}$, with M' the set of squarefree products of primes p_j such that each j lies outside the range of g restricted to the largest of them,

$$H'' = \{H(m)\colon m \notin M'\}, \text{ and } S(k) = \Big(\bigwedge_{m \in M_k \cap M'} \sim H(m) \to \bigwedge_{j \leqslant k} Q^{(1)}(p_{g(j)}, 1, 0)\Big),$$

where M_k is the set of squarefree products of the first k primes. Since the sets of numbers involved are obviously recursive, $\boldsymbol{T}_f$ is decidable. But note that $\boldsymbol{T}_f$ is a proper extension of $\boldsymbol{R}_f[B]$, for, the sentence $\sim S(k)$ is valid in $\mathfrak{Q} \times \mathfrak{C}_{p_{g(k)}}$ which is a model of $\boldsymbol{R}_f[B]$ but not of $\boldsymbol{T}_f$. As to $\boldsymbol{T}_i$, we find that $\boldsymbol{T}_i = \boldsymbol{T}_f^* = \boldsymbol{T}[\sim H'] = \boldsymbol{T}|\sim D \cup \sim S|$, and it is easy to see that the intersection ranges over a sequence of uniformly decidable theories. Thus we have all the required decidabilities and that (26) holds follows from the fact that $\boldsymbol{T}_{fi} = \partial\boldsymbol{T} = \boldsymbol{T}_f[\sim H] = \boldsymbol{T}_f[\{Q^{(1)}(p_{g(n)}, 1, 0)\colon n \in N\}]$. For, indeed, if $m \notin R(g)$, then $\boldsymbol{T}_{fi}[\sim Q^{(1)}(p_m, 1, 0)]$ has the model $\prod_{k \notin R(g)} \mathfrak{C}_{p_k}$, and so $\boldsymbol{T}_{fi}$ is not co-axiomatisable.

Proposition 5. *The theory of finite elementary Abelian p-groups extended by the set of sentences*

$$A(n, m) = \Big(\bigwedge_{j \leqslant n} \sim Q^{(1)}(p_{g(n)}, 1, j) \to \sim Q^{(1)}(p_{g(n)}, 1, n+m)\Big), \quad n \geqslant 1, \quad m \geqslant 1,$$

has property (25).

Proof. Let $\boldsymbol{R}$ be the extension of $\boldsymbol{AG}_f[\sim H(1)]$ by the set B of all sentences $\sim Q^{(1)}(p, 1, 0) \to Q^{(1)}(p, 2, 0) \wedge Q^{(1)}(q, 1, 0)$, where p and q range over all pairs of distinct primes. $\boldsymbol{R}$ is the theory of finite elementary p-groups. We have $\boldsymbol{R} = \boldsymbol{R}_f$, $\boldsymbol{R}_i = \boldsymbol{R}_{fi}$, $\boldsymbol{R}_f^* = \partial\boldsymbol{R} = \boldsymbol{L}$ and $\boldsymbol{R}$ is clearly fully decidable. As a matter of fact, as a set of basic sentences for $\boldsymbol{R}$ one may choose the union of the two sets $H = \{H(p)\colon p \in \Lambda\}$ and $Q = \{Q^{(1)}(p, 1, m)\colon p \in \Lambda, m \geqslant 1\}$, each of which is disjoint, and $\boldsymbol{R}_i$ is the intersection of the complete theories $\boldsymbol{R}_i[H(p)]$. For $\boldsymbol{T} = \boldsymbol{R}[A]$ we obtain the following

$$\boldsymbol{T}_f = \boldsymbol{R}[\{H(p_{g(n)}) \to \bigvee_{j \leqslant n} Q^{(1)}(p_{g(n)}, 1, j)\colon n \geqslant 1\}] = \tau\{\mathfrak{C}_{p_k}^m\colon m \geqslant 1, k \notin R(g \restriction m)\},$$

$$\boldsymbol{T}_i = \boldsymbol{R}_i = \boldsymbol{R}[\{H(p) \to \sim Q^{(1)}(p, 1, n)\colon p \in \Lambda, n \geqslant 0\}] = \tau\{\mathfrak{C}_p^\omega\colon p \in \Lambda\},$$

$$\boldsymbol{T}_f^* = \boldsymbol{T}_i[\{\sim H(p_m)\colon m \notin R(g)\}] = \bigcap_{m \in R(g)} \boldsymbol{T}_i[H(p_m)] = \tau\{\mathfrak{C}_{p_m}^\omega\colon m \in R(g)\},$$

$$\boldsymbol{T}_{fi} = \boldsymbol{T}_i[\{\sim H(p_m)\colon m \in R(g)\}] = \bigcap_{m \notin R(g)} \boldsymbol{T}_i[H(p_m)] = \tau\{\mathfrak{C}_{p_m}^\omega\colon m \notin R(g)\},$$

$$\partial\boldsymbol{T} = \boldsymbol{T}[\sim H \cup \sim Q] = \tau(\mathfrak{Q}).$$

From this it is clear that $\boldsymbol{T}_f$, $\boldsymbol{T}_i$, $\boldsymbol{T}$ and $\partial\boldsymbol{T}$ are decidable, but that $\boldsymbol{T}_f^*$ is not axiomatisable, while $\boldsymbol{T}_{fi}$ is not co-axiomatisable.

We observe that this example gives an illustration of how a degenerate diagram may unfold upon extension. The Lindenbaum algebra for $\boldsymbol{R}$ is atomic with the set Q of atoms, $\boldsymbol{R} = \boldsymbol{R}_f$ and to $\boldsymbol{R}_i = \boldsymbol{R}_{fi}$ corresponds the intersection of the filters generated by the sets $\sim Q_p$. The algebra for $\boldsymbol{T}$ is obtained by "deleting" the atoms $Q^{(1)}(p_{g(n)}, 1, n+m)$, and the set H splits now relative, to $\boldsymbol{T}$ into an r.e. co-basis for $\boldsymbol{T}_f^*$ relative to $\boldsymbol{T}_i$ and a co-r.e. co-basis for $\boldsymbol{T}_f$ which is also a co-basis for $\boldsymbol{T}_{fi}$ relative to $\boldsymbol{T}_i$. Neither of these theories has a recursive co-basis but $\boldsymbol{T}_f$ still has a recursive co-axiomsystem. $\boldsymbol{T}$ is still atomic, and the set of atoms splits into the set $\{Q^{(1)}(p_m, 1, n)\colon m \notin R(g \restriction n)\}$ of diagrams of finite models and the set $\{H(p_{g(n)}) \wedge \bigwedge_{j \leqslant n} \sim Q^{(1)}(p_{g(n)}, 1, j)\colon n \geqslant 1\}$ which is a $\boldsymbol{T}$-co-basis for $\boldsymbol{T}_f^*$. The union $\partial\boldsymbol{T} = \boldsymbol{T}_f\boldsymbol{T}_f^*$ coincides now with the complete and consistent extension obtained by negating all basic sentences, which always plays a special role.

Proposition 6. *The theory of all Abelian groups that are either infinite or of the form $\mathfrak{C}_2^m \times \mathfrak{C}_{p_{g(m)}}$, has property* (23).

Proof. Let us denote this theory by $\boldsymbol{T}$. It is decidable by virtue of Lemma 9, for it is obtained from the extension $\boldsymbol{AG}_i = \boldsymbol{AG}[\sim E]$, where E is the familiar disjoint sequence of sentences $E(n)$ stating that there are exactly $n+1$ elements, by weakening $\sim E$ with the sequence U, where $U(n) = \tau(\mathfrak{C}_2^m \times \mathfrak{C}_{p_{g(m)}})$ if $n = 2^m \cdot p_{g(m)} - 1$, for some m, and $\sim E(n)$ otherwise. It is clear that this sequence is recursive. However, the sentence $Q^{(1)}(p_n, 1, 0)$ is valid in $\boldsymbol{T}_f$ if and only if $n \neq 0$ and n does not belong to the range of g and thus $\boldsymbol{T}_f$ is not axiomatisable. It remains to verify that all the remaining theories are decidable. For $\boldsymbol{T}_i$ this is clear since $\boldsymbol{T}_i = \boldsymbol{AG}_i$. $\boldsymbol{T}_{fi}$, on the other hand, is complete and axiomatisable; in fact, it is obviously the theory of the group $\mathfrak{C}_2^\omega$. Thus, by Corollary 3, we have $\boldsymbol{T}_f^* = \boldsymbol{T}_i$ and of course $\partial\boldsymbol{T} = \boldsymbol{T}_{fi}$, and so these theories are decidable too. We note, however, that $\boldsymbol{T}_f^*$ and with it $\boldsymbol{T}_i$, cannot have a recursive co-basis, but only recursive co-axiomsystems, since otherwise $\boldsymbol{T}_f$ would have to be decidable.

Proposition 7. $\boldsymbol{T} = \boldsymbol{AG}[A]$ *has property* (19) *if*

$$A(n) = \big(H(2n) \to Q^{(1)}(2, 1, g(n))\big), \quad n \geqslant 1.$$

Proof. We observe that $\boldsymbol{T}$ is a recursive weakening of $\boldsymbol{AG}_f^* = \boldsymbol{AG}[\sim H]$ by the sequence B, where $B(2n-1) = H(2n-1)$ and $B(2n) = Q^{(1)}\big((2, 1, g(n)\big)$. Thus $\boldsymbol{T} = \boldsymbol{AG}[\sim H \vee B]$ and $\boldsymbol{T}_i = \boldsymbol{AG}_i[\sim H \vee B]$, and it follows by Lemma 9 that both $\boldsymbol{T}$ and $\boldsymbol{T}_i$ are decidable. By Lemma 10 we have $T_f^* = AG_f^* = T[\sim H]$, so that T_f^* is decidable too and $T_f = T|H|$. Since g is assumed to be one-one it follows that

$$T_f = AG_f[\{H(2n) \leftrightarrow Q^{(1)}(2,1,g(n)):\ n \geqslant 1\} \cup \cup \{\sim Q^{(1)}(2,1,m):\ 1 \leqslant m \notin R(g)\}]$$

and similarly for T_{fi} with AG_f replaced by AG_{fi}. Since the group $\mathfrak{C}_2^{g(n)} \times \times \mathfrak{C}_{2n}^{\omega}$ is a model for T_f as well as for T_{fi}, $\sim Q^{(1)}(2,1,m)$ is a theorem of either theory if and only if $1 \leqslant m \notin R(g)$ and we find that neither theory is axiomatisable. However, $\partial T = AG[\{\sim Q^{(1)}(2,1,m):\ m \geqslant 1\}]$, and is thus decidable, so that T has all the desired properties.

Proposition 8. *$T = AG[B \cup A]$ has property* (11) *if B and A are the sets consisting of all sentences*

$$B(q,k,n,p) = \big(\sim D(q,k,n) \to Q^{(1)}(p,1,0)\big)$$

and

$$A(m) = \big(\bigwedge_{j \leqslant m} \sim E(j) \to Q^{(1)}(p_{g(m)},1,0)\big), \qquad m \geqslant 1.$$

Proof. This then is an example of a decidable theory for which both T_i and T_{fi} are undecidable, while ∂T, T_f and T_f^* are decidable. In fact, the latter two extensions will turn out to be strongly decidable in the sense that they have recursive T-co-bases. Let us denote the sets $\{Q^{(1)}(p_{g(m)},1,0):\ m \geqslant 1\}$ and $\{Q^{(1)}(p,1,0):\ p \in A\}$ by Q_g and Q_1 respectively. From Lemma 12 (ii) it follows that, whenever S is a co-infinite subset of Q_1, then $(AG[S])_f = AG_f[S]$; so that in particular $(AG[Q_g])_f = AG_f[Q_g]$. On the other hand, $(AG[B])_f = AG_f$ and $(AG[B][Q_g])_f = (AG[Q_g])_f$ because $AG[B] \subset AG_f$. Thus $(AG[B][Q_g])_f = (AG[B])_f[Q_g]$ and, since the premises of the sentences $A(m)$ obviously form a decreasing sequence, we can apply 10(i) to $AG[B] \subset AG[B][A] < AG[B][Q_g]$ so as to obtain $(AG[B][A])_f = (AG[B])_f[A]$, i.e., $T_f = AG_f[A] = T[D]$ and $T_f^* = T|\sim D|$. Because of our choice of B it follows immediately that $T_f^* \supseteq T[\sim E(0), Q_1]$, and, recalling what the sentences D are, one sees that $T_f[\sim E(0), Q_1] = AG[\sim E(0), Q_0]$ and is thus complete. Now Corollary 3 shows that $T_f^* = T[\sim E(0), Q_1]$. Therefore T_f and T_f^* are both axiomatisable and have recursive T-co-bases. It should be observed however, that from this alone does not follow automatically their decidability, since the decidability of T has not yet been established. And indeed, if one represents T_f as the intersection of the theories $T[\sim Q^{(1)}(p,1,0)]$ and $T[E(0)]$, one does use, as will follow from the decidability of T, a sequence of uniformly decidable theories, but not obviously so. Note that the set of primes for which the corresponding theory is a finite extension of AG is not recursive. However, we can use the facts that $T_f = \bigcap T[E(n)]$, and that $T[E(n)] = AG[E(n) \wedge \bigwedge_{j \leqslant n} Q^{(1)}(p_{g(n)},1,0)]$, from which the co-axiomatisability of T_f follows. As to T_f^*, the co-basis itself obviously yields a co-axiomsystem because $T[\sim D(p,k,0)] = AG[Q_1, \sim Q^{(2)}(p,k,0)]$ and $T[\sim D(p,k,n)] = AG[Q_1, Q^{(2)}(q,k,n)]$, for $n \geqslant 1$, and the theory $AG[Q_1]$ is decidable according to earlier remarks. Having thus established the decidability of T_f and T_f^* we finally have arrived at the decidability of T, and recall that ∂T, being the additive theory of the rationals, is decidable as well. It remains to show that neither T_i nor T_{fi} is decidable. But it is clear that the axioms A are so chosen that $Q^{(1)}(p_n,1,0)$ is a theorem of T_i and so also of T_{fi} whenever $n \in R(g)$, while $\mathfrak{C}_{p_n}^{\omega}$ is a model for T_{fi} and thus also for T_i whenever $n \notin R(g)$. In fact $T_{fi} = AG_{fi}[Q_g]$.

Proposition 9. *The theory T of all groups of the form $\mathfrak{C}_2^n \times \mathfrak{C}_3^{g(n)}$ or $\mathfrak{C}_2^{\omega} \times \mathfrak{C}_3^m$, with $0 \neq m \notin R(g)$ and $n \geqslant 1$, has property* (5).

Proof. Let $R = AG[H(6)]$, $T = R[\langle Q^{(1)}(2,1,n+1) \leftrightarrow Q^{(1)}(3,1,g(n+1))\rangle]$, and set $Q_2 = \{Q^{(1)}(2,1,n):\ n \geqslant 1\}$ and $Q_3 = \{Q^{(1)}(3,1,n):\ n \geqslant 1\}$. It is clear that $\sim Q^{(1)}(3,1,k)$ is a theorem of T_i if and only if $k \in R(g)$ and a theorem of T_f if and only if $k \notin R(g)$, so that T_i is not co-axiomatisable while T_f is not axiomatisable and $T[\sim Q_3] \subseteq T_{fi}$. But $T[\sim Q_3]$ coincides with the theory $R[\sim Q_2 \cup \sim Q_3]$, which is the theory of the group $\mathfrak{C}_2^{\omega} \times \mathfrak{C}_3^{\omega}$. Therefore $T_{fi} = T[\sim Q_3]$ and T_{fi} is complete and decidable. From this we immediately obtain $T_f^* = T_i$ and $\partial T = T_{fi}$, as well as $T = \bigcap T[Q^{(1)}(3,1,k)]$, where k ranges over all positive integers. We note that the set Q_3 splits into two sets, one of which is an r.e. co-basis for T_f while the other is a co-r.e. but not r.e. co-basis for T_f^*. Nevertheless this set yields a co-axiomsystem for T, because each of the extensions of T by a sentence of Q_3 is complete and hence decidable, thus establishing the decidability of T which is all we needed to complete our proof. In fact, each theory $T[Q(3,1,k)]$ coincides with the extension of $R[Q(3,1,k)]$ by the recursive sequence A, where $A(n)$ is $Q^{(1)}(2,1,n)$ or $\sim Q^{(1)}(2,1,n)$ according as $k = g(n)$ or not. We observe that those theories for which $k \in R(g)$ are finitely axiomatisable. Again, although the set of finitely axiomatisable ones among them is not recursive, they are uniformly axiomatisable and we have a situation as described in the discussion of Definition 5. This then, is an example of a theory which is decidable despite the fact that neither T_f nor T_f^* nor T_i is decidable.

In view of Lemma 11 this finally completes the proof of Theorem 2 and with it settles the question that motivated this paper. It seems in order to mention at this point that this article can be viewed from two, so to speak, polar, vantage points. Its purpose may be seen in the two theorems, and these then give justification to the general machinery introduced. On the other hand, one may well find the main interest and content in the lemmas and generalities and would, in this case, consider the theorems and the propositions that lead up to them as illustrations. As a matter of fact, it would be quite feasible to find ad hoc proofs of a model theoretic nature for the propositions, if those were the exclusive aim. But, as mentioned before, we were here principally interested in shedding some light on the recursive structure of the Lindenbaum algebra of a decidable theory. It should be observed that, as long as our examples are viewed as extensions of $\boldsymbol{AG}$, they can indeed be viewed in this light since the set of finite complete extensions of $\boldsymbol{AG}$ coincides with the set of theories of individual finite Abelian groups, and so the theories $\boldsymbol{T}_f$, $\boldsymbol{T}_i$ etc. are "canonically" associated with $\boldsymbol{T}$ via the Lindenbaum algebra of $\boldsymbol{AG}$. However, if they are taken out of context, they must be seen as examples of Lindenbaum algebras with a distinguished recursive set of generators.

§ 6. Remarks. In this investigation we have been concerned with theories $\boldsymbol{T}$ whose language $\boldsymbol{L}$ contains equality as a logical constant, so that the extension $\boldsymbol{T}_i = \boldsymbol{T}[\sim E]$ is canonically associated with $\boldsymbol{T}$. Any theory $\boldsymbol{T}$ belonging to a language without equality coincides with its theory of infinite models, and $\boldsymbol{T}_f$ is then the intersection of all finitely axiomatisable complete extensions of $\boldsymbol{T}$ that have finite models. The diagram shrinks to $(\boldsymbol{T}, \boldsymbol{T}_f, \boldsymbol{T}_f^*, \partial\boldsymbol{T})$.

For languages with, as well as without, equality it may be of some interest to analyse the relations in the diagram that is obtained by replacing E by the set C of all complete sentences, i.e., of sentences s such that the theory $[s]$ is complete and consistent. To our knowledge it is not known whether, for a fixed language—e.g., the language $\boldsymbol{L}$ with equality and one binary predicate symbol—the set C is recursively enumerable. Therefore we do not know whether Corollary 4 of § 2 carries over to $\boldsymbol{T}_c = \bigcap_C \boldsymbol{T}[s]$, the theory of all C-models of $\boldsymbol{T}$, and hence whether the analogue of Theorem 1 (ii) holds. In view of this and of the remarks following Corollary 4, as well as in view of the fact that there exists a finitely axiomatisable consistent and decidable theory, the consistency of which can not be proved in arithmetic (cf. 5, Corollary 3.6), we raise the obviously related questions.

PROBLEM 4. *Is the set C of complete and consistent sentences of L recursively enumerable?*

PROBLEM 4′. *Does there exist a finitely axiomatisable complete and consistent theory whose consistency cannot be proved in Peano's arithmetic?*

If, for a fixed theory $\boldsymbol{T}$, K is the set of sentences s such that $\boldsymbol{T}[s]$ is complete, then the theories $\boldsymbol{T}[\sim K]$ and $\boldsymbol{T}_k = \bigcap_K \boldsymbol{T}[s]$ and with them the further three extensions, are canonically associated with the Lindenbaum algebra of $\boldsymbol{T}$. In case $\boldsymbol{T}$ is finitely axiomatisable, $\boldsymbol{T}_k$ of course coincides with $\boldsymbol{T}_c$, so that an analysis of the pair $\boldsymbol{T}[\sim C]$, $\boldsymbol{T}_c$, is of special interest for finitely axiomatisable theories $\boldsymbol{T}$.

We repeat the question whether Theorem 1 can be strengthened for finitely axiomatisable theories $\boldsymbol{T}$; in particular,

PROBLEM 5. *Does there exist a finitely axiomatisable decidable theory for which the theory of infinite models is undecidable?*

PROBLEM 6. *Does there exist a finitely axiomatisable decidable theory for which the theory of finite models is undecidable?*

We observe that a negative answer to either problem would lead to a weak form of hereditary undecidability. More precisely, if Problem 6 has a negative answer (for Problem 5 the situation is completely analogous), and $\boldsymbol{T}$ is undecidable and is the theory of finite models of a finitely axiomatisable theory, then every finitely axiomatisable subtheory of $\boldsymbol{T}$ is undecidable, since every subtheory of $\boldsymbol{T}$ must have an undecidable theory of finite models. On the other hand, if every undecidable theory of the form $[a]_f$ were hereditarily undecidable in the usual stronger sense of the word, then it would, of course, follow that the answer to Problem 6 is negative. This hereditary undecidability would certainly be a very strong property. Still, it is a remarkable fact (cf. [2]), that every known undecidability proof of a theory of the form $[a]_f$ for an undecidable theory $[a]$ leads even to a proof of the recursive inseparability of the set of $[a]$-finitely refutable sentences from the set of logically valid sentences, so that it may not be unreasonable to consider the possibility that every undecidable theory of the form $[a]_f$ is at least hereditarily undecidable under the further assumption that $[a]$ be undecidable.

PROBLEM 7. *Does there exist a finitely axiomatisable theory $\boldsymbol{T}$ with an undecidable theory $\boldsymbol{T}_f$ of finite models, and a decidable theory $\boldsymbol{R}$ such that $\boldsymbol{T} \subset \boldsymbol{R} \subset \boldsymbol{T}_f$?*

Note that these problems are not directly amenable to the methods of [5], for a transformation, as in [5], of an axiomatisable theory into a finitely axiomatisable one does not preserve complete sentences, and an isomorphism of theories as in [5] need not preserve the sentences $E(n)$.

Finally we recall that there are finitely axiomatisable undecidable theories $\boldsymbol{T}$ with non-trivial decidable theory $\boldsymbol{T}_f$. An example is the theory of Abelian cancellation semigroups, as shown in [7]. In view of Corollary 6 the analogous situation for $\boldsymbol{T}_i$ is impossible; however, it is easy to construct an infinitely axiomatisable undecidable theory $\boldsymbol{T}$ with decidable theory $\boldsymbol{T}_i$ by adjoining to a decidable theory that has models of all finite cardinalities the set of all sentences $\sim E(g(n))$ where g is a recursive function with non-recursive range.

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On a class of subalgebras of $C(X)$ with applications to $\beta X \setminus X$.

by

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W. Rudin has proved that, assuming the continuum hypothesis, $\beta\mathbf{N}\setminus\mathbf{N}$ has a dense subset of 2^c P-points. A similar theorem of N. J. Fine and L. Gillman states that, assuming the continuum hypothesis, $\beta\mathbf{R}\setminus\mathbf{R}$ has a dense subset of remote points in $\beta\mathbf{R}$. It is the purpose of this paper to unify these results by giving a more general method of finding such points.

Specifically, for a completely regular space X, we define a class of subalgebras of $C(X)$ called *β-subalgebras*. Examples of β-subalgebras include $C(X)$ itself and $C^*(X)$. With each β-subalgebra A of $C(X)$ we associate a (possibly empty) set of points in $\beta X \setminus X$ called *A-points*. We show that, under the continuum hypothesis and with reasonable restrictions on A and X, $\beta X\setminus X$ has a dense subset of 2^c A-points. The Rudin theorem is then obtained by observing that the P-points of $\beta\mathbf{N}\setminus\mathbf{N}$ are precisely the $C^*(\mathbf{N})$-points, and the Fine-Gillman theorem follows from the fact that the remote points in $\beta\mathbf{R}$ are precisely the $C(\mathbf{R})$-points.

Our method considerably simplifies the Fine–Gillman proof of the existence of remote points in $\beta\mathbf{R}$ but does not have the power of their method. Using their method, we show the existence of remote points in $\beta\mathbf{R}$ which are not P-points of $\beta\mathbf{R}\setminus\mathbf{R}$. We conclude by investigating a β-subalgebra H of $C(\mathbf{N})$ previously studied by R. M. Brooks. We correct Brooks's characterization of the maximal ideals in H and show that his characterization holds precisely for the ideals M^p where p is a P-point of $\beta\mathbf{N}\setminus\mathbf{N}$ (equivalently, where p is an H-point).

1. Preliminaries. The basic reference for this paper will be the Gillman and Jerison text [5]; the terminology and notation will, with only a few exceptions, be that of [5].

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