# A note on irreducible separation

by

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**1.** It is well known that an irreducible cutting of a connected metric space between two of its points must be a closed set. The corresponding result for general topological spaces is only a little more complicated and is easy to prove, as this note will show. The fact that it is not well known is hard to account for. Perhaps it is not quite a simple enough result to be of much use, or perhaps it is just hard to guess what the theorem should be.

Our procedure will be to prove a somewhat more general theorem, obtaining the proposition we have just been talking about as an obvious special case.

**2.** Let us say that $M$ *divides* topological space $X$ between two points $a$ and $b$ provided that $X-M$ is the union of two separated sets one containing $a$ and the other $b$. Let us say that $M$ is an *irreducible separating set* (i.s.s.) (of $X$ between $a$ and $b$) provided that $M$ divides $X$ between $a$ and $b$ but that no proper subset of $M$ divides $X$ between $a$ and $b$. (Examples: the empty set is an i.s.s. of any disconnected space; any irreducible cutting of a connected space $X$ is an i.s.s. of $X$; any separating point (in the sense of Whyburn [2]) of a space $X$ is an i.s.s. of $X$.)

**3.** The following lemma generalizes the well-known fact ([1], p. 23) that each point of an irreducible cutting of a connected $T_1$ space is a limit point of both „halves" of its complement.

LEMMA. *If $M$ is an irreducible separating set of topological space $X$ between $a$ and $b$ so that $X-M$ is the union of two separated sets $H$ and $K$ with $a \in H$ and $b \in K$, and if $p \in M$, then either $p \in \overline{H} \cap \overline{K}$ and $\{p\}$ has no limit point outside $M$, or $p$ is in the exterior of both $H$ and $K$ and $\{p\}$ has a limit point in each of $H$ and $K$.*

This lemma is easily proved by using the following two obvious remarks in alternation.

3.1. *If $H$ and $K$ are two separated subsets of topological space $X$, then no point $p$ can be both a limit point of $K$ and such that $\{p\}$ has a limit point in $H$.*

3.2. *If $M$ is an* i.s.s. *of topological space $X$ between points $a$ and $b$, and if $H$ is a clopen subset of $X-M$ such that $a \in H$ but $b \notin H$, then each point $p$ of $M$ either is a limit point of $H$ or is such that $\{p\}$ has a limit point in $H$.*

**4.** THEOREM. *Any irreducible separating set $M$ of a topological space $X$ is the union of two disjoint sets* (*not always non-empty*) *one of which is open and the other closed.*

Proof. The closed set is $\bar{H} \cap \bar{K}$. For a point $p$ in $M-(\bar{H} \cap \bar{K})$, the lemma guarantees a limit point of $\{p\}$ in $H$ and another in $K$. Then the intersection $W$ of suitable neighborhoods of each satisfies

$$p \in W \subset M-(\bar{H} \cap \bar{K}) .$$

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**References**

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[2] G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Pub. 28, Providence, R. I., 1942.

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# Prime $z$-ideal structure of $C(\mathbf{R})$

by

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**Introduction.** In the study of the ideal structure of the ring $C(X)$ of all real-valued continuous functions on a topological space $X$, a special role is played by the class of $z$-ideals. A *$z$-ideal* is an ideal that is maximal with respect to the sets of zeros of its members. Maximal ideals are $z$-ideals, and every $z$-ideal is an intersection of prime ideals. These and other basic facts concerning the algebraic structure of $C(X)$ are found in the Gillman and Jerison text *Rings of Continuous Functions* [GJ]. Early results concerning the *prime* ideal structure of $C(X)$ are summarized and extended in [GJ], Chapter 14. For example, a prime ideal is contained in a unique maximal ideal; in fact, the prime ideals containing a given prime ideal form a chain. Later results are found in [GJ$_1$], [FG], and [K].

Of special interest is the family of *prime $z$-ideals*. For example, minimal prime ideals are $z$-ideals, and prime $z$-ideals have interesting connections with the topology of the space. In the case of a completely regular Hausdorff space $X$, prime $z$-ideals in $C(X)$ are related to convergence problems in the Stone–Čech compactification $\beta X$.

In this work we consider the real line $\mathbf{R}$ and examine the prime $z$-ideal structure of $C(\mathbf{R})$. However, some basic results are obtained in Part I for a completely regular Hausdorff space. The main results for the real line are obtained in Part II. The main effort is directed toward the determination of the order types of chains of prime $z$-ideals in $C(\mathbf{R})$.

We find that the prime $z$-ideal structure contrasts greatly with the prime ideal structure. For example, although every maximal chain of prime ideals, of cardinal greater than one, has cardinal at least $2^{\aleph_1}$ and contains $\eta_1$-sets, we shall demonstrate, assuming the continuum hypothesis, the existence of maximal chains of prime $z$-ideals in $C(\mathbf{R})$ of all cardinals $\mathfrak{m}$ with $1 \leqslant \mathfrak{m} \leqslant \aleph_0$. In fact, we characterize all countable decreasing well-ordered maximal chains; they exist precisely for all countable ordinals

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