If we now define, for $I \subset N$ containing 1, $L^{(I)}$ as the product of all lattices in L_I, each taken as many times as it appears as a factor of L, then we get clearly

$$L = \underset{I}{\mathrm{P}} L^{(I)}$$

and this proves theorem I.

Remark. If in this proof one replaces the word "lattice" by "quasilattice", and do the same in the statement of the theorem, then we get a representation theorem for distributive n-quasilattices. However, the following characterization of distributive n-quasilattices seems to be simpler:

THEOREM II. *An algebra* $\mathfrak{A} = (X; o_1, \ldots, o_n)$, $n \geqslant 2$, *is a distributive n-quasilattice if and only if it is the sum of a direct system of distributive n-lattices.*

(For the definition of the sum of direct systems of algebras, see [2].)

Proof. The sufficiency is nearly trivial (cf. theorem 3 of [1]). The necessity follows from theorem 3 of [2], as the operation $f_{1,2,\ldots,n}(x, y)$ satisfies the conditions characterizing the partition functions, which follows from (1), lemma 2, (vi), lemmas 3 and 4 and (v).

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Some remarks on sums of direct systems of algebras

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0. Introduction. In this paper we give some additional remarks concerning the notion of a sum of direct system (with the least upper bound property) of abstract algebras defined in [1]. At first we recall the following definition:

Let $\mathcal{A}$ be a direct system of abstract algebras of a fixed similarity type without nullary fundamental operations, indexed by elements of a partially ordered set I, the ordering relation of which has the least upper bound property. Moreover, we assume (which is not an essential restriction) that the carriers of the algebras $\mathfrak{A}_i$ ($i \in I$) of this system are mutually disjoint. The sum $S(\mathcal{A})$ of the system $\mathcal{A}$ is an abstract algebra of the same similarity type as the algebras $\mathfrak{A}_i$, the carrier of which is the *sum* of the carriers A_i of all algebras of the system $\mathcal{A}$ and whose fundamental operations are defined by

$$F_t(x_1, \ldots, x_n) = F_t\big(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_n,i_0}(x_n)\big),$$

where $x_1 \in A_{i_1}, \ldots, x_n \in A_{i_n}$, $i_0 = \text{l.u.b.}\,(i_1, \ldots, i_n)$, $\{F_t\}$ is the set of fundamental operations of the algebras in the system $\mathcal{A}$, and φ_{ij} are the canonical homomorphisms of $\mathcal{A}$.

Let us also recall the definition of a P-function (partition function) of a given abstract algebra $\mathfrak{A} = (A, F)$ without nullary fundamental operations.

A mapping $f\colon A^2 \to A$ is called a *P-function* if it satisfies the following conditions:

(1) $$f(x, x) = x,$$

(2) $$f\big(x, f(y, z)\big) = f\big(x, f(z, y)\big),$$

(3) $$f\big(f(x, y), z\big) = f\big(x, f(y, z)\big),$$

(4) $$f\big(F(x_1, \ldots, x_n), y\big) = F\big(f(x_1, y), \ldots, f(x_n, y)\big),$$

(5) $$f\big(F(x_1, \ldots, x_n), x_k\big) = F(x_1, \ldots, x_n) \quad (1 \leqslant k \leqslant n),$$

$$(6)\qquad f\big(y, F(x_1, \ldots, x_n)\big) = f\big(y, F\big(f(y, x_1), \ldots, f(y, x_n)\big)\big),$$

$$(7)\qquad f\big(y, F(y, y, \ldots, y)\big) = y,$$

where $F(x_1, \ldots, x_n)$ is an arbitrary n-ary fundamental operation of $\mathfrak{A}$.

The connections between the notion of a sum of a direct system of algebras and the notion of a P-function are formulated in theorem II in [1], where it is shown that to every P-function f of an algebra there corresponds a representation of this algebra as a sum of a direct system of algebras, in which $f(x, y) = x$, and conversely.

1. Now we shall give some more properties of the sum of a direct system of algebras and of P-functions.

LEMMA 1. *An operation $f(x, y) = x\circ y$ is a P-function of the algebra $\mathfrak{A} = (X; (F_t)_{t\in T})$ if and if it satisfies the formulas* (1)-(3) *and the formulas*

$$(8)\qquad F_t(x_1, \ldots, x_n)\circ y = F_t(x_1, \ldots, x_{k-1}, x_k\circ y, x_{k+1}, \ldots, x_n) \qquad (k = 1, 2, \ldots, n)$$

and

$$(9)\qquad x_1\circ x_2\circ \ldots\circ x_n\circ F_t(x_1, \ldots, x_n) = x_1\circ x_2\circ \ldots\circ x_n.$$

Proof. Sufficiency. From (8) and (9) we obtain the following equalities:

$$F_t(x_1, \ldots, x_n)\circ y = F_t(x_1, \ldots, x_n)\circ \underbrace{y\circ y\circ \ldots\circ y}_{n \text{ times}} = F_t(x_1\circ y, \ldots, x_n\circ y),$$

$$F_t(x_1, \ldots, x_n)\circ x_k = F_t(x_1, \ldots, x_{k-1}, x_k\circ x_k, x_{k+1}, \ldots, x_n) = F_t(x_1, \ldots, x_n),$$

$$\begin{aligned} y\circ F_t(x_1, \ldots, x_n) &= y\circ F_t(x_1, \ldots, x_n)\circ x_1\circ \ldots\circ x_n \\ &= y\circ x_1\circ \ldots\circ x_n\circ F_t(x_1, \ldots, x_n) = y\circ x_1\circ \ldots\circ x_n \\ &= (y\circ x_1)\circ (y\circ x_2)\circ \ldots\circ (y\circ x_n) \\ &= (y\circ x_1)\circ \ldots\circ (y\circ x_n)\circ F_t(y\circ x_1, \ldots, y\circ x_n) \\ &= y\circ F_t(y\circ x_1, \ldots, y\circ x_n)\circ x_1\circ \ldots\circ x_n \\ &= y\circ F_t(y\circ x_1, \ldots, y\circ x_n), \end{aligned}$$

which prove (4), (5) and (6). Equality (7) follows from (1) and (9) by identification of variables.

Nesessity. By (4) and (5) we have

$$\begin{aligned} F_t(x_1, \ldots, x_{k-1}, x_k\circ y, x_{k+1}, \ldots, x_n) &= F_t(x_1, \ldots, x_{k-1}, x_k\circ y, x_{k+1}, \ldots, x_n)\circ (x_k\circ y) \\ &= F_t(x_1\circ x_k\circ y, x_2\circ x_k\circ y, \ldots, x_n\circ x_k\circ y) \\ &= F_t(x_1, \ldots, x_n)\circ x_k\circ y = F_t(x_1, \ldots, x_n)\circ y \end{aligned}$$

proving (8), and similarly, using (1), (2), (3), (6) and (7) we get

$$\begin{aligned} x_1\circ \ldots\circ x_n\circ F_t(x_1, \ldots, x_n) &= x_1\circ \ldots\circ x_n\circ F_t(x_1\circ \ldots x_n, \ldots, x_1\circ \ldots\circ x_n) \\ &= x_1\circ \ldots\circ x_n, \end{aligned}$$

which proves (9).

THEOREM I. *If in an algebra $\mathfrak{A} = \big(X; (F_t)_{t\in T}\big)$ there exist two P-functions $x\circ_1 y$ and $x\circ_2 y$ both of them being terms in two variables in $\mathfrak{A}$, then $x\circ_1 y = x\circ_2 y$ for all $x, y\in X$.*

Note that the trivial P-function $f(x, y) = x$ does not, in general, satisfy the assumptions of this theorem. If it, however, does, as in the case of groups, where $x = xyy^{-1}$, then this theorem shows that then every algebraic P-function is trivial (and from Theorem 1 from [1] it follows that then even every P-function is trivial).

Proof. Formula (8) of Lemma 1 implies that for every term F in two variables of $\mathfrak{A}$ equation (5) is satisfied, and so, under our assumptions we get

$$(x\circ_1 y)\circ_2 y = x\circ_1 y \quad\text{ and }\quad (x\circ_2 y)\circ_1 y = x\circ_2 y$$

but by (1) and (4) left-hand sides of the two last equalities are equal, and the equality $x\circ_1 y = x\circ_2 y$ results.

LEMMA 2. *If $x\circ_1 y$ and $x\circ_2 y$ satisfy* (1)-(3) *and are mutually distributive (on the both sides), then the following equalities hold:*

$$x\circ_1 (x\circ_2 y)\circ_1 (x\circ_2 y\circ_2 z) = x\circ_1 (x\circ_2 y\circ_2 z),$$

$$x\circ_1 (x\circ_2 z)\circ_1 (x\circ_2 y\circ_2 z) = x\circ_1 (x\circ_2 y\circ_2 z).$$

Proof. We prove the first equality, the proof of the second being analogous. We have $(x\circ_1 y)\circ_2 (x\circ_1 z) = x\circ_1 (y\circ_2 z)$ and $(x\circ_1 y)\circ_2 (x\circ_1 z) = x\circ_1 (x\circ_2 z)\circ_1 (y\circ_2 x)\circ_1 (y\circ_2 z)$, hence

$$x\circ_1 (y\circ_2 z) = x\circ_1 (x\circ_2 z)\circ_1 (y\circ_2 x)\circ_1 (y\circ_2 z).$$

Putting in the last equality $x\circ_2 y$ in the place of y we obtain

$$x\circ_1 (x\circ_2 y\circ_2 z) = x\circ_1 (x\circ_2 y)\circ_1 (x\circ_2 z)\circ_1 (x\circ_2 y\circ_2 z).$$

This formula in turn implies

$$\begin{aligned} x\circ_1 (x\circ_2 y)\circ_1 (x\circ_2 y\circ_2 z) &= x\circ_1 (x\circ_2 y\circ_2 z)\circ_1 (x\circ_2 y) \\ &= \big(x\circ_1 (x\circ_2 y)\circ_1 (x\circ_2 z)\circ_1 (x\circ_2 y\circ_2 z)\big)\circ_1 (x\circ_2 y) \\ &= x\circ_1 (x\circ_2 y)\circ_1 (x\circ_2 z)\circ_1 (x\circ_2 y\circ_2 z) = x\circ_1 (x\circ_2 y\circ_2 z), \end{aligned}$$

as needed.

LEMMA 3. *If $x\circ y$ is a P-function of the algebra $\mathfrak{A} = \big(X; (F_t)_{t\in T}\big)$, then*

$$F_t(x_1, \ldots, x_n)\circ F_t(y_1, \ldots, y_n) = F_t(x_1\circ y_1, \ldots, x_n\circ y_n).$$

Proof. By (1), (2), (5), (8) and (9) we have

$$\begin{aligned} F_t(x_1, \dots, x_n) \circ F_t(y_1, \dots, y_n) &= F_t(x_1, \dots, x_n) \circ F_t(y_1, \dots, y_n) \circ y_1 \circ y_2 \circ \dots \circ y_n \\ &= F_t(x_1, \dots, x_n) \circ y_1 \circ \dots \circ y_n \circ F_t(y_1, \dots, y_n) \\ &= F_t(x_1, \dots, x_n) \circ y_1 \circ \dots \circ y_n \\ &= F_t(x_1 \circ y_1, \dots, x_n \circ y_n) . \end{aligned}$$

THEOREM II. *Let $\mathfrak{A} = \big(X; (F_t)_{t \in T}\big)$ and let $T = T_1 \cup T_2$ $(T_1, T_2 \neq \emptyset)$. If $x \circ_1 y$ and $x \circ_2 y$ are two operations defined in X which satisfy equalities (1)-(4) and (6), are mutually distributive and the operation $\circ_i$ satisfies conditions (5) and (7) for $F = F_t$ with $t \in T_i$ $(i = 1, 2)$, then the operation $x \circ y = x \circ_1 (x \circ_2 y)$ is a P-function for $\mathfrak{A}$.*

Proof. Condition (1) is trivially satisfied. We shall prove now that (3) is also true. In fact, we have

$$\begin{aligned} (x \circ y) \circ z &= \big(x \circ_1 (x \circ_2 y)\big) \circ_1 \Big(\big(x \circ_1 (x \circ_2 y)\big) \circ_2 z\Big) \\ &= x \circ_1 (x \circ_2 y) \circ_1 (x \circ_2 z) \circ_1 (x \circ_2 y \circ_2 z) = x \circ_1 (x \circ_2 y \circ_2 z) , \end{aligned}$$

hence we get, using Lemma 2

(10) $$(x \circ y) \circ z = x \circ_1 (x \circ_2 y \circ_2 z)$$

and

$$x \circ (y \circ z) = x \circ_1 \Big(x \circ_2 \big(y \circ_1 (y \circ_2 z)\big)\Big) = x \circ_1 (x \circ_2 y) \circ_1 (x \circ_2 y \circ_2 z) = x \circ_1 (x \circ_2 y \circ_2 z),$$

whence

(11) $$x \circ (y \circ z) = x \circ_1 (x \circ_2 y \circ_2 z) .$$

From (10) and (11) condition (3) follows immediately and, moreover, as $\circ_2$ satisfies (2), it follows that $\circ$ satisfies (2) as well.

We turn now to condition (4). Let $t \in T_1$. In this case we get

$$\begin{aligned} F_t(x_1, \dots, x_n) \circ_1 \big(F_t(x_1, \dots, x_n) \circ_2 y\big) &= F_t(x_1, \dots, x_n) \circ_1 F_t(x_1 \circ_2 y, \dots, x_n \circ_2 y) \\ &= F_t\big(x_1 \circ_1 (x_1 \circ_2 y), \dots, x_n \circ_1 (x_n \circ_2 y)\big) \\ &= F_t(x_1 \circ y, \dots, x_n \circ y) \end{aligned}$$

(by Lemma 3, which has to be applied to the algebra $\big(X; (F_t)_{t \in T_1}\big)$).

In the case of $t \in T_2$ the proof is similar.

To prove (5), take $t \in T_1$ (when $t \in T_2$, the proof follows the same lines) and observe that

$$\begin{aligned} F_t(x_1, \dots, x_n) \circ_1 \big(F_t(x_1, \dots, x_n) \circ_2 x_k\big) &= F_t(x_1, \dots, x_n) \circ_2 \big(F_t(x_1, \dots, x_n) \circ_1 x_k\big) \\ &= F_t(x_1, \dots, x_n) \circ_2 F_t(x_1, \dots, x_n) = F_t(x_1, \dots, x_n) . \end{aligned}$$

Moreover (still under assumption $t \in T_1$), we have

$$\begin{aligned} y \circ F_t(x_1, \dots, x_n) &= y \circ_1 \big(y \circ_2 F_t(x_1, \dots, x_n)\big) \\ &= y \circ_1 \big(y \circ_2 F_t(y \circ_2 x_1, \dots, y \circ_2 x_n)\big) \\ &= y \circ_2 \big(y \circ_1 F_t(y \circ_2 x_1, \dots, y \circ_2 x_n)\big) \\ &= y \circ_2 \Big(y \circ_1 F_t\big(y \circ_1 (y \circ_2 x_1), \dots, y \circ_1 (y \circ_2 x_n)\big)\Big) \\ &= y \circ_1 \big(y \circ_2 F_t(y \circ x_1, \dots, y \circ x_n)\big) = y \circ F_t(y \circ x_1, \dots, y \circ x_n), \end{aligned}$$

thus proving (6). The proof for $t \in T_2$ is similar.

Finally we prove (7) and once more we restrict ourselves to the case $t \in T_1$. We have then

$$\begin{aligned} y \circ F_t(y, \dots, y) &= y \circ_1 \big(y \circ_2 F_t(y, \dots, y)\big) \\ &= y \circ_2 \big(y \circ_1 F_t(y, \dots, y)\big) = y \circ_2 y = y , \end{aligned}$$

and so the theorem is proved.

Marczewski [2] defined the set $S(\mathfrak{A})$, for a given algebra $\mathfrak{A}$, as the set of all natural numbers n for which there exist algberaic operations in $\mathfrak{A}$, depending on exactly n variables.

We prove now

THEOREM III. *If $\mathcal{A}$ is a non-trivial (i.e. consisting of at least two algebras) direct system of algebras without nullary fundamental operations and of the same similarity class, then either $S\big(S(\mathcal{A})\big) = \{1\}$ or $S\big(S(\mathcal{A})\big) = \{1, 2, 3, \dots\}$.*

Proof. We shall prove that in $S(\mathcal{A})$ every term defines an algebraic operation which depends on every variable occurring explicitly in this term. In fact, let I be the partially ordered set of indices of algebras occurring in the system $\mathcal{A}$. As $|I| \neq 1$, we can find $i \neq j$ in I such that $i \leqslant j$. If now $f(x)$ is a term with one variable and $f'(x)$ is the algebraic operation in the algebras in $\mathcal{A}$ defined by $f(x)$, then for $a \in A_i$ we have $f'(a) \in A_i$ and for $b \in A_j$ we have $f'(b) \in A_j$, hence $f'(a) \neq f'(b)$, as the carriers of algebras in $\mathcal{A}$ are disjoint, and so $f'(x)$ is not constant.

If $f(x_1, \dots, x_n)$ is a term with n variables, and $f'(x_1, \dots, x_n)$ is the algebraic operation defined by f, then for $k = 1, 2, \dots, n$ we have for $a \in A_i$, $b \in A_j$

$$f'(a, a, \dots, a) \in A_i , \qquad f'(\underbrace{a, \dots, a}_{k-1 \text{ times}}, b, a, \dots, a) \in A_j$$

and so f' has to depend on all its n variables.

Observe now that if all algebras in the system $\mathcal{A}$ are unary, then $S(\mathcal{A})$ is also unary, and so $S\big(S(\mathcal{A})\big) = 1$. If, however, there is a fundamental operation of $k > 1$ variables, say $f(x_1, \dots, x_k)$, then $s(x, y) = f_k(x, y, \dots, y)$

is a term with two variables, and clearly by formal superposition of $s(x, y)$ with itself we get terms of an arbitrary number n of variables. As we have seen, those terms define algebraic operations depending on n variables, and so in this case $S(S(\mathcal{A})) = \{1, 2, 3, \ldots\}$.

2. We give now an application of Theorem II, namely, a representation theorem for a class of binary algebras.

Let now $\mathfrak{A} = (X; o_1, o_2)$, where o_i are binary operations satisfying the following conditions:

$$x\, o_i\, x = x\,, \tag{12}$$

$$(x\, o_i\, y)\, o_i\, z = x\, o_i\, (y\, o_i\, z)\,, \tag{13}$$

$$x\, o_i\, y\, o_i\, z = x\, o_i\, z\, o_i\, y\,, \tag{14}$$

$$(x\, o_i\, y)\, o_j\, z = (x\, o_j\, z)\, o_i\, (y\, o_j\, z)\,, \tag{15}$$

$$x\, o_j\, (y\, o_i\, z) = (x\, o_j\, y)\, o_i\, (x\, o_j\, z) \qquad (i, j = 1, 2)\,, \tag{16}$$

$$(x\, o_1\, y)\, o_2\, y = (y\, o_1\, x)\, o_2\, x\,. \tag{17}$$

Note that these conditions imply the algebras $(X; o_i)$ are semigroups of the type considered by Yamada and Kimura [3].

We shall need the following equality which is a consequence of (12)-(17):

$$x\, o_i\, y\, o_i\, (x\, o_j\, y) = x\, o_i\, y = x\, o_i\, y\, o_i\, (y\, o_j\, x)\,. \tag{i}$$

In fact, $x\, o_i\, y = (x\, o_i\, y)\, o_j\, (x\, o_i\, y) = (x\, o_j\, x)\, o_i\, (x\, o_j\, y)\, o_i\, (y\, o_j\, x)\, o_i\, (y\, o_j\, y)$ and so

$$\begin{aligned} x\, o_i\, y\, o_i\, (x\, o_j\, y) &= \big(x\, o_i\, y\, o_i\, (y\, o_j\, x)\, o_i\, (x\, o_j\, y)\big)\, o_i\, (x\, o_j\, y) \\ &= x\, o_i\, y\, o_i\, (y\, o_j\, x)\, o_i\, x\, o_j\, y = x\, o_i\, y\,. \end{aligned}$$

The second part of (i) follows similarly.

LEMMA 4. *Formulas* (12)-(17) *imply* $(x\, o_1\, y)\, o_2\, z = (x\, o_2\, z)\, o_1\, y$.

In fact, using (i) we get

$$\begin{aligned} (x\, o_1\, y)\, o_2\, x &= (x\, o_2\, x)\, o_1\, (y\, o_2\, x) = x\, o_1\, (y\, o_2\, x) = x\, o_1\, x\, o_1\, (y\, o_2\, x) \\ &= x\, o_1\, (y\, o_2\, x)\, o_1\, x = x\, o_1\, (x\, o_2\, y)\, o_1\, y = x\, o_1\, y\, o_1\, (x\, o_2\, y) = x\, o_1\, y\,, \end{aligned}$$

and similarly, using also Lemma 2, we get

$$\begin{aligned} (x\, o_1\, y)\, o_2\, z &= (x\, o_2\, z)\, o_1\, (y\, o_2\, z) = (x\, o_2\, z)\, o_1\, (y\, o_2\, z)\, o_1\, y \\ &= (x\, o_2\, z)\, o_1\, (x\, o_2\, z)\, o_1\, y\, o_1\, (y\, o_2\, z) \\ &= (x\, o_2\, z)\, o_1\, y\, o_1\, (y\, o_2\, z)\, o_1\, (y\, o_2\, x\, o_2\, z)\, o_1\, (x\, o_2\, z) \\ &= (x\, o_2\, z)\, o_1\, y\, o_1\, (y\, o_2\, x\, o_2\, z)\, o_1\, (x\, o_2\, z) \\ &= (x\, o_2\, z)\, o_1\, y\, o_1\, (x\, o_2\, z) = (x\, o_2\, z)\, o_1\, y\,. \end{aligned}$$

LEMMA 5. *The operation* $xy = (x\, o_1\, y)\, o_2\, y$ *is associative.*

Proof. Using Lemma 2, Lemma 4 and (i) we get

$$(xy)z = \Big(\big((x\, o_1\, y)\, o_2\, y\big)\, o_1\, z\Big)\, o_2\, z = (x\, o_1\, y\, o_1\, z)\, o_2\, y\, o_2\, z$$

and also

$$\begin{aligned} x(yz) &= \big(x\, o_1\, ((y\, o_1\, z)\, o_2\, z)\big)\, o_2\, \big((y\, o_1\, z)\, o_2\, z\big) \\ &= (x\, o_1\, y\, o_1\, z)\, o_2\, (x\, o_1\, z)\, o_2\, (y\, o_1\, z)\, o_2\, z \\ &= (x\, o_1\, y\, o_1\, z)\, o_2\, \big((x\, o_2\, y)\, o_1\, z\big)\, o_2\, z \\ &= (x\, o_1\, y\, o_1\, z)\, o_2\, (x\, o_1\, z)\, o_2\, y\, o_2\, z \\ &= \big((x\, o_1\, z)\, o_1\, y\big)\, o_2\, (x\, o_1\, z)\, o_2\, y\, o_2\, z \\ &= (x\, o_1\, z\, o_1\, y)\, o_2\, y\, o_2\, z \\ &= (x\, o_1\, y\, o_1\, z)\, o_2\, y\, o_2\, z\,. \end{aligned}$$

Consider now the algebra $\mathfrak{P}^* = (S_1 \times S_2; o_1, o_2)$, where each of the sets S_1, S_2 is a semilattice with respect to the operation xy, and the fundamental operations o_1 and o_2 are defined by

$$[x_1, y_1]\, o_1\, [x_2, y_2] = [x_1x_2, y_1]\,, \qquad [x_1, y_1]\, o_2\, [x_2, y_2] = [x_1, y_1y_2]\,.$$

THEOREM IV. *An algebra* $\mathfrak{P} = (X; o_1, o_2)$ *belongs to the equational class determined by equations* (12)-(17) *if and only if it is the sum of a direct system of algebras which are subalgebras of suitable algebras of the form* $\mathfrak{P}^*$.

Proof. The sufficiency is trivial. To prove the necessity observe that the operation o_i $(i = 1, 2)$ is a P-function in the algebra $(X; o_i)$, hence from (15), (16) and Theorem II it follows that $x\, o_1\, (x\, o_2\, y)$ is a P-function for the algebra $\mathfrak{P}$. Now we may apply Theorem III of [1] to obtain that $\mathfrak{P}$ is the sum of a direct system of algebras, say $\{\mathfrak{P}_i\}_{i \in I}$, where in each algebra $\mathfrak{P}_i$ in addition to equalities (12)-(17) the equality

$$x\, o_1\, (x\, o_2\, y) = x \tag{18}$$

is also satisfied.

In every algebra $\mathfrak{P}_i$ we define the relations R_1 and R_2 by means of

$$x R_j y \text{ if and only if } x\, o_j\, y = x \quad \text{and} \quad y\, o_j\, x = y \qquad (j = 1, 2)\,.$$

Clearly, R_1 and R_2 are equivalence relations; moreover, they are congruences, because if $x R_j y$ and $u R_j v$ (j is one of the numbers $1, 2$), then for $k = 1, 2$ we have

$$\begin{aligned} &(x\, o_k\, u)\, o_j\, (y\, o_k\, v) \\ &\quad = (x\, o_j\, y)\, o_k\, (x\, o_j\, v)\, o_k\, (u\, o_j\, y)\, o_k\, (u\, o_j\, v) \\ &\quad = x\, o_k\, (x\, o_j\, v)\, o_k\, (u\, o_j\, y)\, o_k\, u\, o_k\, (x\, o_j\, y\, o_j\, u\, o_j\, v)\, o_k\, (u\, o_j\, x\, o_j\, y\, o_j\, v) \\ &\quad = x\, o_k\, u\, o_k\, (x\, o_j\, u)\, o_k\, (u\, o_j\, x) = x\, o_k\, u\,. \end{aligned}$$

Every equivalence class of the relation R_1 has at most one point in common with a given equivalence class of the relation R_2, as from xR_jy $(j=1,2)$ it follows that $x=(x\,o_1\,y)\,o_2\,y$ and $y=(y\,o_1\,x)\,o_2\,x$, which in view of (17) gives $x=y$. This shows that the algebra $\mathfrak{P}_i$ can be isomorphically imbedded in the product $(\mathfrak{P}_i/R_1)\times(\mathfrak{P}_i/R_2)$. Our theorem will be proved if we show that the algebras $\mathfrak{S}_{ij}=\mathfrak{P}_i/R_j$ $(j=1,2)$ are semilattices. It is enough to do this for $\mathfrak{S}_{i1}$. In this algebra the operation $x\,o_2\,y$ is trivial, i.e. equal to x, by (17), and Lemma 4 and the operation $x\,o_1\,y$ is equal to $xy=(x\,o_1\,y)\,o_2\,y$. Using Lemma 5, and formulas (12) and (17) we obtain our assertion.

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