A universal null set which is not concentrated

by

R. B. Darst (Lafayette, Ind.)

In 1914, N. Lusin [3] established, assuming the continuum hypothesis, the existence of an uncountable subset E of the interval $I = \{x;\ x \text{ real},\ 0 \leqslant x \leqslant 1\}$ with the following property.

(1) *If N is a nowhere dense subset of I, then $E \cap N$ is, at most, a countable set*: $\operatorname{card}(E \cap N) \leqslant \aleph_0$.

Thereafter a hierarchy of properties, related to property (1), which subsets E of I might possess arose. These are discussed in Chapter 7 of [2] where an abundant supply of references can be found. We shall list several of these, as well as others, below.

(2) *If $\{x_i\}$ is a sequence of numbers, dense in I, and O_i is an open set containing x_i, then*

$$\operatorname{card}(E - \bigcup_i O_i) \leqslant \aleph_0 .$$

(3) *There is a sequence $\{y_i\}$ of numbers, dense in I, such that if O_i is an open set containing y_i, then* $\operatorname{card}(E - \bigcup_i O_i) \leqslant \aleph_0$.

(4) *If $\{\varepsilon_i\}$ is a sequence of positive numbers, then there exists a sequence $\{x_i\}$ of numbers such that*

$$E \subset \bigcup_i N(x_1, \varepsilon_i) , \quad \textit{where} \quad N(x, \varepsilon) = \{y;\ x - \varepsilon/2 < y < x + \varepsilon/2\} .$$

(5) *If φ is a homeomorphism of I onto I, then the image, $\varphi(E)$, of E has Lebesgue measure zero.*

(6) *If μ is a non-negative non-atomic finite Baire measure on I, then* $\mu(E) = 0$.

The following implications are clear or well known: $(1) \leftrightarrow (2) \rightarrow (3) \rightarrow$ $\rightarrow (4) \rightarrow (5) \leftrightarrow (6)$, $(5) \leftrightarrow (6)$ being established by Lebesgue.

It is easy to see that, in (5), φ need only be a monotone continuous mapping of I onto I and, hence, that properties (5) and (6) are invariant under continuous monotone mappings. While properties (1) to (4) are invariant under continuous mappings, it does not appear to be known whether property (5) is invariant under continuous functions of bounded variation. This latter question would, of course, be settled by showing either that $(5) \rightarrow (4)$ or that if E has property (5) and each of φ_1 and φ_2

is a continuous non-decreasing function on I with $\varphi_1(0) = \varphi_2(0) = 0$, then the set $\{\varphi_1(x) - \varphi_2(x): x \in E\}$, transformed affinely to lie in I, has property (5). By giving an example of a Hamel basis with property (1), the author has shown [1] that if E has property (1), then $\{x-y: x, y \in E\}$ need not have property (5).

In order to show that (3) does not imply (1), it suffices to observe that N. Lusin's argument [3] for the existence of an uncountable set E with property (3) works in a Cantor set C contained in I with the relative topology: there is an uncountable subset E of C such that if $\{y_i\}$ is a countable dense subset of C and O_i is an open set containing y_i, $i = 1, 2, \ldots$, then $\operatorname{card}(E - \bigcup_i O_i) \leqslant \aleph_0$.

The purpose of this note is to construct an example to show that (5) does not imply (3). In our construction we need to assume the continuum hypothesis.

Construction. Let $\{x^a\}$ and $\{\mu_a\}$ be well orderings of the sequences $x = \{x_i\}$ of elements of I and the (non-trivial) non-atomic Baire measures μ on I, where each a has countably many predecessors. Let F_a be a first category F-sigma which supports μ_a so that $M_a = \bigcup_{\beta \leqslant a} F_\beta$ is a first category F-sigma. Denote by V the set of non-negative non-atomic Baire measures ν on I such that $\nu(I) = 1$. Let J_0 be a Cantor set in $I - M_0$, let ν_0 be an element of V which lives on J_0, and let $N^0 = \{N_i^0\}$ be a sequence of segments such that $x_i^0 \in N_i^0$ and $\sum_i \nu_0(N_i^0) < 1$. There is a Cantor set K_0 in $J_0 - \bigcup_i N_i^0$. Let S_0 be an uncountable subset of K_0 which satisfies (1) with respect to the space K_0. Suppose that J_β, ν_β, N^β, K_β, and S_β have been obtained for $\beta < a$ such that J_β is a Cantor set in $I - M_\beta \cup (\bigcup_{\gamma < \beta} K_\gamma)$, ν_β is an element of V which lives on J_β, N^β is a sequence of segments N_i^β such that $x_i^\beta \in N_i^\beta$ and $\sum_i \nu_\beta(N_i^\beta) < 1$, K_β is a Cantor set in $J_\beta - \bigcup_i N_i^\beta$, and S_β is an uncountable subset of K_β which satisfies (1) with respect to the space K_β. Then it is clear how to obtain S_a. Let $E = \bigcup_a S_a$. If $x = x^a$, then $S_a \subset (I - \bigcup_i N_i^a)$. If $\mu = \mu_a$, then $\bigcup_{\beta \geqslant a} S_\beta$ is a subset of $I - F_\beta$, which is a set of μ_a measure zero, and $\mu_a(S_\beta) = 0$ for each β and, hence, $\mu(E) = 0$.

References

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PURDUE UNIVERSITY

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Polynomial factors of light mappings on an arc

by

Sam W. Young (Salt Lake City, Utah)

Introduction. In this paper we characterize light mappings of an arc onto an arc by a factorization property. It is shown that a mapping of an arc onto an arc is light if and only if it is topologically equivalent to a real valued continuous function f of $[0, 1]$ onto $[0, 1]$ such that f can be factored $f = Pg = P(g)$ where P is a polynomial and g is arbitrarily near the identity. Only techniques of classical real variables are employed.

DEFINITION 1. If f is a mapping (continuous function), then f is *light* if and only if for each x in the range of f, $\overset{-1}{f}(x)$ is totally disconnected.

The class of light real-valued continuous functions on an interval includes nowhere differentiable functions and nowhere monotone functions. The latter type was treated, for example, by Garg in [1]. There are also continuous functions f such that each inverse set $\overset{-1}{f}(x)$ is a Cantor set. An interesting example of a function having all three of the properties just mentioned was described by Jolly in [2].

DEFINITION 2. If f is a continuous function of $[a, b]$ onto $[c, d]$, then $f = f_1 f_2$ is a *factorization* of f, which means that there exists an interval $[a', b']$ such that f_2 is a continuous function of $[a, b]$ onto $[a', b']$ and f_1 is a continuous function of $[a', b']$ onto $[c, d]$ and for each $x \in [a, b]$,

$$f(x) = f_1(f_2(x)) .$$

THEOREM 1. *If f is a continuous light function of $[a, b]$ onto $[c, d]$ and $\varepsilon > 0$, there exists a factorization $f = Pg$ such that P is a polynomial of $[a, b]$ onto $[c, d]$ and g is a continuous function of $[a, b]$ onto $[a, b]$ such that*

$$|g(x) - x| < \varepsilon \quad \text{for all } x \in [a, b] .$$

We will first establish four lemmas.

DEFINITION 3. If f is a continuous function, $V(f) = \{t$: there exists an open interval Ω containing t such that $f(x) - f(t)$ does not change sign on $\Omega \cap [t, \infty) \cap$ domain of f or on $\Omega \cap (-\infty, t] \cap$ domain of $f\}$.