Then if A' is $H \cap T(f^{-1}(A))$ and B' is $H \cap T(f^{-1}(B))$, A' and B' are disjoint closed subsets of G, neither A' nor B' separates two points of $T(F)$ (which has $k+2$ points) in G, and $A' \cup B'$ separates each two points of $T(F)$ in G. This contradicts Lemma B, since the coherence of G is $\leqslant k$. Consequently, there are two points of $T(F)$ that are not separated in G by G' and therefore belong to the same quasicomponent Q of $G-G'$.

From Lemma 1, Q is connected. Since T is monotone, $T^{-1}(Q)$ is connected. Then $f(T^{-1}(Q))$ is a connected subset of $Y-(A \cup B)$ that contains two points of F and this involves a contradiction. It follows that the coherence of Y is less than or equal to k.

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On a method of construction of abstract algebras

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1. In this note we consider abstract algebras with finitary operations without nullary fundamental operations (¹) and of a fixed type. First we recall the definition of a direct system of algebras (see [3], chapter 3):

(i) *I is a given poset (partially ordered set) whose ordering relation is denoted by $\leqslant$.*

(ii) *For each $i \in I$ an algebra $\mathfrak{A}_i = \langle A_i \colon \langle F_t^{(i)} \rangle_{t \in T} \rangle$ is given, all algebras $\mathfrak{A}_i$ being of the same type.*

(iii) *For each pair i, j of elements of I with $i \leqslant j$ a homomorphism $\varphi_{ij} \colon \mathfrak{A}_i \to \mathfrak{A}_j$ is given. The resulting set of homomorphisms must satisfy the following conditions:*

(a) *$i \leqslant j \leqslant k$ implies $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, and*

(b) *φ_{ii} is the identity map for $i \in I$.*

The system $\langle I, \langle \mathfrak{A}_i \rangle_{i \in I}, \langle \varphi_{ij} \rangle_{i \leqslant j;\, i,j \in I} \rangle$ is called a *direct system* of algebras.

We shall consider only direct systems $\mathcal{A}$ with the l.u.b.-property, i.e. systems which satisfy additionally the condition:

(iv) *The ordering relation of I induces a partial order with the least upper bound property* (²).

For every such direct system $\mathcal{A}$ we define an algebra $\mathfrak{A} = S(\mathcal{A})$ which we shall call the *sum* of the direct system $\mathcal{A}$.

We may clearly assume that the carriers of the algebras $\mathfrak{A}_i$ are mutually disjoint, as otherwise we could obtain this by taking isomorphic copies.

(¹) This is not a serious restriction. In fact, if a fundamental operation F_t is nullary, then one can replace it by a unary operation $F_t'(x) = F_t$, without essential changes in the algebraic structure of the algebra in question.

(²) We recall that an ordered set has the *least upper bound property* if every two of its elements have a least common upper bound.

The carrier of the algebra $S(\mathcal{A})$ will be equal to $\bigcup_{i\in I} A_i$, and the fundamental operations of $S(\mathcal{A})$ are defined by

$$F_t(x_1, \ldots, x_n) = F_t^{(i_0)}(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_n,i_0}(x_n))$$

where $i_0 = \text{l.u.b.}(i_1, \ldots, i_n)$ and $x_r \in A_{i_r}$ $(r = 1, 2, \ldots, n)$.

Note that the set I is a semilattice under the binary operation $i_1 i_2 = \text{l.u.b.}(i_1, i_2)$. The notion of the sum of a direct system was considered in the case of semigroups by various authors (see e.g. [1], [4] chapter 8 and [9]). As far as we know, the general definition does not appear in literature. The notion of the direct limit of algebras is closely connected with our sum; in fact, the direct limit is a homomorphic image of the sum.

Let σ_1 and σ_2 be terms in an algebra. We shall say that the equation $\sigma_1 = \sigma_2$ is *regular* if the set of free variables occurring in σ_2 and σ_1 is the same. Now we prove

THEOREM I. *If $\mathcal{A}$ is a direct system of algebras with the* l.u.b.*-property, containing at least two algebras, then in the algebra $S(\mathcal{A})$ all regular equations satisfied in all algebras of $\mathcal{A}$ are satisfied, whereas every other equation is false in $S(\mathcal{A})$.*

Proof. The first part follows easily from the definition of fundamental operations in $S(\mathcal{A})$ and the properties of homomorphisms. To prove the second part consider an irregular equation which is satisfied in $\mathfrak{A} = S(\mathcal{A})$, and a fortiori in each $\mathfrak{A}_i$, say

$$\sigma_1 = \sigma_2, \tag{1}$$

and let X_i be the set of free variables occuring in σ_i $(i = 1, 2)$. Since $X_1 \neq X_2$, we may assume without restriction that, say, $x_j \in X_1 \setminus X_2$. Since $|I| \neq 1$, we can find i_1, i_2 in I with $i_1 \leqslant i_2$ and $i_1 \neq i_2$. Choose $a \in A_{i_1}$, $b \in A_{i_2}$ and put in (1) $x_j = b$, $x_i = a$ $(i \neq j)$. Then evidently $\sigma_1 \in A_{i_2}$, $\sigma_2 \in A_{i_1}$; thus $\sigma_1 \neq \sigma_2$ since the sets A_{i_1} and A_{i_2} are disjoint.

As an immediate consequence of this theorem we get the following

COROLLARY 1. *An equational class of algebras which has no nullary fundamental operations is closed under the formation of $S(\mathcal{A})$ if and only if the defining equations of this class are all regular.*

Since evidently the class of all groups is not closed under the formation of $S(\mathcal{A})$ (a group is never a disjoint sum of its subgroups except the trivial case), we also get the following

COROLLARY 2. *The class of all groups cannot be defined as an equational class without nullary fundamental operations, all defining equations being regular.*

Now let $\mathfrak{A} = \langle A; \langle F_t\rangle_{t\in T}\rangle$ be an algebra without nullary fundamental operations and let $f\colon A^2 \to A$ be a function, not necessarily algebraic in $\mathfrak{A}$. We shall call f a *partition function* for the algebra $\mathfrak{A}$ (or, shortly, *P-function*) if it satisfies the following formulas:

$$f(f(x, y), z) = f(x, f(y, z)), \tag{1.1}$$

$$f(x, x) = x, \tag{1.2}$$

$$f(x, f(y, z)) = f(x, f(z, y)), \tag{1.3}$$

$$f(F_t(x_1, \ldots, x_{n(t)}) y) = F_t(f(x_1, y), \ldots, f(x_{n(t)}, y)), \tag{1.4}$$

$$f(y, F_t(x_1, \ldots, x_{n(t)})) = f(y, F_t(f(y, x_1), \ldots, f(y, x_{n(t)}))), \tag{1.5}$$

$$f(F_t(x_1, \ldots, x_{n(t)}), x_k) = F_t(x_1, \ldots, x_{n(t)}) \qquad (1 \leqslant k \leqslant n(t)), \tag{1.6}$$

$$f(y, F_t(y, \ldots, y)) = y. \tag{1.7}$$

The following theorem demonstrates the connection between P-functions for the algebra $\mathfrak{A}$ and the possible representations of $\mathfrak{A}$ in the form $\mathfrak{A} = S(\mathcal{A})$.

THEOREM II. *To every* P*-function f for the algebra $\mathfrak{A}$ there corresponds a representation $\mathfrak{A} = S(\mathcal{A})$ obtained as follows:*

Divide A into disjoint subsets A_i $(i \in I)$ putting two elements a, b of A into the same set A_i if and only if $f(a, b) = a$ and $f(b, a) = b$. The sets A_i are seen to be closed under fundamental operations of A, and so let $\mathfrak{A}_i = \langle A_i; \langle F_t\rangle\rangle_{t\in T}$. In the set I of indices we introduce the relation "$\leqslant$" defining $i_1 \leqslant i_2$ if and only if there exist $a \in A_{i_1}$, $b \in A_{i_2}$ such that $f(b, a) = b$. It turns out that this definition is consistent and the relation obtained gives I the structure of a poset with l.u.b. *Finally define the mappings $\varphi_{i_1,i_2}\colon A_{i_1} \to A_{i_2}$ for $i_1 \leqslant i_2$ by putting $\varphi_{i_1,i_2}(a) = f(a, b)$ where b is an arbitrary element of A_{i_2}. The mappings so defined are homomorphisms and the system*

$$\mathcal{A} = \langle I, \langle \mathfrak{A}_i\rangle_{i\in T}, \langle \varphi_{ij}\rangle_{i\leqslant j;\, i,j\in I}\rangle$$

is a direct system of algebras for which $\mathfrak{A} = S(\mathcal{A})$. Conversely, every representation $\mathfrak{A} = S(\mathcal{A})$ can be obtained by this construction by starting with a suitable P*-function f.*

Moreover, the correspondence between P*-functions for $\mathfrak{A}$ and representations of $\mathfrak{A}$ in the form $\mathfrak{A} = S(\mathcal{A})$ is one-to-one.*

Proof. The consistency of the definitions of A_i's, their disjointness, the required properties of the ordering relation and the mappings can easily be obtained from theorem I in [10]. We must now prove that each A_i is closed under F_t $(t \in T)$ and that the mappings are homomorphisms.

Let $a_1, \ldots, a_{n(t)} \in A_i$ and $b = F_t(a_1, \ldots, a_{n(t)})$. Then $f(b, a_1) = f(F_t(a_1, \ldots, a_{n(t)}), a_1) = b$ by (1.6) and, moreover,

$$f(a_1, b) = f(a_1, F_t(a_1, \ldots, a)_{n(t)}) = f(a_1, F_t(f(a_1, a_1), \ldots, f(a_1, a_{n(t)})))$$
$$= f(a_1, F_t(a_1, \ldots, a_1)) = a_1$$

by (1.7) hence $b \in A_i$.

Let $i_1 \leqslant i_2$, $a_1, \ldots, a_{n(t)} \in A_{i_1}$ and $b_j = \varphi_{i_1,i_2}(a_j) = f(a_j, c)$, where c is an arbitrary element of A_{i_2}. Then

$$\begin{aligned} F_t(\varphi_{i_1,i_2}(a_1), \ldots, \varphi_{i_1,i_2}(a_{n(t)})) &= F_t(b_1, \ldots, b_{n(t)}) \\ &= F_t(f(a_1, c), \ldots, f(a_n, c)) \\ &= f(F_t(a_1, \ldots, a_{n(t)}), c) \\ &= \varphi_{i_1,i_2}(F_t(a_1, \ldots, a_{n(t)})) , \end{aligned}$$

which implies that the mappings φ_{i_1,i_2} are homomorphisms. It follows that $\mathcal{A} = \langle I, \langle \mathfrak{A}_i \rangle_{i\in I}, \langle \varphi_{i_1,i_2} \rangle_{i_1 \leqslant i_2;\, i_1 i_2 \in I} \rangle$ is a well-defined direct system of algebras with l.u.b. It remains to prove the equality $\mathfrak{A} = S(\mathcal{A})$, which will be established if the following identity holds:

$$F_t(x_1, \ldots, x_{n(t)}) = F_t(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_{n(t)},i_0}(x_{n(t)}))$$

where $x_k \in A_{i_k}$ and $i_0 = \text{l.u.b.}(i_1, \ldots, i_{n(t)})$.

Observe that with $y \in A_{i_0}$ we have

$$\begin{aligned} F_t(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_{n(t)},i_0}(x_{n(t)})) &= F_t(f(x_1, y), \ldots, f(x_{n(t)}, y)) \\ &= f(F_t(x_1, \ldots, x_{n(t)}), y) , \end{aligned}$$

and so it suffices to prove that $F_t(x_1, \ldots, x_{n(t)}) \in A_{i_0}$.

In fact, let $F_t(x_1, \ldots, x_{n(t)}) \in A_j$. Then by (1.6) we get $j \geqslant i_0$, and (1.5) with (1.7) imply

$$\begin{aligned} f(y, F_t(x_1, \ldots, x_{n(t)})) &= f\big(y, F_t(f(y, x_1), \ldots, f(y, x_{n(t)}))\big) \\ &= f(y, F_t(y, \ldots, y)) = y; \end{aligned}$$

thus (1.6) implies $i_0 \geqslant j$, whence $i_0 = j$, and the proof is complete.

The converse part of our theorem and the fact that the correspondence between P-functions and representations of $\mathfrak{A}$ as $S(\mathcal{A})$ is one-to-one can easily be checked.

Now let $\mathfrak{A}$ be an algebra belonging to an equational class K whose defining equations are all regular. Let $g(x, y)$ be a term of $\mathfrak{A}$ and let K^* be the equational class defined by the equations of the class K to which the equation $g(x, y) = x$ has been added. From theorems I and II we get the following

THEOREM III. *The term $g(x, y)$ defines a P-function for $\mathfrak{A}$ if and only if $\mathfrak{A}$ is representable as the sum of a direct system of algebras from the class K^*.*

We leave the simple proof to the reader.

2. Examples.

2.1. Let $\mathfrak{P}$ be an idempotent semigroup satisfying (1.3). Such semigroups are called *left-normal* in [10]. It is easy to verify that $x \cdot y$ defines a P-function for $\mathfrak{P}$, whence by theorem III $\mathfrak{P}$ is the sum of a direct system of trivial algebras, i.e. of algebras with the fundamental operation $f(x, y) = x$. This result was proved by Yamada and Kimura in [10], theorem I.

2.2. An algebra $\mathfrak{Q} = (X; +, \cdot)$, where $+$ and $\cdot$ are both binary, idempotent, commutative and associative operations satisfying $(x+y)z = xz+yz$ and $x+yz = (x+y)(x+z)$, is called a *distributive quasilattice* (cf. [6]). It was proved in [6] that the operation $x \circ y = x+xy$ satisfies (1.1)-(1.7) and so is a P-function for $\mathfrak{Q}$. Thus theorem III shows that $\mathfrak{Q}$ is the sum of a direct system of distributive lattices. (Cf. theorem IV of [6].)

2.3. An algebra $\mathfrak{D} = (X; d(x_1, \ldots, x_n))$ which satisfies the axioms: $d(x, \ldots, x) = x$ and $d(d(x_{11}, \ldots, x_{1n}), \ldots, d(x_{n1}, \ldots, x_{nn})) = d(x_{11}, x_{22}, \ldots, x_{nn})$ is called an *n-dimensional diagonal algebra*. (Cf. [7] for further properties and a representation theorem.) From a theorem of Liapin (see [4], p. 108) it follows that a semigroup satisfying $xyx = x$ is a 2-dimensional diagonal algebra.

Now let $\mathfrak{M} = (X; \cdot)$ be an idempotent semigroup satisfying the condition $xyzt = xzyt$. (This condition (in the form $(xy)(zt) = (xz)(yt)$ when associativity is not assumed) was studied for binary operations, not necessarily associative, by various authors (see e.g. [5], [8]) and was called *mediality*, the *entropic law* or *abelianity*.) It is easy to verify that the operation $x \circ y = xyx$ is a P-function for $\mathfrak{M}$, and so by theorem III we find that $\mathfrak{M}$ is the sum of a direct system of 2-dimensional diagonal algebras. (3)

2.4. Let $\mathfrak{S} = (X; f(x_1, \ldots, x_n))$ be an algebra whose single fundamental operation f is idempotent and symmetric (i.e. $f(x_1, \ldots, x_n) = f(x_{i_1}, \ldots, x_{i_n})$ where $i_1, \ldots, i_n$ is an arbitrary permutation of $1, 2, \ldots, n$) and satisfies the following generalization of the associative law:

$$\text{(A)}\quad \begin{aligned} & f(f(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) \\ &\quad = f(x_1, f(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) = \ldots \\ &\quad = f(x_1, \ldots, x_{n-1}, f(x_n, \ldots, x_{2n-1})) . \end{aligned}$$

It is easy to check that the operation $x \circ y = f(x, y, y, \ldots, y)$ is a P-function for $\mathfrak{S}$, and so by theorem III $\mathfrak{S}$ is the sum of a direct system of algebras with one n-ary fundamental operation f which is idempotent,

(3) This result was proved by Yamada and Kimura in [10], theorem 6.8.

symmetric, satisfies (A) and, moreover, $f(x, y, y, \dots, y) = x$. The last class of algebras can be described completely, owing to the following result of K. Urbanik (whose proof we reproduce with his kind permission) (For similar results see also [2].):

Let $\mathfrak{A} = (X; f(x_1, \dots, x_n))$ $(n \geqslant 3)$ be an algebra such that the operation f is symmetric, satisfies (A) *and, moreover, $f(x, y, \dots, y) = x$. Then in the set X it is possible to define a binary operation $+$ such that $(X; +)$ is an abelian group, $(n-1)a = 0$ for all $a \in X$ and finally $f(x_1, \dots, x_n) = x_1 + x_2 + \dots + x_n$.*

Proof. Define $x+y = f(x, y, 0, \dots, 0)$ where 0 is an arbitrary, but fixed element of X. The commutativity of $+$ is a consequence of the symmetry of f, and $x+0 = x$ follows from the identity $f(x, y, \dots, y) = x$. Moreover,

$$\begin{aligned}(x+y)+z &= f(f(x, y, 0, \dots, 0), z, 0, \dots, 0)\\ &= f(x, f(y, 0, \dots, 0, z), 0, \dots, 0)\\ &= f(x, f(y, z, 0, \dots, 0), 0, \dots, 0)\\ &= x+(y+z),\end{aligned}$$

whence $(X, +)$ is a commutative semigroup with a unit. Now put $-x = f(x, x, \dots, x, 0, 0)$. Then clearly

$$\begin{aligned}x+(-x) &= f(x, f(x, \dots, x, 0, 0), 0, \dots, 0)\\ &= f(f(x, \dots, x, 0), 0, \dots, 0) = f(0, \dots, 0) = 0;\end{aligned}$$

consequently $(X, +)$ is an abelian group.

Now we shall prove the formula

$$(*)\qquad f(x_1, x_2, \dots, x_k, 0, \dots, 0) = x_1 + x_2 + \dots + x_k \qquad (k = 1, 2, \dots, n)$$

using induction in k. For $k = 1$ the formula is contained in our assumptions. Assume $1 \leqslant k < n$ and the formula $(*)$ for this k. Then

$$\begin{aligned}x_1 + \dots + x_{k+1} &= f(x_1, f(x_2, \dots, x_{k+1}, 0, \dots, 0), 0, \dots, 0)\\ &= f(f(x_1, \dots, x_{k+1}, 0, \dots, 0), 0, \dots, 0)\\ &= f(x_1, \dots, x_{k+1}, 0, \dots, 0),\end{aligned}$$

as we wanted.

From $(*)$ the identity $(n-1)a = 0$ follows immediately, and our proof is complete.

2.5. Let $\mathfrak{A}$ be an arbitrary abstract algebra without nullary fundamental operations and consider the system $\mathcal{A}$ of all subalgebras of $\mathfrak{A}$ with the natural order by inclusion and with injections as homomorphisms.

To define the sum $S(\mathcal{A})$ one has to replace the algebras of $\mathcal{A}$ by disjoint isomorphic copies. It would be interesting to find a characterization of algebras which can be represented in the form $S(\mathcal{A})$ where $\mathcal{A}$ is the system of all subalgebras of a suitable algebra. (This construction was considered in [3], § 36.)

2.6. Let $\mathfrak{B} = (X; \cdot, h(x))$ where $x \cdot x = x$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $x \cdot y \cdot z = x \cdot z \cdot y$, $h(h(x)) = x$, $h(x) \cdot y = h(x \cdot y)$, $y \cdot h(x) = y \cdot h(y \cdot x)$, $y \cdot h(y) = y$. This algebra is not idempotent and satisfies the assumptions of theorem III, whence it is the sum of a direct system of algebras of the form $(Y; h(x))$, where $h(h(x)) = x$ and $x \cdot y = x$.

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