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Existence of 2-segments in 2-metric spaces

by

Edward Z. Andalafte* and **Raymond W. Freese** (St. Louis, Missouri)

1. Introduction. It is known that in a certain class of metric spaces the existence of a metric segment joining each two points of a set is equivalent to the existence of a between-point for each pair of its distinct points. In particular it has been shown that each two distinct points of a complete, convex metric space are end-points of a metric segment of the space ([1], pp. 41-43, [2], [3], p. 89).

A similar question can be considered in a 2-metric (area metric) space ([3], [4]), and the purpose of this paper is to prove that under suitable definitions of completeness and convexity, each three points with non-vanishing 2-metric (area) of any suitably chosen 2-metric space are vertices of a 2-segment, an isometric image under the 2-metric of a closed Euclidean triangle.

2. Preliminary notions. By a *2-metric space* is meant a set S of objects $a, b, c, \ldots, p, q, r, \ldots$ and a function pqr, called the *2-metric* or *area*, on ordered triples of points of S into the non-negative real numbers, satisfying

(1) *If $p, q \in S$, there is a point $r \in S$ with $pqr \neq 0$, and*

(2) *Each four points of S are 2-congruently embeddable in the 3-dimensional Euclidean space E_3, i.e., if $p, q, r, s \in S$ there are points $p', q', r', s' \in E_3$ and a 1-1 area-preserving correspondence between the quadruples.*

A triple p, q, r of points of a 2-metric space is said to be *linear* provided $pqr = 0$. A 1-1 area-preserving function between subsets of 2-metric spaces is called a *2-congruence*. The expression $T \subset_2 E_3$ indicates that the set T is 2-congruent with a subset of E_3, and the notation

$$p_1, p_2, \ldots, p_n \approx_2 p'_1, p'_2, \ldots, p'_n$$

indicates that the n-tuples are 2-congruent in the given order.

Several notions of betweenness can be defined in a 2-metric space. The most useful in the following is the notion of *linear betweenness*, $B(p, q, r)$, defined to mean $p \neq q \neq r \neq p$ and for each t in S, $tpq + tqr$

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$= tpr$. A subset of a 2-metric space is said to be *linearly 2-convex* provided for each pair p, r of its distinct points, it contains a point q satisfying $B(p, q, r)$. A weaker but more natural notion of betweenness is the notion of *interior* of a triple, defined as follows. A point p of a 2-metric space S is said to be *weakly interior* to $q, r, s \in S$, denoted $p\bar{I}qrs$, provided $pqr+prs+pqs = qrs$. If none of the areas involved vanishes we say that p is *strictly interior* to q, r, s and write $pIqrs$. A quadruple of distinct points is called *triadic* provided one of its points is weakly interior to the remaining triple, pairs of whose points are called the *sides* of the triadic quadruple.

Similarly we can define a number of notions of convergence of a sequence of points. A sequence $\{p_n\}$ of points of a 2-metric space is called *weakly 2-convergent* to p in S provided $\lim p_n pt = 0$ for each point t of S. The sequence $\{p_n\}$ is said to be a *2-Cauchy sequence* provided for some non-linear triple $a, b, c \in S$ we have

$$\lim ap_m p_n = \lim bp_m p_n = \lim cp_m p_n = 0 \quad (m, n \to \infty).$$

A simple example shows that a weakly 2-convergent sequence need not be 2-Cauchy, and that the 2-metric is not necessarily a continuous function relative to the weak 2-convergence topology. The notion of strong 2-convergence is therefore introduced as follows. The sequence $\{p_n\} \subset S$ is said to be *strongly 2-convergent* to $p \in S$ provided

(1) $\{p_n\}$ is weakly 2-convergent to p, and

(2) for each point $q \in S$ and each sequence $\{q_n\}$ weakly convergent to q, we have $\lim pp_m q_n = 0$ $(m, n \to \infty)$.

It is now noted that every strongly 2-convergent sequence is a 2-Cauchy sequence, and that relative to the strong 2-convergence topology, the 2-metric function is continuous. More surprising is the fact that in any 2-metric space with continuous 2-metric, strong 2-convergence and weak 2-convergence are equivalent. Clearly in Euclidean spaces the notions of weak 2-convergence, strong 2-convergence, and metric convergence are equivalent, as are the notions of 2-Cauchy and metrically Cauchy sequences.

Defining a 2-metric space S as *2-complete* provided every 2-Cauchy sequence of its points is strongly 2-convergent to a point of the space, and defining a 2-segment as above, the main result may now be stated.

THEOREM. *Let S be a 2-complete, linearly 2-convex 2-metric space in which each 5 points containing a triadic quadruple and a point linear with one side are 2-congruent with 5 points of E_3. Then each three points a, b, c of S with $abc \neq 0$ are vertices of a 2-segment.*

In the remainder of the paper $[a'b'c']$, $[a'b']$, and $(a'b')$ will denote respectively the closed Euclidean triangle and the closed and open Euclidean segments with the given points as vertices or end-points, and $L(a'b')$ will denote the Euclidean line determined by the distinct points a', b'.

3. Weak 2-segments. In order to prove the main theorem a weakened form is first proved, as Lemma 1, in terms of which results are obtained permitting strengthening of the conclusions to obtain the desired result.

LEMMA 1. *Let S be a 2-complete, linearly 2-convex 2-metric space in which each 5 points consisting of a triadic quadruple and a point linear with one side are 2-congruent with 5 points of E_3. Then for each three points a, b, c of S such that $abc \neq 0$, there exist points a', b', c' of E_2 with $a'b'c' = abc$ and a 1-1 mapping g on a subset of S containing a, b, c, onto the closed triangle $[a'b'c']$ satisfying:*

(i) *$g(a) = a'$, $g(b) = b'$, $g(c) = c'$, and*

(ii) *If x and y are in the domain of g, $D(g)$, then $axy = a'x'y'$, $bxy = b'x'y'$, $cxy = c'x'y'$, where $x' = g(x)$, $y' = g(y)$.*

Proof. Let $a, b, c \in S$ and choose $a', b', c' \in E_2$ as vertices of an equilateral triangle whose area $a'b'c' = abc$. The function g is defined as follows. Let f_1: $a \to a'$, $b \to b'$, $c \to c'$, $D(f_1) = \{a, b, c\}$, $R(f_1) = \{a', b', c'\}$. Suppose inductively that functions $f_1, \ldots, f_n$ have been defined. To define f_{n+1} partition $[a'b'c']$ into 4^n non-overlapping half-closed triangles and denote this partition by I_n. Let Σ_n be the set of functions f on subsets of S satisfying (i) and (ii), with $R(f) \subset [a'b'c']$, which are extensions of f_n. Letting $N_n(f)$ be the number of triangles of partition I_n with points common to $R(f)$ we choose $f_{n+1} \in \Sigma_n$ such that $N_n(f_{n+1}) \geqslant N_n(f)$ for all $f \in \Sigma_n$. Now the sequence $f_1, f_2, \ldots, f_n, \ldots$ uniquely determines a function f which is an extension of each f_n, and with $D(f) \bigcup D(f_n) \subset S$, and $R(f) = \bigcup R(f_n) \subset [a'b'c']$.

The function f is now extended to the closure of $D(f)$ and $R(f)$. Let $p \in D'(f)$. Then there is a sequence $\{p_n\}$ of pairwise distinct points of S, strongly 2-convergent to p. Consider $\{p'_n\}$ where $p'_n = f(p_n)$. Since $\{p_n\}$ is strongly 2-convergent, it is 2-Cauchy, and $\lim ap_m p_n = \lim bp_m p_n = \lim cp_m p_n = 0$, implying $\lim a'p'_m p'_n = \lim b'p'_m p'_n = \lim c'p'_m p'_n = 0$ which imply (in E_2) that $\{p'_n\}$ is 2-Cauchy and converges to a limit, call it p'. Now p' is uniquely determined by p, for if a sequence $\{\bar{p}_n\} \subset D(f)$ strongly 2-converges to p, then the assumption that $\{\bar{p}'_n\}$ is strongly 2-convergent to $p'' \neq p'$ leads to a contradiction by consideration of the sequence $p_1, \bar{p}_1, p_2, \bar{p}_2, \ldots$, and use of properties of subsequences. Thus we define $g(p) = p'$. Now g is 1-1 in $D'(f)$ for let $p' \in R'(f)$. Then a sequence $\{p'_n\} \subset R(f)$ is strongly 2-convergent to p', and is thus 2-Cauchy, and the related sequence $\{p_n\} \subset S$ is 2-Cauchy (relative to a, b, c). Hence by 2-completeness $\{p_n\}$ is strongly 2-convergent to $p \in D'(f)$, uniquely determined, as before. Hence g: $p \to p'$ is 1-1 in $D'(f)$.

We define $g(x) = f(x)$ for x in $D(f)$, and hence g is an extension of f, $D(g) = \bar{D}(f) \subset S$, $R(g) = \bar{R}(f) \subset [a'b'c']$. By continuity of the 2-metric g satisfies (ii). We note also that if $p \in D(g)$, $apb + bpc + apc = abc$.

It remains to be shown that $R(g) = [a'b'c']$. We observe first that $[a'b']$, $[a'c']$, $[b'c'] \subset R(g)$. For if the contrary were assumed, some point say of $[b'c']$ would not be in $R(g)$ and by closure of $R(g)$, points $p', q' \in [b'c']$ could be found with $p', q' \in R(g)$, $(p'q') \cap R(g) = \emptyset$. Let $p, q \in S$ be the corresponding points. Then by linear 2-convexity there is a point s in S such that $B(p, s, q)$. Let s' denote the unique point of $(p'q')$ such that $a'p's' = aps$, $a's'q' = asq$, and $p's'q' = 0$. Clearly $s' \notin R(g)$. It follows that $B(p', s', q')$, which implies, by familiar properties of linear betweenness, $B(b', s', c')$ as well as $abs + asc = abc$. Also since $a, b, p, q, s \subset_2 E_3$, and $aps = a'p's'$, $aqs = a'q's'$, it follows that

$$a, b, p, q, s \approx_2 a', b', p', q', s'$$

and hence $abs = a'b's'$, $acs = a'c's'$.

We define $g^*(x) = g(x)$ if $x \in D(g)$, $g^*(s) = s'$. We must now show that g^* satisfies (ii). It suffices to show that for $x \in D(g)$, $asx = a's'x'$, $bsx = b's'x'$, $csx = c's'x'$. Now $x\bar{I}abc$, so $a, b, c, s, x \subset_2 E_3$, and since

$$axb + bxc + axc = abc,$$

it follows that if $a \to a'$, $b \to b'$, $c \to c'$ then $x \to x' = g(x)$ and since $abs = a'b's'$, $acs = a'c's'$, that

$$a, b, c, s, x \approx_2 a', b', c', s', x'.$$

Thus g^* satisfies (ii). Consider now a decomposition I_n of $[a'b'c']$ such that the triangle of I_n containing s' contains no point of $R(g)$. This can certainly be done, since $R(g)$ is closed. Then $N_n(g^*) \geqslant N_n(f_{n+1}) + 1$, contrary to the choice of f_{n+1}. Hence each point of $[a'b']$, $[a'c']$, $[b'c']$ is in $R(q)$.

To complete the proof that $R(g) = [a'b'c']$, suppose there is a point x' interior to $[a'b'c']$, $x' \notin R(g)$. Then a circle can be found with center x' whose interior contains no point of $R(g)$ but whose circumference contains a point q' of $R(g)$. At least one of the segments $[a'q']$, $[b'q']$, $[c'q']$, say $[b'q']$, contains a point of the circumference distinct from q'. Let p' be the first point of $[b'q'] \cap R(g)$ encountered in proceeding from q' to b'. Then $p', q' \in R(g)$, $(p'q') \cap R(g) = \emptyset$. As before, letting p, q be inverse images of p', q', there is a point s of S with $B(p, s, q)$, and a point s' is uniquely determined in $[a'b'c']$ so that $p's'q' = 0$, $a'p's' = aps$, $a'q's' = aqs$, and a function g^* is again defined so that $g^*(s) = s'$, $g^*(x) = g(x)$ if $x \in D(g)$.

Again we must show that g^* satisfies (ii). Let t' be the intersection of $L(b'q')$ with $[a'c']$, and t its inverse image in S. Now $atc = 0$, and since $b'p'q' = 0 = p'q't'$, we have $bpq = 0 = pqt$ and (since each quadruple is embeddable in E_3) $bpt = 0 = bqt$. But since $psq = 0$, consideration of b, p, s, q yields $bsp = 0 = bsq$ and applying to b, p, s, t we have $pst = 0 = bst$. Thus $bst = b's't'$. Again $a, p, s, q, t \subset_2 E_3$ and as before by the choice of s' we have

$$a, p, q, s, t \approx_2 a', p', q', s', t'$$

so $ast = a's't'$. Applying the same procedure to a, b, p, s, q yields $abs = a'b's'$, so we thus obtain that

$$abt = a'b't' = a'b's' + a's't' = abs + ast.$$

To complete the proof that g^* satisfies (ii), let $x \in D(g)$. Again we must show $asx = a's'x'$, $bsx = b's'x'$, $csx = c's'x'$. Suppose $x'\bar{I}a'b't'$. Then $x\bar{I}abt$ and $a, b, t, s, x \subset_2 E_3$, and indeed $a, b, t, s, x \approx_2 a', b', t', s', x'$, so that $asx = a's'x'$, $bsx = b's'x'$. To show that $csx = c's'x'$, we observe that $x'\bar{I}a'c'u'$, or $x'\bar{I}b'c'u'$, where u' is the intersection of $L(c's')$ and $[a'b']$. As we have seen, $u' \in R(g)$ and letting u be its inverse image, we have

$$a, b, c, u \approx_2 a', b', c', u'.$$

Further, since $abs = a'b's'$ and $ast = a's't'$, from $B(a, t, c)$ we can show that $s\bar{I}abc$, so that $a, b, c, u, s \subset_2 E_3$, and indeed

$$a, b, c, u, s \approx_2 a', b', c', u', s'$$

and $csx = c's'x'$ follows. Similarly if $x'\bar{I}b'c'u'$, and thus g^* satisfies (ii). Again a contradiction is reached by partitioning $[a'b'c']$ so that the triangle containing s' contains no point of $R(g)$. Thus $R(g) = [a'b'c']$ and the proof is complete.

The function g will be referred to hereafter as a *weak 2-isometry*.

4. Properties of the weak 2-isometry. We are now in a position to develop further properties of the weak 2-isometry g which will permit strengthening of the conclusion of Lemma 1.

LEMMA 2. *Under the hypotheses and notation of Lemma* 1, *if* $p, q, r \in D(g)$ *then* $p'q'r' = 0$ *if and only if* $pqr = 0$.

Proof. If $p'q'r' = 0$, denote by s' an intersection of $L(p'r')$ with one side of $[a'b'c']$. The labelling is chosen so that $s' \in [a'b']$ and $B(s', p', q')$. If $B(a', s', b')$ then $asb = 0$ where s is the inverse image of s', and $p\bar{I}abq$. Hence $a, s, b, p, q \subset_2 E_3$ and it follows that

$$a, s, b, p, q \approx_2 a', s', b', p', q'.$$

Thus $spq = s'p'q' = 0$. Similarly $sqr = 0$, and since $p, q, r, s \subset_2 E_3$, we have $pqr = 0$. If $p' = s'$ the result follows as above, as well as if $s' = b'$, or $s' = a'$.

Conversely, if $pqr = 0$ and $p'q'r' \neq 0$, then if p', q', r' are not on the sides of $[a'b'c']$, some side, say $[a'b']$, is intersected by at least two of the lines $L(p'q')$, $L(q'r')$, and $L(p'r')$. Suppose for instance that $L(p'q')$

and $L(p'r')$ intersect $[a'b']$ at u' and v' respectively. Then $u'p'q' = 0 = v'p'r'$ and by the first part of the proof $upq = 0 = vpr$, where u and v are inverse images of u' and v' under g. Then consideration of quadruple p, q, r, u yields $pru = 0$, and in p, r, u, v we must have $puv = 0$. But $buv = 0$, so $bup = 0$ and from $bua = 0$ it follows that $bpa = 0$. Hence $b'p'a' = 0$, contrary to the choice of p'.

The proof is similar if one or more of p', q', r' are on sides of $[a'b'c']$, unless the three points are on different sides of $[a'b'c']$. In this case if $p' \in [a'b']$, $q' \in [a'c']$, $r' \in [b'c']$, let x' denote the intersection of $[a'r']$ and $[p'q']$. Then from $p'x'q' = 0$ we have $pxq = 0$ which with $pqr = 0$ yields $pxr = 0$. But $axr = 0$ so $apr = 0$, yielding with $apb = 0$ the result $arb = 0$ from which it follows by Lemma 1 that $a'r'b' = 0$, contrary to the choice of r'.

LEMMA 3. *Under the hypotheses and notation of Lemma* 1, *if* $p, q, r \in D(g)$ *such that* $L(p'q')$ *intersects* $[a'b']$ *and* $[a'c']$, *and if* $apq = apr + arq$, *then for each* x *in* $D(g)$

$$xpq = xpr + xrq .$$

Proof. By hypothesis, $L(p'q')$ intersects sides $[a'b']$ and $[a'c']$ of $[a'b'c']$, say in points s' and t' respectively. Let s and t be their respective inverse images. The labelling may be so chosen that $B(p', q', t')$.

If $x'\bar{I}a'p'c'$ it follows using Lemma 1 that $x\bar{I}apc$ and since $a, p, c, t, x \subset_2 E_3$ and $apx = a'p'x'$, $acx = a'c'x'$, and $pcx = p'c'x'$, we have further

$$a, p, c, t, x \approx_2 a', p', c', t', x'$$

and $ptx = p't'x'$. Now $x'\bar{I}p't'c'$ or $x'\bar{I}a'p't'$, and it follows that $x\bar{I}ptc$ or $x\bar{I}apt$ respectively. In the former case since $p, c, t, q, x \subset_2 E_3$ we have indeed

$$p, c, t, q, x \approx_2 p', c', t', q', x'$$

and hence $xpq = x'p'q'$, and a similar argument yields the same result if $x'\bar{I}a'p't'$. Similarly $xpr = x'p'r'$. Repeating the argument using r in place of p, we see that if $x'\bar{I}a'r'c'$ then $xrq = x'r'q'$, and the desired result follows. A similar argument yields the result if $x'\bar{I}a'r'b'$.

The only remaining points z' of $R(g)$ satisfy $z'\bar{I}b'r'c'$. We must show that $zpq = zpr + zrq$ where z is the inverse image of such a z'. Consider a line through z' parallel to $L(p'q')$. For at least one x' on this line, we have $x'\bar{I}a'r'c'$ or $x'\bar{I}a'r'b'$. Consider $[z'x']$ for such an x'. For each $y' \in [z'x']$ the inverse images must satisfy either $ypq = ypr + yrq$, $ypr = ypq + yqr$, or $yrq = yrp + ypq$. The proof is completed by partitioning $[z'x']$ into two sets

$$T_1 = \{y' \in [z'x'] |\ ypq = ypr + yrq\} \quad \text{and}$$
$$T_2 = \{y' \in [z'x'] |\ ypq - ypr = -yqr \text{ or } ypq - yqr = -ypr\}$$

and showing that $T_2 = \emptyset$.

Assertion 1. *T_1 and T_2 are closed sets.*

For if y' were a limit point of T_2 a sequence $\{y'_n\} \subset T_2$ could be found converging to y', i.e., with $y_n pq \neq y_n pr + y_n rq$, and a subsequence, hereafter denoted by $\{y'_n\}$, could be chosen so that all members add in the same way with p, q, r, e.g., $y_n pq - y_n pr = -y_n qr$. Hence since $\{y'_n\}$ converges, it is 2-Cauchy relative to a', b', and c', and thus $\{y_n\}$ is 2-Cauchy relative to a, b, and c, and thus converges to some $y^* \in D(g)$. But then $y^{*\prime} = g(y^*)$ is a limit of $\{y'_n\}$ relative to a', b', and c', and hence $y^{*\prime} = y'$. Thus $\{y_n\}$ converges to y and hence $ypq \neq ypr + yrq$, and $y' \in T_2$. Similarly T_1 is closed.

Assertion 2. $T_1 \cap T_2 = \emptyset$.

For if there were a common point, say y', then both $ypq = ypr + yrq$ and either $ypq - ypr = -yqr$ or $ypq - yqr = -ypr$ would hold, and thus $ypr = 0$ or $yqr = 0$. Assuming the former, then $y'p'r' = 0$, contrary to the choice of x' on the line through z' parallel to $L(p'r')$. Similarly $yqr \neq 0$.

The desired result then follows immediately, for if T_2 were non-empty $[z'x']$ would be expressed as the union of non-null, closed, disjoint sets, contrary to the connectedness of Euclidean segments. This completes the proof of the lemma.

5. Existence of 2-segments.

Making use of the preceding lemmas the main theorem now follows readily.

THEOREM. *Let S be a 2-complete, linearly 2-convex 2-metric space in which each 5 points containing a triadic quadruple and a point linear with one side are 2-congruent with 5 points of E_3. Then each three points a, b, c of S with $abc \neq 0$ are vertices of a 2-segment.*

Proof. It suffices to show that for each p, q, r of $D(g)$, $p'q'r' = pqr$. Suppose that p', q', r', a' are triadic, and choose the labelling so that $p'\bar{I}a'q'r'$. Then there is a point x' in $R(g)$ such that $B(a', p', x')$ and $B(q', x', r')$. Letting x denote the inverse image of x' we have from $B(q', x', r')$, using Lemma 1, that $aqr = aqx + axr$, and by Lemma 3 $pqr = pqx + pxr$. But from $B(a', p', x')$ and Lemma 3 we have $aqx = aqp + pqx$ and $arx = arp + prx$. Thus

$$\begin{aligned} pqr = pqx + pxr &= aqx - aqp + arx - arp \\ &= a'q'x' - a'q'p' + a'r'x' - a'r'p' \\ &= p'q'r', \end{aligned}$$

the desired result. A similar procedure is followed if p', q', r', a' are atriadic, and the proof is complete.

In conclusion, a simple counterexample shows that the requirement that S be linearly 2-convex cannot be replaced by the requirement of the existence of a weak interior point or a strict interior point for each of its triples.

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UNIVERSITY OF MISSOURI AT ST. LOUIS
ST. LOUIS UNIVERSITY

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On the equivalence of an exhaustion principle and the axiom of choice

by

T. Neubrunn (Bratislava)

Introduction. An interesting and very general abstract formulation of the exhaustion principle used in measure theory was given in the paper [1]. The aim of this note is to give, in a direct way, an abstract formulation of the following simple form of the exhaustion principle and to show that it is equivalent to the axiom of choice.

Theorem I (Measure exhaustion theorem). *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a finite measure. Then there exists a set $P \in \mathfrak{M}$ such that $E \subset \mathfrak{M}$, $E \subset X-P$, implies $\mu(E) = 0$.*

Notation. We shall use the notation according to [2]. Let us explain some further symbols which we shall use in the paper.

(a) S will denote a fixed set and $\mathfrak{m}$ a cardinal number such that $\mathfrak{m} \leqslant \overline{\overline{S}}$ ($\overline{\overline{A}}$ denotes the cardinal number of the set A).

(b) $R \subset S \times S$ will be a relation (see [4], p. 54), xRy means $\langle x, y\rangle \in R$, x non Ry means $\langle x, y\rangle \notin R$.

(c) $Y \subset S$ will be a non-void set. $\hat{S}_Y$ stands for a system of subsets E of Y for the elements of which the following is true:

$$x \in E,\ y \in E,\ x \neq y \quad \Rightarrow \quad xRy \text{ or } yRx\,,$$
$$x \in E \ \Rightarrow \ x \operatorname{non} Rx\,.$$

(d) $\varphi^{(\mathfrak{m})}$ stands for an S-valued function with the domain consisting of all $E \in \hat{S}_Y$ for which $\overline{\overline{E}} \leqslant \mathfrak{m}$.

(e) The function $\varphi^{(\mathfrak{m})}$ and the relation R fulfil the following condition:

$$y \in S,\ \varphi^{(\mathfrak{m})}(E)Ry \ \Rightarrow \ xRy \text{ for each } x \in E\,.$$

Now we shall formulate an abstract form of Theorem I.

Principle of exhaustion. *Let S, Y, $\hat{S}_Y$, R and $\varphi^{(\mathfrak{m})}$ fulfil assumptions* (a)-(e). *Let $\overline{\overline{E}} \leqslant \mathfrak{m}$ and $\varphi^{(\mathfrak{m})}(E) \in Y$ for each $E \in \hat{S}_Y$ and let there exist at least one $x \in Y$ for which $x \operatorname{non} Rx$. Then there exists $z \in Y$ such that, for each $y \in Y$, zRy implies yRy.*