Thus, we shall obtain 3^{n+1} congruent perfect sets $P_{t_1,\ldots,t_n,t_{n+1}}$ for which conditions (i)-(iii) are fulfilled.

Having supposed $F_n = \bigcup_{t_1,\ldots,t_n} P_{t_1,t_2,\ldots,t_n}$, we obtain $E \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$

Let $F = \bigcap_{n=1}^{\infty} F_n$. From conditions (i)-(iii) it is easy to see that F is a quite symmetric perfect set.

COROLLARY 1. *For every linear set E of positive Lebesgue measure there exists a perfect set P such that $P+P \subseteq E$.*

COROLLARY 2. *If a closed set E has 0 for a metric density point, then there exists a perfect set P such that $\mathfrak{D}(P) \subseteq E$.*

We take a quite symmetric subset F which is symmetric with respect to 0. There exists a perfect set P such that $P+P = F$.

Since P is also symmetric with respect to 0, we have $P-P = P+ +P^{(0)} = P+P$ and $\mathfrak{D}(P) = (P-P) \cap [0, \infty] = F \cap [0, \infty)$. Thus, $\mathfrak{D}(P) \subset F \subseteq E$.

Corollary 1 solves a problem stated by J. Mycielski in § 4.2 of [1]. J. Mycielski informs us, while this paper was still in preparation, that he found a more direct proof of Corollaries 1 and 2 which is to appear in [2]. I am greatly thankful to him.

References

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Reçu par la Rédaction le 1. 6. 1966

On distributive quasi-lattices

by

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Introduction. In the present paper we are concerned with certain algebras which we call *distributive quasi-lattices* (DQL). These algebras generalize distributive lattices and the main result proved here is a representation theorem which essentially reduce their structure to that of a certain direct spectrum of distributive lattices. To get this we need first analogous results on certain semigroups called here *idempotent quasi-Abelian semigroups.*

Finally, we characterize the independence in idempotent quasi-Abelian semigroups and in distributive quasi-lattices.

§ 1. Idempotent quasi-Abelian semigroups. Let us consider an algebra $\mathfrak{P} = \langle X; \circ \rangle$ in which the fundamental operation $\circ$ satisfies the following axioms:

$$(x \circ y) \circ z = x \circ (y \circ z), \tag{1}$$

$$x \circ x = x, \tag{2}$$

$$x \circ y \circ z = x \circ z \circ y. \tag{3}$$

In view of (1) and (2) $\mathfrak{P}$ is an idempotent semigroup. Such semigroups will be called *idempotent right quasi-Abelian semigroups* (QAS).

If $\circ$ fulfils (1), (2), and

$$x \circ y \circ z = y \circ x \circ z, \tag{3'}$$

then $\mathfrak{P}$ is called an *idempotent left quasi-Abelian semigroup*. In the sequel we deal only with idempotent right quasi-Abelian semigroups. The left case is dual.

We shall give now some examples of such semigroups.

EXAMPLES. 1. Let a be a positive real number. Let the set X of the algebra $\mathfrak{P}$ be $\{x\colon\ 0 < |x| \leqslant a\}$.

Operation $\circ$ will be defined as follows: If $\operatorname{sgn} x = \operatorname{sgn} y$ then $x \circ y = x$; in the remaining cases, $x \circ y = |x|$. It is easy to see that here $\circ$ depends on both variables.

2. Let X be a set of complex numbers different from zero. For two numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ operation $\circ$ will be defined as follows:

$$z_1 \circ z_2 = \frac{z_1}{|z_1|} \max\{|z_1|, |z_2|\} .$$

3. Let I be an arbitrary set $|I| \geqslant 2$ and $X = \{\langle a, A\rangle : a \in A \subseteq I, |A| < \aleph_0\}$. Operation $\circ$ will be defined as follows: $\langle a, A\rangle \circ \langle b, B\rangle = \langle a, A \cup B\rangle$. This is a QAS.

(i) *$x \circ x_1 \circ x_2 \circ \ldots \circ x_n = x \circ x_{i_1} \circ x_{i_2} \circ \ldots \circ x_{i_n}$, where $(i_1, \ldots, i_n)$ is an arbitrary permutation of $(1, 2, \ldots, n)$.*

This follows from (1), (2), (3).

(ii) ([4]) *The algebra $\mathfrak{P}$ is a* QAS *iff it has the form*

$$(\bigcup_{T \in T} T; \circ) \quad \textit{where} \quad T_1 \cap T_2 = 0 \quad \textit{for} \quad T_1 \neq T_2 ,$$

the set T is partially ordered with l.u.b. *by a certain relation $\leqslant^*$, for every T^1 and T_2, where $T_1 \leqslant^* T_2$, there exists a mapping φ_{T_1,T_2} transforming the set T_1 into the set T_2 such that*

$$\varphi_{T,T}(x) = x , \quad \varphi_{T_2,T_3}(\varphi_{T_1,T_2}(x)) = \varphi_{T_1,T_3}(x) ,$$

and if $a \in T_1$, $b \in T_2$ then $a \circ b = \varphi_{T_1,T_{12}}(a)$, where $T_{12} =$ l.u.b. (T_1, T_2).

(iii) *If there exists in an algebra $\mathfrak{A}$ an operation satisfying* (1), (2), (3) *and $\mathfrak{A}$ is not a one-element algebra but it is an idempotent algebra (i.e. each algebraic operation is idempotent), then the operation $\circ$ is either trivial, i.e. $x \circ y = x$, or depends on both variables.*

Indeed, if $\circ$ does not depend on both variables then, because of idempotency of the algebra, it can only be $x \circ y = x$ or $x \circ y = y$. In the second case, however, there is $x \circ y = x \circ x \circ y$, whence, $y = x \circ y = y \circ x = x$.

§ 2. Distributive quasi-lattices.

2.1. Definition and simple properties. An algebra $\mathfrak{Q} = (X; +, \cdot)$, whose fundamental operations satisfy the axioms

(1) $$x + x = x ,$$

(2) $$x \cdot x = x ,$$

(3) $$x + y = y + x ,$$

(4) $$x \cdot y = y \cdot x ,$$

(5) $$(x+y)+z = x+(y+z) ,$$

(6) $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

will be called a *quasi-lattice*. (In the sequel we shall also write xy for $x \cdot y$.)

If, moreover, the operations $+$ and $\cdot$ satisfy the axioms

(7) $$x(y+z) = xy + xz ,$$

(8) $$(x+y)(x+z) = x + yz ,$$

then the algebra $\mathfrak{Q}$ will be called a *distributive quasi-lattice* (DQL). (In the formulation of the above axioms we assume that $\cdot$ links stronger than $+$.) A quasi-lattice is a generalization of a lattice and a DQL is a generalization of a distributive lattice, since lattices fulfill (1)-(6). Lattices fulfill the additional condition

(9) $$x(x+y) = x + xy = x$$

called *axiom of absorption.*

Notice that if in a DQL the operations $+$ and $\cdot$ are equal, then axioms (1)-(8) are reduced to the axioms of a semi-lattice, i.e. of an algebra of sets with addition. The examples of DQL given below are not distributive lattices, and hence this generalization is essential.

EXAMPLES: 1. Let, for positive integers x and y, $[x, y]$ denote the least positive common multiple of x and y, and let (x, y) denote the greatest common divisor of those numbers. Let us consider an algebra $\mathfrak{A} = (X; +, \cdot)$, where the set X is a set of all integers, zero excluded, and operations $+$ and $\cdot$ are defined as follows $x+y = -[|x|, |y|]$ when $x, y < 0$ and $x+y = [|x|, |y|]$ in the remaining cases; $x \cdot y = -(|x|, |y|)$ when $x, y < 0$ and $xy = (|x|, |y|)$ in the remaining cases. It is easily seen that $\mathfrak{A}$ is a DQL which does not satisfy (9) (since we have for instance $(-1)+(-1)\cdot(1) = (-1)+(1) = (1) \neq (-1)$).

2. Let J be the interval $[0, 1]$ and let $\boldsymbol{F}$ be a set of all nowhere vanishing continuous real functions defined on J. We define an algebra $\mathfrak{A} = (\boldsymbol{F}; \cup, \cap)$ by putting $f_1 \cup f_2 = \max\{|f_1|, |f_2|\}$ if one of the functions f_1, f_2 is positive, and $f_1 \cup f_2 = -\max\{|f_1|, |f_2|\}$ if $f_1, f_2 < 0$; $f_1 \cap f_2 = \min\{|f_1|, |f_2|\}$ if at least one of the functions f_1 and f_2 is positive, and finally, $f_1 \cap f_2 = -\min\{|f_1|, |f_2|\}$ if $f_1, f_2 < 0$. It is easy to check that the algebra $\mathfrak{A}$ is a DQL, which does not satisfy (9).

3. Let S be an arbitrary set with $|S| \geqslant 2$. Let us consider an algebra $\mathfrak{Q} = (2^S \times 2^S; +, \cdot)$, where the operation $+$ is defined by $\langle A_1, B_1\rangle + \langle A_2, B_2\rangle = \langle A_1 \cup A_2, B_1 \cup B_2\rangle$ and $\cdot$ by $\langle A_1, B_1\rangle \cdot \langle A_2, B_2\rangle = \langle A_1 \cap A_2, B_1 \cup B_2\rangle$. It is easy to check that $\mathfrak{Q}$ is a DQL which does not satisfy (9). ((9) fails if B_1 and B_2 are such that $B_1 \subset B_2$ and $B_1 \neq B_2$.)

LEMMA 1. *In a* DQL *the following weakened laws of absorption are fulfilled*

(10) $$x + y + xy = x + y ,$$

(11) $$xy(x+y) = xy ,$$

(12) $$x+xy+yz = x+yz\,,$$

(13) $$x(x+y)(y+z) = x(y+z)\,.$$

Proof. It suffices to prove (12), since (13) is dual to (12) and (10) and (11) can be obtained from (12) and (13), respectively, by identifying variables y and z. Because of (8), (7), (5), (2), and (4) we have $x+yz = (x+y)(x+z) = (x+y)x+(x+y)z = x+xy+xz+yz$, whence $x+xy+yz = (x+yz)+xy = x+xy+xz+yz+xy = x+xy+xz+yz = x+yz$, i.e. (12) holds.

THEOREM 1. *Replacing in the set of axioms* (1)-(8) *axiom* (2) *by formula* (10), *we obtain an equivalent set of axioms. Similarly, replacing axiom* (1) *by axiom* (11), *we obtain an equivalent set of axioms.*

We shall prove the first part of Theorem 2. The proof of the second part is dual. By Lemma 1 it suffices to show that (1), (3)-(8) and (10) imply (2). We have $x = x+x = x+x+x\cdot x = x+x\cdot x = (x+x)(x+x) = x\cdot x$.

LEMMA 2. *In a* DQL *the operation of two variables* $x\odot y = x+xy$ *satisfies axioms* (1)-(3) *of* § 1.

Proof. Because of (1)-(8) and (11) we have $(x\odot y)\odot z = (x+xy)+(x+xy)z = x+xy+xz+xyz = x+xyz$, $x\odot(y\odot z) = x+x(y+yz) = x+xy+xyz = x+xyz$, i.e. $(x\odot y)\odot z = x\odot(y\odot z)$, thus (1) of § 1 is satisfied. Formula (2) of § 1 is obvious. Arguing similarly as in the proof of associativity, we obtain $x\odot y\odot z = x+xyz$ and $x\odot z\odot y = x+xzy$, but by (4) the right-hand sides of the last two equalities are equal and (3) of § 1 follows.

LEMMA 3. *The following formulae hold*

$$(x+y)\odot z = (x\odot z)+(y\odot z)\,,$$
$$(x\cdot y)\odot z = (x\odot z)(y\odot z)\,.$$

Proof. We have $(x+y)\odot z = (x+y)+(x+y)\cdot z = x+y+xz+yz = (x+xz)+(y+yz) = (x\odot z)+(y\odot z)$, $(x\cdot y)\odot z = xy+xyz = (xy+xyz)+(xyz+xyz\cdot z) = x(y+yz)+xz(y+yz) = (x+xz)(y+yz) = (x\odot z)(y\odot z)$.

LEMMA 4. *If in a* DQL *the operations* $+$ *and* $\cdot$ *are distinct, then either* $x\odot y = x$ (*thus* $\odot$ *is trivial*) *or* $\odot$ *depends on both variables and is different from each of the operations* $+$ *and* $\cdot$.

Proof. Let us observe that from the assumption of the lemma it follows that our DQL has more than one element. If $\odot$ does not depend on both variables, then because of (iii) it must be $x\odot y = x$. If $\odot$ depends on both variables, then $\odot$ is distinct from $+$ and $\cdot$, because by (10) in the first case we would get $x+y = x+y+xy = (y+x)+xy = y+yx+xy = xy+y = xy+(xy)y = xy$ and in the second case $x+y = x++y+xy = (x+xy)+y = xy+y = y+yx = yx$, which contradicts the supposition.

THEOREM 2. *If there exists in a* DQL *an element* 1 *such that* $1+x = 1$ *and* $1\cdot x = x$ *for every* x *or there exists an element* 0 *such that* $0+x = x$ *and* $0\cdot x = 0$ *for every* x, *then our* DQL *is a distributive lattice.*

Proof. We shall prove the first part of Theorem 3; the proof of the second part is dual. We have $x = 1\cdot x = (1+y)x = 1\cdot x+xy = x+xy$ and we get (9).

2.2. Representation Theorem.

THEOREM 3. *An algebra* $\mathfrak{Q}$ *is a distributive quasi-lattice iff it has the form* $(\bigcup_{T\in \mathbf{T}} T;\ +,\ \cdot)$, *where* $T\cap T' = 0$ *for* $T\neq T'$; $\mathbf{T}$ *is partially ordered with* l.u.b. *by a certain relation* $\leqslant^*$; *each of the sets* T *is a distributive lattice, with respect to* $+$ *and* $\cdot$; *for each pair* T *and* T', *where* $T\leqslant^* T'$, *there exists a homomorphism* $\varphi_{T,T'}\colon T\to T'$ *such that* $\varphi_{T,T}(x) = x$, $\varphi_{T_2,T_3}(\varphi_{T_1,T_2}(x)) = \varphi_{T_1,T_3}(x)$ *and if* $a\in T_1$, $b\in T_2$ *then*

(14) $$a+b = \varphi_{T_1,T_{12}}(a)+\varphi_{T_2,T_{12}}(b)$$

and

(15) $$ab = \varphi_{T_1,T_{12}}(a)\cdot\varphi_{T_2.T_{12}}(b)\,,$$

where l.u.b.$(T_1, T_2) = T_{12}$.

Proof. We have first to prove that algebras of this form are DQL. Notice that $\varphi_{T_1,T_2}(x)\in T_2$. Let us check, for example, formulae (3) and (7). The proofs of the remaining formulae are analogous. We put

$$\text{l.u.b.}\,(T_2, T_3) = T_{23}\,, \qquad \text{l.u.b.}\,(T_1, T_2, T_3) = T_{123}\,.$$

Let $a\in T_1$, $b\in T_2$, $c\in T_3$; we have

$$a+b = \varphi_{T_1,T_{12}}(a)+\varphi_{T_2,T_{12}}(b) = \varphi_{T_2,T_{12}}(b)+\varphi_{T_1,T_{12}}(a) = b+a\,.$$

Thus (3) holds. Further, we have

$$\begin{aligned} a(b+c) &= a\big(\varphi_{T_2,T_{23}}(b)+\varphi_{T_2,T_{23}}(c)\big)\\ &= \varphi_{T_1\text{l.u.b.}T_1,T_{23}}(a)\cdot\varphi_{T_{23}\text{l.u.b.}T_{23},T_1}\big(\varphi_{T_2,T_{23}}(b)+\varphi_{T_3,T_{23}}(c)\big)\\ &= \varphi_{T_1\text{l.u.b.}T_1,T_{23}}(a)\cdot\Big(\varphi_{T_{23},T_{123}}\big(\varphi_{T_2,T_{23}}(b)+\varphi_{T_{23},T_{123}}(\varphi_{T_3,T_{23}}(c))\big)\Big)\\ &= \varphi_{T_1,T_{123}}(a)\big(\varphi_{T_2,T_{123}}(b)+\varphi_{T_3,T_{123}}(c)\big)\\ &= \varphi_{T_1,T_{123}}(a)\cdot\varphi_{T_2,T_{123}}(b)+\varphi_{T_1,T_{123}}(a)\cdot\varphi_{T_3,T_{123}}(c)\,. \end{aligned}$$

Similarly we prove that

$$ab+ac = \varphi_{T_1,T_{123}}(a)\cdot\varphi_{T_2,T_{123}}(b)+\varphi_{T_1,T_{123}}(a)\cdot\varphi_{T_3,T_{123}}(c)\,.$$

Thus (7) follows.

We shall now show that each DQL $\mathfrak{Q} = (X; +, \cdot)$ is of the required form. For this purpose let us consider the operation $x+xy$. Because of Lemma 2 the algebra $(X; x+xy)$ is a idempotent quasi-Abelian semigroup. Let us denote the operation $x+xy$ by $x\odot y$. Thus $(X, \odot)$ has a form described in (ii). Hence $\odot$ is trivial in each of the sets T, i.e. in each of those sets the formula $x\odot y = x+xy = x$ holds, i.e. in each of the sets T the axiom of absorption (9) is satisfied. It is easy to verify that the sets T are closed under $+$ and $\cdot$, Thus in this case T is a distributive lattice. Now we have to show that in our case the mappings $\varphi_{T,T'}(x)$ considered in (ii) are homomorphisms. $\varphi_{T,T'}(x)$ for $x \in T$ and $p \in T'$ has been defined by the formula $\varphi_{T,T'}(x) = x\odot p$. Thus the fact that $\varphi_{T,T'}$ is a homomorphism follows from Lemma 3. It remains to prove (14) and (15) which reduces to $a+b = (a\odot b)+(b\odot a)$ and $a\cdot b = (a\odot b)(b\odot a)$. By (10) we have $(a\odot b)+(b\odot a) = a+ab+b+ba = a+b+ab = a+b$. Similarly $(a\odot b)(b\odot a) = (a+ab)(b+ba) = ab$, q.e.d.

§ 3. Independence in idempotent quasi-Abelian semigroups and in distributive quasi-lattices. In the present section we shall characterize independent sets in QAS and DQL. Notice that if a QAS has more than one element and the operation $\circ$ does not depend on both variables, then because of (iii) of § 1, we have $x\circ y = x$. Our QAS is then a trivial algebra in which each subset is obviously independent. If $\circ$ depends on both variables and the formula $x\circ y = y\circ x$ holds, then this is a semi-lattice. G. Szász proved in [3] that a subset J of a semi-lattice $(S; +)$ is independent iff it does not contain different elements $a_1, \ldots, a_n$ satisfying the relation $a_n \leqslant a_1+a_2+\ldots+a_{n-1}$. Thus we have to consider only the case where $\circ$ depends on both variables, and $x\circ y \neq y\circ x$.

THEOREM 4. *If in a QAS $\mathfrak{P}$ the operation $\circ$ depends on both variables, and $x\circ y \neq y\circ x$, then a subset J in $\mathfrak{P}$ is independent if and only if it does not contain different elements $a_1, \ldots, a_s$ for which at least one of the following relations is satisfied.*

(1) $$a_1\circ\ldots\circ a_{s-1} = a_1\circ\ldots\circ a_{s-1}\circ a_s\,,$$

(2) $$a_1\circ a_2\circ\ldots\circ a_s = a_2\circ a_1\circ\ldots\circ a_s\,.$$

Proof. Because of axioms (1)-(3) of § 1 and because of (i) of § 1 every n-ary algebraic operation g in $\mathfrak{P}$ can be presented in the form

$$g(x_1, \ldots, x_n) = f^n_{i_1,\ldots,i_p}(x_1, \ldots, x_n) = x_{i_1}\circ x_{i_2}\circ\ldots\circ x_{i_p}\,,$$

where $1 \leqslant p \leqslant n$, $1 \leqslant i_k \leqslant n$ $(k = 1, \ldots, p)$ and all i_k are different, and the operation $f^n_{i_1,\ldots,i_p}$ does not depend on the order of the indices $i_2, \ldots, i_p$.

We shall prove that if (i) $\{i_1, \ldots, i_p\} \neq \{j_1, \ldots, j_q\}$ or (ii) $\{i_1, \ldots, i_p\} = \{j_1, \ldots, j_q\}$ but $i_1 \neq j_1$ then

(3) $$f^n_{i_1,\ldots,i_p}(x_1, \ldots, x_n) = f^n_{j_1,\ldots,j_q}(x_1, \ldots, x_n)$$

does not hold.

Indeed, in case (i) there exists an r such that $i_r \notin \{j_1, \ldots, j_q\}$ or there exists an r' such that $j_{r'} \notin \{i_1, \ldots, i_p\}$. Let, for instance, $i_r \notin \{j_1, \ldots, j_q\}$. Then putting x instead of the variable x_{i_r} and y instead of the remaining variables, we obtain from (3) one of the equalities $x\circ y = x$, $y\circ x = x$ and each of them contradicts the assumption that $\circ$ depends on both variables. In case (ii) let us substitute in (3) x for x_{i_1}, y for x_{j_1}, and x for the remaining variables. We shall obtain either $x = y$ or $x\circ y = y\circ x$. Both the equalities contradict the assumption.

Suppose now that J is dependent. Thus there exist in J different elements $b_1, \ldots, b_n$ such that

$$f^n_{i_1,\ldots,i_p}(b_1, \ldots, b_n) = f^n_{j_1,\ldots,j_q}(b_1, \ldots, b_n)$$

and either (i) or (ii) holds.

In case (i) let r be a number such that $i_r \in \{j_1, \ldots, j_q\}$; then $b_{i_1}\circ b_{i_2}\circ\ldots\circ b_{i_p} = b_{j_1}\circ\ldots\circ b_{j_q}$, whence $b_{i_1}\circ\ldots\circ b_{i_p} = b_{i_1}\circ\ldots\circ b_{i_p}\circ b_{i_r} = b_{j_1}\circ\ldots\circ b_{j_q}\circ b_{i_r}$. Hence $b_{j_1}\circ\ldots\circ b_{j_q}\circ b_{i_r} = b_{j_1}\circ\ldots\circ b_{j_q}$, i.e. $b_{j_r} \leqslant_* b_{j_1}\circ\ldots\circ b_{j_q}$; consequently, a relation of the form (1) holds.

In case (ii) a formula of the form (2) obviously holds (see § 1, (i)).

To complete the proof let us assume that J has different elements $a_1, \ldots, a_s$ such that either (1) or (2) holds. Then

$$f^s_{1,2,\ldots,s}(a_1, \ldots, a_s) = f^s_{1,2,\ldots,s-1}(a_1, \ldots, a_s)$$

or

$$f^s_{1,\ldots,s}(a_1, a_2, \ldots, a_s) = f^s_{2,1,3,\ldots,s}(a_1, a_2, \ldots, a_s)\,.$$

Since both equalities cannot be identities in $\mathfrak{P}$, then the set J is dependent, q.e.d.

Let us characterize now the independence in a DQL. A family $\boldsymbol{F}$ of subsets of $\{1, 2, \ldots, n\}$ will be called a *family of relatively incomparable sets* if $S^* \in \boldsymbol{F}$, where $S^* = \bigcup_{S\in\boldsymbol{F}} S$ and $\boldsymbol{F}\setminus\{S^*\}$ is a family of sets incomparable by inclusion.

THEOREM 5. *Every algebraic operation $f(x_1, \ldots, x_n)$ of a DQL can be written in the form*

$$p_{\boldsymbol{F}}(x_1, \ldots, x_n) = \sum_{S\in\boldsymbol{F}} \prod_{j\in S} x_j\,,$$

where $\boldsymbol{F}$ is a family of relatively incomparable subsets of the set $\{1, \ldots, n\}$.

Proof. Because of distributivity of multiplication with respect to addition it is clear that $f(x_1, \ldots, x_n)$ can be presented in the form of a sum of products

$$p_G(x_1, \ldots, x_n) = \sum_{T \in G} \prod_{i \in T} x_i .$$

Formula (10) of Section 2.1 can easily be generalized as follows

$$x_1 + \ldots + x_p = x_1 + x_2 + \ldots + x_p + x_1 x_2 \ldots x_p . \tag{4}$$

Putting in (4) $x_i = x_1^i \ldots x_{l_i}^i$, we obtain

$$\begin{aligned} x_1^1 \cdot \ldots \cdot x_{l_1}^1 + \ldots + x_1^i \cdot \ldots \cdot x_{l_1}^i + \ldots + x_1^p \cdot \ldots \cdot x_{l_p}^p \\ = x_1^1 \cdot \ldots \cdot x_{l_1}^1 + \ldots + x_1^p \cdot \ldots \cdot x_{l_p}^p + x_1^1 \cdot \ldots \cdot x_{l_1}^1 \cdot \ldots \cdot x_1^p \cdot \ldots \cdot x_{l_p}^p . \end{aligned} \tag{5}$$

Formula (5) means that to the sum of products we can always add the product of all products. Hence, to the family $\boldsymbol{G}$ in p_G we can always add the set $S^* = \bigcup_{T \in G} T$. Let us assume that the set S^* has been added. Putting in (12) of Section 2.1 $x = x_1 \ldots x_k$, $y = x_1 \cdot x_2 \cdot \ldots \cdot x_k \cdot \ldots \cdot x_{k+l}$, and $z = x_{k+1} \cdot \ldots \cdot x_{k+l+m}$, we obtain

$$x_1 \ldots x_k + x_1 \cdot \ldots \cdot x_k \cdot \ldots \cdot x_{k+l} + x_1 \cdot \ldots \cdot x_{k+l+m} = x_1 \cdot \ldots x_k + x_1 \cdot \ldots \cdot x_{k+l+m} . \tag{6}$$

It follows that if in $\boldsymbol{G}$ $T_1 \subset T_2 \subset S^*$, then T_2 can be removed. Repeating all this we finally obtain a family $\boldsymbol{F}$ of relatively incomparable sets, q.e.d.

If in a DQL $x + y = x \cdot y$, then it is a semi-lattice, and in semi-lattices the independent sets have been characterized by Szász (as mentioned above).

If $+$ and $\cdot$ are distinct then because of Lemma 4 operation $\odot$ is either trivial or depends on both variables and is different from $+$ and $\cdot$. In the first case our DQL is a distributive lattice and we have the following theorem of E. Marczewski (see [2]): A subset J of a distributive lattice is dependent if and only if it does not contain different elements $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ for which $a_1 \cdot \ldots \cdot a_m \leqslant b_1 + \ldots + b_n$ holds. Thus it remains to consider non-trivial $\odot$.

In order to formulate the condition of independence we define a relation $(a \leqslant b) \overset{\mathrm{df}}{\Longleftrightarrow} (a + b = b)$. It is easy to see that $\leqslant$ is a partial order.

THEOREM 6. *If in a* DQL *the operations $+$ and $\cdot$ are different and the operation $\odot$ depends on both variables, then a subset J of this* DQL *is independent iff it does not contain different elements $a_1, \ldots, a_{m_1}$, $b_1, \ldots, b_{m_2}$ $(m_1 \geqslant 1, m_2 \geqslant 1)$ satisfying at least one of the relations*

$$a_1 a_2 \ldots a_{m_1} b_1 b_2 \ldots b_{m_2} \leqslant b_1 + \ldots + b_{m_2} , \tag{7}$$

$$a_1 \cdot \ldots \cdot a_{m_1} \leqslant a_1 \cdot \ldots \cdot a_{m_1}(b_1 + \ldots + b_{m_2}) . \tag{8}$$

Proof. Let us assume that J is dependent. Then there exist different elements $a_1, \ldots, a_n \in J$ and two different families $\boldsymbol{F}$ and $\boldsymbol{F}'$ of relatively incomparable subsets of the set $\{1, \ldots, n\}$ and such that

$$p_{\boldsymbol{F}}(a_1, \ldots, a_n) = p_{\boldsymbol{F}'}(a_1, \ldots, a_n) . \tag{9}$$

Let $S^* = \bigcup_{S \in \boldsymbol{F}} S$ and $T^* = \bigcup_{T \in \boldsymbol{F}'} T$. Since $\boldsymbol{F} \neq \boldsymbol{F}'$, we have either

(i) $$S^* \neq T^*$$

or

(ii) $$S^* = T^* \quad \text{and} \quad \boldsymbol{F} \setminus \{S^*\} \neq \boldsymbol{F}' \setminus \{T^*\} .$$

In case (i) we have $S^* \setminus T^* \neq 0$ or $T^* \setminus S^* \neq 0$.

Suppose that $S^* \setminus T^* \neq 0$ (the other case is dual). We can assume that

$$T^* \subset S^* \quad \text{and} \quad T^* \neq S^* , \tag{10}$$

since otherwise, multiplying (9) by $\prod_{j \in T^* \setminus S^*} a_j$ we can obtain (10) indeed. Then applying formulae (4) and (5), we get

$$\prod_{j \in S^*} a_j \leqslant p_{\boldsymbol{F}}(a_1, \ldots, a_n) = p_{\boldsymbol{F}'}(a_1, \ldots, a_n) \leqslant \sum_{j \in T^*} a_j .$$

Because of (10) the last relation yields one of the form (7).

In case (ii), by virtue of a lemma of Marczewski (see [2]) we have either

(a) *There exists in the set $\boldsymbol{F} \setminus \{S^*\}$ a set S_0 such that $T \setminus S_0 \neq 0$ for each $T \in \boldsymbol{F}' \setminus \{T^*\}$*

or

(b) *There exists in the set $\boldsymbol{F}' \setminus \{T^*\}$ a set T_0 such that $S \setminus T_0 \neq 0$ for each $S \in \boldsymbol{F} \setminus \{S^*\}$.*

Because of symmetry it suffices to consider (a). We can assume that

$$S_0 \subset T \quad \text{and} \quad S_0 \neq T \quad \text{for every} \quad T \in \boldsymbol{F}' , \tag{11}$$

since otherwise multiplying both sides of (9) by $\prod_{j \in S_0} a_j$ we can obtain (11) indeed. Then we can get

$$\begin{aligned} \prod_{j \in S_0} a_j \leqslant p_{\boldsymbol{F}}(a_1, \ldots, a_n) = p_{\boldsymbol{F}'}(a_1, \ldots, a_n) \\ = \Big(\prod_{j \in S_0} a_j\Big) \cdot p_{\boldsymbol{F}''}(a_1, \ldots, a_n) \leqslant \prod_{j \in S_0} a_j \sum_{T' \in \boldsymbol{F}''} \sum_{j \in T'} a_j , \end{aligned} \tag{12}$$

where $F'' = \{T\setminus S_0\colon\ T \in F'\}$. In the last equality of (12) we took advantage of the possibility of putting the common factor before parenthesis, and in the last inequality we used formulae (4) and (5). Relation (12) obviously yields one of the form (8).

To conclude the proof let us assume that for some different elements $a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}$ of J either (7) or (8) holds. We shall show that J is then dependent in DQL.

Formula (7) can be written in the form $a_1 \ldots a_{m_1} b_1 \ldots b_{m_2} + b_1 + \ldots + b_{m_2} = b_1 + \ldots + b_{m_2}$; if J was independent this would mean that the following identity holds

$$x_1 \ldots x_{m_1} y_1 \ldots y_{m_2} + y_1 + \ldots + y_{m_2} = y_1 + \ldots + y_{m_2}\,.$$

Then putting $x_i = x$, $y_j = y$ we obtain $xy + y = y$ contrary to the assumption that $\odot$ is non-trivial.

Inequality (8) can be written in the form

$$a_1 \ldots a_{m_1} + a_1 \ldots a_{m_1}(b_1 + \ldots + b_{m_2}) = a_1 \ldots a_{m_1}(b_1 + \ldots + b_{m_2})\,.$$

If J was independent this would mean that the following identity holds

$$x_1 \ldots x_{m_1} + x_1 \ldots x_{m_1}(y_1 + \ldots + y_{m_2}) = x_1 \ldots x_{m_1}(y_1 + \ldots + y_{m_2})\,.$$

By applying identification like in the previous case we obtain $x + xy = xy$ contrary to Lemma 4; q.e.d.

References

[1] E. Marczewski, *Independence and homomorphism in abstract algebras*, Fund. Math. 50 (1961), pp. 45-61.

[2] — *Concerning independence in lattices*, Colloq. Math. 10 (1963), pp. 21-23.

[3] G. Szász, *Marczewski's independence in lattices and semi-lattices*, Colloq. Math. 10 (1963), pp. 15-20.

[4] M. Yamada and N. Kimura, *Note on idempotent semigroups. II*, Proc. Jap. Acad. 34 (1958), pp. 110-112.

Reçu par la Rédaction le 7. 6. 1965