On a family of AR-sets

by

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K. Borsuk has recently constructed [3] a family of $2^{\aleph_0}$ 2-dimensional compact AR-sets such that none of them contains a 2-dimensional closed subset homeomorphic to a subset of an other set. As an application of this family it has been shown that there is no universal 2-dimensional AR-set, and that a 3-dimensional cube Q^3 has no r-neighbour on the left.

In the present note we shall show that these results can be extended to every finite dimension, and, with a slight modification, to an infinite dimension. The constructions and the proofs are suitably adapted constructions and proofs of [3].

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1. Zone. Let E^n denote the n-dimensional Euclidean space and E^∞ the Hilbert space. Let Δ be an n-dimensional simplex in E^{n+k}, where k is a natural number or the infinity. Let us denote by E^{*k} the orthogonal complementary space to the hyperspace E^n of Δ in E^{n+k}, and by $l_1, l_2, \ldots, l_k$ the orthogonal basis of the space E^{*k}. Giving a sequence $\{\varepsilon_i\}$ of positive numbers such that $\varepsilon_i < 1/2i$, $i = 1, 2, \ldots, k$, let us denote by L_i the segment with length $2\varepsilon_i$ with the centre in the barycentric centre b_Δ of Δ, and in the directions of vector l_i. By the *k-dimensional zone* of the simplex Δ we understand the minimal convex subset of E^{n+k} containing the set Δ and all segments L_i. It will be denoted by $Z^k(\Delta, \{\varepsilon_i\})$.

In this paper we use only the case where $k = 1$ or $k = \infty$.

Let us consider a homogeneously n-dimensional $(n > 2)$ polytope $P \subset E^{n+k}$ with a triangulation T. By the *boundary* of P we understand the union $P^{\cdot}$ of $(n-1)$-dimensional simplexes of T which are incident exactly with one n-dimensional simplex of T, and by the *edge* of P the closure P^* of the set of all points of P having a neighbourhood in P which cannot be disconnected by a simple arc. Obviously the notions of boundary and edge are independent of the choice of triangulation T.

One can easily see that for ε_i sufficiently small the common part of the zones of different simplexes of T coincides with the common part of the boundaries of those simplexes. The sequence $\{\varepsilon_i\}$ satisfying this condition is said to be *suitable* for the triangulation T. Let $\{\varepsilon_i\}$ be the sequence suitable for the triangulation T. By the *k-dimensional zone* (k a natural number or the infinity) of the triangulation T we shall understand the polytope

$$Z^k(T, \{\varepsilon_i\}) = \bigcup_{\Delta \in T} Z^k(\Delta, \{\varepsilon_i\}).$$

Evidently the polytope P is a deformation retract of the zone $Z^k(T, \{\varepsilon_i\})$.

2. Construction of a finite-dimensional membrane. Given a sequence $\{n_k\}$ of natural numbers >1, let us assign to each natural number $k \geqslant 1$ a polytope P_k, its triangulation T_k and a positive number $\varepsilon^{(k)}$ satisfying the following conditions:

(1_k) *P_k is a homogeneously n-dimensional polytope ($n > 2$) in E^{n+1} which is an AR-set with the boundary $P^{\cdot}_k = P^{\cdot}_1$.*

(2_k) *The edge P^*_k of P_k is a subset of $P_k - P^{\cdot}_k$ and its components are rectilinear segments.*

(3_k) *All simplexes of triangulation T_k of P_k have the diameter $<1/k$. For every point $x \in P_k - P^*_k$ the union of all simplexes of T_k containing x is homeomorphic to an n-dimensional ball.*

(4_k) *$\varepsilon^{(k)}$ is suitable for the triangulation T_k, $\varepsilon^k < 1/k$; for $k \geqslant 2$ if $\Delta^{\wedge} \in T_{k-1}$ and $\Delta \in T_k$ is such n-simplex that $\Delta \subset Z(\Delta^{\wedge}, \varepsilon^{(k-1)})$, then*

$$Z(\Delta, \varepsilon^{(k)}) \not\subseteq Z(\Delta^{\wedge}, \varepsilon^{(k-1)}).$$

Let P_1 denote a polytope in E^{n+1} homeomorphic to an n-dimensional simplex ($n > 2$), T_1 its triangulation with diameters of simplexes <1, and $\varepsilon^{(1)}$ a number <1 suitable for the triangulation T_1.

Let us assume that we have defined the polytope P_k, its triangulation T_k and the number $\varepsilon^{(k)}$ in such a manner that conditions $(1_k), \ldots, (4_k)$ are satisfied. For each n-dimensional simplex Δ of T_k, let us consider a system consisting of n_k n-simplexes $\Delta_1, \Delta_2, \ldots, \Delta_{n_k}$ lying in the interior of the simplex Δ and such that b_Δ is their common vertex and that $\Delta_i \cap \Delta_j = b_\Delta$ for $i \neq j$. Let a_Δ be a point lying on the axis L of $Z(\Delta, \varepsilon^{(k)})$ at a distance $\varepsilon^{(k)}/2$ from b_Δ. Consider the system of $(n+1)n_k$ n-dimensional simplexes $\Delta'_1, \Delta'_2, \ldots, \Delta'_{(n+1)n_k}$; their vertices are: a_Δ and n vertices of the simplex Δ_i, $i = 1, 2, \ldots, n_k$.

One can easily see that the polytope

$$R_\Delta = (\Delta - \bigcup_{i=1}^{n_k} \Delta_i) \cup (\bigcup_{j=1}^{(n+1)n_k} \Delta'_j)$$

is homogeneously n-dimensional and is a deformation retract of the zone $Z(\Delta, \varepsilon^{(k)})$. We set $P_{k+1} = \bigcup_{\Delta \in T_k} R_\Delta$. The set P_{k+1} is said to be a *modification set* of the polytope P_k corresponding to the triangulation T_k and to the number n_k.

As T_{k+1} we choose an arbitrary triangulation of P_{k+1} with simplexes of diameter $<1/(k+1)$ and as $\varepsilon^{(k+1)}$ the number satisfying (4_{k+1}). It is easily seen by the same argument as in [1] that P_{k+1} and its triangulation T_{k+1} satisfy conditions (1_{k+1}), (2_{k+1}) and (3_{k+1}).

The construction implies that the edge P^*_m coincides with the segments $\overline{a_\Delta b_\Delta}$, where $\Delta \in T_k$, $k < m$. Since $\overline{a_\Delta b_\Delta}$ is a common part of nn_k simplexes, we shall say that it is a *segment of ramification of order nn_k*.

It follows from (4_k) that

$$Z(T_{k+1}, \varepsilon^{(k+1)}) \subset Z(T_k, \varepsilon^{(k)}), \quad k = 1, 2, \ldots,$$

i.e. the sequence of the polytopes $\{Z(T_k, \varepsilon^{(k)})\}$ is decreasing.

Every space X homeomorphic to the set

$$P(\{n_k\}) = \bigcap_{k=1}^{\infty} Z(T_k, \varepsilon^{(k)})$$

will be called a *membrane* corresponding to the sequence $\{n_k\}$. The polytope P_1 will be called the *base* of membrane X, the boundary $P^{\cdot}_1$ of base P_1 will be called the *boundary of membrane* X and will be denoted by $X^{\cdot}$.

As in [1] we can prove that every membrane with a base homeomorphic to an n-dimensional simplex is an n-dimensional AR-set.

3. Construction of an infinite-dimensional membrane. Let W be a polytope and T a triangulation of W. We shall say that the polytope W is *strongly connected in the dimension m* if, given two simplexes Δ and Δ' of dimension $\geqslant m$ of T, there exists a sequence of simplexes $\Delta = \Delta_1, \Delta_2, \ldots, \Delta_s = \Delta'$ of T such that for Δ_i, Δ_{i+1} one of them is the face of the other and $\dim \Delta_i \geqslant m$, $i = 2, \ldots, s-1$. Obviously this property is independent of the choice of the triangulation T. One can easily see that if an n-dimensional polytope with a triangulation T is strongly connected in the dimension m, then its $(n-1)$-skeleton, that is the union of all $(n-1)$-simplexes of T, is strongly connected in the dimension $(m-1)$.

Now, given a sequence $\{n_k\}$ of natural numbers >1, let us assign to each natural number $k \geqslant 1$ a polytope P'_k, its triangulation T_k and a sequence $\{\varepsilon^{(k)}_\nu\}$ of positive numbers satisfying the following conditions:

($1'_k$) *P'_k is a homogeneously $(q+k)$-dimensional ($q \geqslant 3$) polytope in E^{q+k+1} and is an AR-set strongly connected in a dimension $\geqslant 3$.*

($2'_k$) *The edge $P^{*\prime}_k$ of P'_k is a union of disjoint rectilinear segments.*

($3'_k$) *The simplexes of the triangulation T_k have diameters $<1/k$. For each point $x \in P'_k - P^{*\prime}_k$ the union of all simplexes of T_k containing x is a homogeneously $(k+q)$-dimensional* AR*-set strongly connected in a dimension $\geqslant 3$.*

($4'_k$) *The sequence $\{\varepsilon_\nu^{(k)}\}$ is suitable for the triangulation T_k, $\varepsilon_\nu^{(k)} < 1/2\nu$, $\varepsilon_\nu^{(k)} < 1/k$, $\nu = 1, 2, \ldots$, $k = 1, 2, \ldots$; if $\Delta^\wedge \in T_{k-1}$ and $\Delta \in T_k$ is a $(q+k)$-dimensional simplex of T_k such that $\Delta \subset Z^\infty(\Delta^\wedge\{\varepsilon_\nu^{(k-1)}\})$, then*

$$Z^\infty(\Delta, \{\varepsilon_\nu^{(k)}\}) \subsetneq Z^\infty(\Delta^\wedge, \{\varepsilon_\nu^{(k-1)}\}).$$

As P'_1 we take a polytope in E^{q+2} homogeneously $(q+1)$-dimensional $(q \geqslant 3)$ which is an AR-set strongly connected in a dimension $\geqslant 3$ satisfying ($3'_1$), as T_1 its triangulation with simplexes of diameter <1; as $\{\varepsilon_\nu^{(1)}\}$ we set a sequence satisfying ($4'_1$).

Let us assume that we have defined the polytope P'_k, its triangulation T_k and the sequence $\{\varepsilon_\nu^{(k)}\}$ in such a manner that conditions ($1'_k$), ..., ($4'_k$) are satisfied. Let $\Delta_1, \Delta_2, \ldots, \Delta_{n_k}$ and $\Delta'_1, \Delta'_2, \ldots, \Delta'_{(q+k+1)n_k}$ denote two systems of $(q+k)$-simplexes defined in the same manner as in the construction of polytope P_{k+1} in the finite-dimensional case. We set

$$R_\Delta = (\Delta - \bigcup_{i=1}^{n_k} \Delta_i) \cup (\bigcup_{j=1}^{(k+q+1)n_k} \Delta'_j),$$

and $R = \bigcup_{\Delta \in T_k} R_\Delta$. Let T'_k be a triangulation of the polytope R with the simplexes of diameter $1/(k+1)$, and let $\varepsilon^{(k+1)}$ be a number suitable for the triangulation T'_k and such that $\varepsilon^{(k+1)} < 1/(k+1)$.

We set $P'_{k+1} = Z(T'_k, \varepsilon^{(k+1)})$ and let T_{k+1} be its triangulation. $Z^\infty(T_k, \{\varepsilon_\nu^{(k)}\})$ is an AR-set and since R is a deformation retract of it, R is also an AR-set and consequently P'_{k+1} is an AR-set. One can easily see that we can choose the triangulations T'_k and T_{k+1} in such a manner that the $(q+k-1)$-dimensional skeleton of T_k is included in the triangulation T_{k+1}. The edge $P^{*\prime}_{k+1}$ is the union of the edge $P^{*\prime}_k$ and all segments of the form $\overline{a_\Delta b_\Delta}$, where Δ is a $(q+k)$-dimensional simplex of T_k. From the construction it follows that the condition ($3'_{k+1}$) is satisfied. We can also choose a sequence $\{\varepsilon_\nu^{(k+1)}\}$ so that ($4'_{k+1}$) is satisfied.

From ($4'_k$) it follows that the sequence of sets $\{Z^\infty(T_k, \{\varepsilon_\nu^{(k)}\})\}$ is decreasing.

Every space X homeomorphic to the set

$$P'(\{n_k\}) = \bigcap_{k=1}^{\infty} Z^\infty(T_k, \{\varepsilon_\nu^{(k)}\})$$

is said to be an *infinite-dimensional membrane* corresponding to the sequence $\{n_k\}$. The polytope P'_1 will be called the *base of the membrane* X, and its boundary P_1 the *boundary of the membrane* X and will be denoted by $X^\cdot$.

From the construction of T_{k+1} it follows that the triangulation T_{k+1} contains the $(q+k-1)$-dimensional skeleton of T_k. Thus $P'(\{n_k\})$ contains the $(q+i-1)$-dimensional skeleton of T_i for $i = 1, 2, \ldots$ and consequently the set $P(\{n_k\})$ has an infinite dimension.

We can say more, namely every point of $P'(\{n_k\})$ has arbitrary small neighbourhoods whose boundaries have finite dimension. Thus $P'(\{n_k\})$ has transfinite dimension ([6]).

The $(q+k+1)$-dimensional simplexes of T_{k+1} contained in $Z^\infty(\Delta, \{\varepsilon_\nu^{(k)}\})$, where Δ is a $(q+k)$-dimensional simplex of T_k form the triangulation $T_{\Delta,k+1}$ of the polytope $Z(T'_k(R_\Delta), \varepsilon^{(k+1)})$, which is an AR-set. It follows that $Z^\infty(T_{\Delta,k+1}, \{\varepsilon_\nu^{(k+1)}\})$ is also an AR-set. Thus there exists a retraction r_Δ^k of the set $Z^\infty(\Delta, \{\varepsilon_\nu^{(k)}\})$ to the set $Z^\infty(T_{\Delta,k+1}, \{\varepsilon_\nu^{(k+1)}\})$. Since $\mathrm{Fr}(\Delta) \subset Z^\infty(T_{k+1}, \{\varepsilon_\nu^{(k+1)}\})$ (the boundary $\mathrm{Fr}(\Delta)$ is taken relatively to the polytope P_k), $r_\Delta^k(X) = x$ for $x \in \mathrm{Fr}(\Delta)$. Setting $r_k(x) = r_\Delta^k(x)$ for $x \in Z^\infty(\Delta, \{\varepsilon_\nu^{(k)}\})$, $\Delta \in T_k$, we infer that the mapping r_k is a retraction of $Z^\infty(T_k, \{\varepsilon_\nu^{(k)}\})$ to $Z^\infty(T_{k+1}, \{\varepsilon_\nu^{(k+1)}\})$ such that for every $\Delta \in T_k$

$$r_k\big(Z^\infty(\Delta, \{\varepsilon_\nu^{(k)}\})\big) = Z^\infty(T_{\Delta,k+1}, \{\varepsilon_\nu^{(k+1)}\}).$$

Let us set $\hat{r_k}(x) = r_k r_{k-1} \ldots r_1(x)$ for $x \in Z^\infty(T_1, \{\varepsilon_\nu^{(1)}\})$. The mapping $\hat{r_k}$ is a retraction of $Z^\infty(T_1, \{\varepsilon_\nu^{(1)}\})$ to $Z^\infty(T_{k+1}, \{\varepsilon_\nu^{(k+1)}\})$, and if $x \in Z^\infty(T_1, \{\varepsilon_\nu^{(1)}\})$ and $\Delta^\wedge$ is a $(q+k+1)$-dimensional simplex of T_{k+1}, such that $\hat{r_k}(x) \in Z^\infty(\Delta^\wedge, \{\varepsilon_\nu^{(k+1)}\})$, then every point $\hat{r_{k+i}}(x)$ for $i = 1, 2, \ldots$, belongs to $Z^\infty(\Delta^\wedge, \{\varepsilon_\nu^{(k+1)}\})$. Since the diameter of the zone $Z^\infty(\Delta^\wedge, \{\varepsilon_\nu^{(k+1)}\}) < 2/(k+1)$, then the sequence $\{\hat{r_k}\}$ converges uniformly to a map r of $Z^\infty(T_1, \{\varepsilon_\nu^{(1)}\})$ to $P'(\{n_k\})$. For every $x \in P'(\{n_k\})$, we have $x \in Z^\infty(T_k, \{\varepsilon_\nu^{(k)}\})$ and $\hat{r_k}(x) = x$ for every $k = 1, 2, \ldots$; consequently r is a retraction of $Z^\infty(T_1\{\varepsilon_\nu^{(1)}\})$ to $P'(\{n_k\})$. Since the zone $Z^\infty(T_1, \{\varepsilon_\nu^{(1)}\})$ is a compact set, we conclude that every infinite-dimensional membrane is a compact AR-set.

4. Bits of a membrane. As in [3], by a *bit of a membrane* X (of finite or infinite dimension) we understand a membrane Y (corresponding to an arbitrary sequence $\{m_k\}$ of naturals $\geqslant 2$) such that $Y \subset X$ and that $Y \cap \overline{X - Y} \subset Y^\cdot$. One can easily see that if a set Q is a union of simplexes of the triangulation T_l homeomorphic to the n-dimensional ball $(n > 2)$ in the finite-dimensional case, and to a homogeneously $(q+l)$-dimensional AR-set strongly connected in the dimension $\geqslant 3$ in the infinite-dimensional case, and if T' denotes the triangulation of Q which consists of simplexes included in T_l, then the constructions of § 2 and § 3 applied only to the simplexes of the triangulations T_{l+k-1} $(k = 1, 2, \ldots)$ lying in $Z^j(T', \{\varepsilon_\nu^{(l)}\})$ $(j = 1, \infty)$ define a set

$$X_Q = P(\{n_k\}) \cap Z^j(T', \{\varepsilon_\nu^{(l)}\}) \quad \text{where} \quad j = 1, \infty,$$

which is a bit of membrane X corresponding to the sequence $\{n_{k+l-1}\}$ and with base Q.

By an *m-membrane* we shall understand a set Y which is a union of m membranes $X_1, \ldots, X_m$ (of finite or infinite dimensions) such that there exists a simple arc L satisfying the condition $X_i \cap X_j = X_i^{\cdot} \cap X_j^{\cdot} = L$, $i \neq j$. The arc L will be called the *edge* of the m-membrane Y and will be denoted by Y^*. By $Y^{\cdot}$ we shall denote the interior of Y^*. The membranes X_i, $i = 1, 2, \ldots, m$, will be called the *wings* of the m-membrane Y. By the *boundary* $Y^{\cdot}$ of the m-membrane Y we shall understand the set $Y^{\cdot} = \bigcup_{i=1}^{m} X_i^{\cdot}$.

By the *m-bit* we shall understand a subset Y of a membrane X which is an m-membrane and $Y \cap \overline{X - Y} \subset Y^{\cdot}$.

We omit the proof of the following lemma because it is completely analogous to the proof which (in the case $n = 2$) is included in [3].

LEMMA 1. *A closed subset Y of an n-dimensional membrane X is n-dimensional if and only if it contains at least one bit of X.*

Obviously every open subset of an infinite-dimensional membrane contains a bit of this membrane.

5. Topological classification of points of a membrane. Let us consider the following subsets of the membrane X (of dimension n or ∞):

X_{I} consists of all points $x \in X$, such that for every $\varepsilon > 0$ there exists a neighbourhood of x in X which is a bit with diameter $< \varepsilon$. The points of X_{I} are said to be *regular points of* X.

X_{II}^m consists of all points $x \in X - X_{\mathrm{I}}$ such that for every $\varepsilon > 0$ there exists a neighbourhood of x in X which is an m-bit with diameter $< \varepsilon$. The points of X_{II}^m are said to be *points of the ramification of order m of the membrane* X.

$X_{\mathrm{III}} = X - X_{\mathrm{I}} - \bigcup_{m=2}^{\infty} X_{\mathrm{II}}^m$. The points of X_{III} are said to be *singular points of* X.

These definitions imply the topological invariance of the sets X_{I}, X_{II}^m and X_{III}. Evidently

$$X = X_{\mathrm{I}} \cup \bigcup_{m-2}^{\infty} X_{\mathrm{II}}^m \cup X_{\mathrm{III}}$$

and the sets X_{I}, $\bigcup_{m=1}^{\infty} X_{\mathrm{II}}^m$ and X_{III} are disjoint.

Let X be an infinite-dimensional membrane (in the finite-dimensional case the argument is analogous) and let $x \in X$. There occurs one of three cases:

(i) There exists a natural l such that x belongs to the $(l+q-1)$-skeleton of triangulation T_l and x does not belong to $P_l^{*\prime}$. It follows that x belongs to the $(j+q-1)$-skeleton of triangulation T_j, and that x does not belong to P_j^* for any $j \geqslant l$. From $(3_j')$ we infer that $x \in X_{\mathrm{I}}$.

(ii) For every $l = 1, 2, \ldots$, the point x belongs to the set

$$\bigcup_{\Delta \in T_l} \big(Z^{\infty}(\Delta, \{\varepsilon_\nu^{(l)}\}) - \Delta^{\cdot}\big);$$

thus $x \in X_{\mathrm{I}}$.

(iii) There exists a natural l such that $x \in P_l^*$. Only in this case x can belong to $\bigcup_{m=l}^{\infty} X_{\mathrm{II}}^m \cup X_{\mathrm{III}}$.

It follows that every point of ramification and every singular point of X belongs to one of the segments of ramification $\overline{a_\Delta b_\Delta}$, and one can easily see that if x belongs to the interior of the segment $\overline{a_\Delta b_\Delta}$, then x is not a singular point. Thus only the end-points of the segments of ramification $\overline{a_\Delta b_\Delta}$ can be singular and consequently the set of singular points is countable.

6. Points of ramification.

LEMMA 2. *There are only two possibilities: either the simple arc disconnects the m-membrane into m components or it does not disconnect it at all.*

Proof. Let us suppose that there exists a simple arc L which disconnects the m-membrane Y. Let us assume at first that the arc L is an irreducible cutting, that is that no subset $L \subsetneq L$ disconnects Y. Let $G_1, G_2, \ldots, G_s$ denote the components of the set $Y - L$. Then L is a common boundary of $G_1, G_2, \ldots, G_s$ ([5], p. 175). Let us show that no regular point of Y belongs to L. Suppose to the contrary that $x \in L \cap Y_{\mathrm{I}}$. Then there exists an arbitrary small neighbourhood which is a bit of Y. We can assume that this neighborhood is of the form X_Q, where Q is an r-dimensional polytope connected in a dimension $\geqslant 3$ if $r > 3$, and Q is homeomorphic to an 3-dimensional ball if $\dim Y = 3$. Since L is the common boundary of the components of $Y - L$, we infer that the set $L \cap X_Q$ disconnects X_Q and since $\mathrm{Fr}(Q) \subset Q^{(r-1)}$ ($Q^{(r-1)}$ denotes the $(r-1)$-skeleton of the polytope Q), the set $L \cap Q^{(r-1)}$ disconnects $Q^{(r-1)}$. But this is impossible because, if $r > 3$, then $Q^{(r-1)}$ is connected in a dimension $\geqslant 2$ and $\dim(L \cap Q^{(r-1)}) < 2$. If $\dim Y = 3$ it is impossible because then L disconnects the set $\mathrm{Fr}(Q)$ homeomorphic to an 2-dimensional sphere.

Thus the arc L contains only the points of ramification or the singular points of Y and, since these points belong to the segments of ramification which are disjoint, we infer that L is included in one of them. If $m = 1$, that is if Y is a membrane, one can see at once that none of the segments of ramification disconnects Y. If $m \geqslant 2$, then by the definition of an m-membrane there exists an edge Y^* which disconnects Y into m components $X_1, X_2, \ldots, X_m$. If there exists in Y an other simple arc L' which disconnects Y into p components and $p \neq m$, then $L' \not\subseteq Y^*$. However, no arc included in the edge disconnects Y.

Now let L be any simple arc in Y. Since Y is an AR-set, L contains an irreducible cutting of Y ([5], pp. 176, 287, 335). If Y is a membrane, that is, if $m = 1$, than L does not disconnect Y, since no irreducible cutting does. If $m > 1$, and if L disconnects Y, then $L \supset Y^*$ and therefore L disconnects Y into k components where $k \geqslant m$. But if $k > m$ then for some i the arc $L \cap Y_i$ disconnects the membrane Y_i which is impossible. Thus $k = m$ and the proof is finished.

Let us put $X_{\mathrm{I}} = X_{\mathrm{II}}^1$.

LEMMA 3. *The sets X_{II}^p and X_{II}^m are disjoint for $p \neq m$.*

Proof. Obviously it suffices to consider the case $p < m$. Suppose to the contrary that $x \in X_{\mathrm{II}}^p \cap X_{\mathrm{II}}^m$. Since $x \in X_{\mathrm{II}}^p$, there exists a neighbourhood Z, which is a p-bit with the wings $Z_1, Z_2, \ldots, Z_p$ and since $x \in X_{\mathrm{II}}^m$, there exists a neighbourhood Z' which is an m-bit with the wings $Z'_1, Z'_2, \ldots, Z'_m$ and such that $Z \supset Z'$. There exists in Z' an arc $Z^{*\prime}$ which disconnects Z' into m components $Z'_1, Z'_2, \ldots, Z'_n$ and since $Z_i \cap Z'_j \neq \emptyset$ for each pair i, j, the arc $Z^{*\prime}$ disconnects Z into at least m components, which by Lemma 2 is impossible because $p < m$.

Lemma 3 implies that if $X = P(\{n_k\})$, then every point lying in the interior of one of the segments $\overline{a_\Delta b_\Delta}$, where Δ is an n-simplex of the triangulation T_k in the finite-dimensional case ($(k+q)$-dimensional simplex of T_k in the infinite-dimensional case) belongs to the set $X_{\mathrm{II}}^{nn_k}$ (to the $X_{\mathrm{II}}^{(k+q)m_k}$ in the infinite-dimensional case). Consequently for each subsequence $\{m_k\}$ of the sequence $\{n_k\}$ and for each open set G of the membrane X, the set $G \cap X_{\mathrm{II}}^{nm_k}$ (the set $G \cap X_{\mathrm{II}}^{(k+q)m_k}$ for $k > k_0$) is of the power $2^{\aleph_0}$. On the other hand, the set $\bigcup_{i \in N} X_{\mathrm{II}}^i \cup X_{\mathrm{III}}$, where N is the set of all naturals which do not belong to the sequence $\{nn_k\}$ (to the sequence $\{(k+q)\cdot n_k\}$), is at most countable because it contains only the end-points of the segments of ramification.

7. Main theorem and corollaries.

THEOREM. *For each n, where n is a natural number or infinity, there exists a function Φ assigning to every real number t an n-dimensional membrane $\Phi(t) \subset E^{n+1}$ in such a manner that, for $t \neq t'$, if n is a finite number then no n-dimensional closed subset of $\Phi(t)$ is homeomorphic to any subset of $\Phi(t')$, and if n is the infinity then no open subset of $\Phi(t)$ is homeomorphic to any subset of $\Phi(t')$ which contains an inner point.*

Proof. If $n = 2$ the theorem was proved in [3]. The proof in the case of $n > 2$, n finite, is completely analogous and will be merely outlined. In the same manner as in [3] we construct a function assigning to every real number t an increasing sequence of natural numbers $\{n_k(t)\}$ such that for $t' < t$ the sequence $\{n_k(t)\}$ contains an infinite sequence $\{m_k\}$ whose terms do not belong to $\{n_k(t')\}$. We set

$$\Phi(t) = P(\{n_k(t)\}).$$

Let us suppose that there exists a homeomorphism h of the subset A of the membrane $\Phi(t)$ to the subset $h(A)$ of the membrane $\Phi(t')$, where A is an n-dimensional closed set. The set A contains a bit Y of the membrane $\Phi(t)$. The points of ramification of order nm_k included in an arbitrary open set of $\Phi(t)$ form a set of the power $2^{\aleph_0}$ while the points of ramification of nm_k included in the membrane $\Phi(t')$ form a set at most countable. Consequently, there exists in the open set $Y - Y^\cdot$ a dense subset R consisting of all points of ramification of order nm_k and such that any point of the set $h(R)$ is neither a singular point nor a point of ramification of order nm_k. Further there exists a point $a \in R$ such that $h(a)$ is an interior point of $h(Y)$. But this is impossible because a is a point of ramification of order nm_k while $h(a)$ is neither a singular point nor a point of ramification of order nm_k.

In the infinite-dimensional case we construct a function assigning to every real number t an increasing sequence of natural numbers $\{n_k(t)\}$ in a little different manner. It is easy to construct an enumeration $\{w_\nu\}$ of all rational numbers $w \in [0, 1]$ such that the set $\{w_{mn}\colon n = 1, 2, \ldots\}$ is dense in the segment $[0, 1]$ for every natural m. Let us define an increasing sequence of natural numbers $\{n_k(t)\}$ by the formula

$$n_k(t) = \min\{n\colon n > n_{k-1}(t),\ |t - w_{(q+k)n}| \leqslant 1/k\}, \qquad t \in [0, 1].$$

It is easy to see that if $t \neq t'$, then the sequence $\{n_k(t)\}$ contains a subsequence $\{m_k\}$ such that the sequence $\{(q+k)m_k\}$ does not belong to $\{n_k(t')\}$. We set $\Phi(t) = P(\{n_k(t)\})$.

Further the proof is the same as in the finite-dimensional case. It suffices only to replace the points of ramification of order nm_k by the points of ramification of order $(q+k)m_k$.

Remark 1. Let $\{\Delta^i\}$, $i = 1, 2, \ldots, j_k$, denotes the set of all n-simplexes (the $(q+k)$-simplexes in the infinite-dimensional case) of the triangulation T_k. Let us assign to each pair (i, k), a number $\nu(i, k) = \bigcup_{\tau=1}^{k-1} j_\tau + i$, $(j_0 = 0)$, and let $\{n_{\nu(i,k)}\}$ be an increasing sequence of natural numbers >1. It is easy to see that if we build the membrane by constructing the modification set on $\Delta^i \in T_k$ by means of $n_{\nu(i,k)}$ simplexes $\Delta_1, \Delta_2, \ldots, \Delta_{n_{\nu(i,k)}}$ that is, if we cut off from every simplex in every path of the construction another number of simplexes, then we obtain a compact AR-set $P(\{n_{\nu(i,k)}\})$ with the following property: if $\dim P(\{n_{\nu(i,k)}\}) = n$, then no two homogeneously n-dimensional different closed subsets of $P(\{n_{\nu(i,k)}\})$ will be homeomorphic, and if $\dim P(\{n_{\nu(i,k)}\}) = \infty$, then no two different open subsets of $P(\{n_{\nu(i,k)}\})$ will be homeomorphic.

Remark 2. The polytopes P_k are smoothly connected ([6], p. 124) in the dimension $(n-1)$, and we can construct a polytope ([6], p. 128) subordinate to the polytope P_k.

Let us denote it by P_k'. Now if $P_{k+1}^{\hat{}}$ denotes the modification set on P_k', then we obtain a sequence of polytopes $\{P_k^{\hat{}}\}$. If we use them for the construction of the membrane in the same manner as in § 2 and, moreover, if we modify in a suitable manner the definition of the subordinate polytope, then we can obtain a family $\Phi(t)$ the elements of which are all irreducible n-dimensional AR-sets [6].

Corollary 1. *Let Y be an arbitrary n-dimensional (infinite-dimensional) ANR-set. There exists a family Ψ consisting of $2^{\aleph_0}$ n-dimensional (infinite-dimensional) ANR-sets such that $Y \in \Psi$ and none of the elements of Ψ contains an n-dimensional closed subset (open subset) homeomorphic to a subset (containing an inner point) of the other element.*

Proof. In the finite-dimensional case it is the consequence of the following theorem [2]:

In an n-dimensional ANR-set every family of n-dimensional subsets which are ANR-sets with the common part of any two of them at most $(n-1)$-dimensional is necessarily at most countable.

From this theorem we infer that the subset of elements of $\Phi(t)$ such that their n-dimensional closed subsets are homeomorphic with the subsets of Y is at most countable. Thus, if we remove these elements from the family $\Phi(t)$, and add the set Y, then we obtain a family Ψ which has the desired property. In the infinite-dimensional case the proof follows and once from the separability of Y.

The theorem cited in the proof of corollary 1 can be used also in the proof of

Corollary 2. *There is no universal n-dimensional AR-set, that is an AR-set which contains all the other n-dimensional AR-sets.*

In the infinite-dimensional case we can formulate this corollary in the following manner:

Corollary 2'. *For an arbitrary infinite-dimensional AR-set X, there exists another infinite-dimensional AR-set Y such that for every injection φ of Y in X the set $\varphi(Y)$ is a non-dense set in X.*

Corollary 3. *The n-dimensional cube has no r-neighbours on the left* [1].

The proof is the same as in [3] in the case of $n = 2$.

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