$\bar{\delta}(b) = 1$ $(a \in A,\ b \in B)$ bzw. (bei Satz 2) $\bar{\gamma}(a) = 1$, $\bar{\delta}(b) \leqslant 1$ $(a \in A,\ b \in B)$. Wir konstruieren $\bar{G}$ komponentenweise. K sei also eine Komponente von G.

Da $\varphi_1 \cup \varphi_2$ bei A gesättigt ist, gibt es keine $a \in A$ mit $\gamma(a) = 0$. Für ein a_0 aus K gelte $\gamma(a_0) = 1$. An a_0 liege eine Kante von φ_1 (o.B.d.A.). Gibt es nun ein a_1 in K, woran keine Kante aus φ_1 stößt, so erschöpft ein Weg von a_0 nach a_1 alle Ecken und Kanten von K; also ist die Zahl der a in K größer als die der b in K, somit $\varphi_1 \cup \varphi_2$ nicht gesättigt bei A. An jedes a von K stößt also eine Kante von φ_1. An jedes b von K stößt eine Kante von φ_1, da ein Weg von a_0 nach b eine ungerade Zahl von Kanten enthält, die abwechselnd zu φ_1 bzw. φ_2 gehören. Die Kanten in K von φ_1 bilden also einen Faktor der Bedingung.

Für jedes a von K gelte $\gamma(a) = 2$. Ist für ein b von K $\delta(b) = 0$, so besteht K nur aus dieser Ecke, und $\varphi_1 \cup \varphi_2$ ist nicht gesättigt bei B. In K gebe es zwei Ecken $b_1 \neq b_2$ mit $\delta(b_1) = \delta(b_2) = 1$. Ein Weg von b_1 nach b_2 erschöpft alle Ecken und Kanten von K. Wie oben (für A) folgt, daß $\varphi_1 \cup \varphi_2$ nicht bei B gesättigt ist. Im Fall von Satz 2 bilden die Kanten von φ_1 in K einen Faktor der Bedingung. Gilt für alle b von K $\delta(b) = 2$, so definieren die Kanten von φ_1 in K einen Faktor der Bedingung. Schließlich gebe es genau ein $b_0 \in B$ in K mit $\gamma(b_0) = 1$. An b_0 stoße eine Kante aus φ_1 (o.B.d.A.). Die Kanten von φ_1 in K bilden dann einen Faktor der Bedingung. Damit sind Satz 1 und Satz 2 bewiesen.

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On regular extensions of operator systems

by

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The aim of this note is to propose a slightly improved version of the extension theorem given by W. Słowikowski in this journal, [8].

I. An *operator system* is an ordered pair $(\boldsymbol{S}, \boldsymbol{X})$, where

a. $\boldsymbol{S}$ is a semigroup with the unit element,

b. $\boldsymbol{X}$ is a commutative group,

c. with each element $A \in \boldsymbol{S}$ there is associated a subgroup $\boldsymbol{G}_A \subset \boldsymbol{X}$, and with each pair $A \in \boldsymbol{S}$, $x \in \boldsymbol{G}_A$ there is associated an element $y \in \boldsymbol{X}$, which is called the *composition* of A and x, $y = Ax$, and

the map $A\colon \boldsymbol{G}_A \to \boldsymbol{X}$ is a homomorphism of $\boldsymbol{G}_A$ onto the whole group $\boldsymbol{X}$.

We do not assume the cancellation law for $\boldsymbol{S}$.

An operator system $(\boldsymbol{S}, \boldsymbol{X})$ is *linear*, if $\boldsymbol{X}$ is a linear space, and $x \in \boldsymbol{G}_A$ implies $\lambda x \in \boldsymbol{G}_A$, and $A(\lambda x) = \lambda Ax$ for every scalar λ.

We may always reduce the theory of linear operator systems $(\boldsymbol{S}, \boldsymbol{X})$ to the theory of ordinary operator systems considering $\boldsymbol{X}$ a group, and enlarging $\boldsymbol{S}$ to include all the operators of multiplication by a scalar.

An operator system is *regular*, if

1. $\boldsymbol{G}_A = \boldsymbol{X}$ for every $A \in \boldsymbol{S}$, and

2. $A(Bx) = (AB)x$, which means that if either member of this equality makes sense then the other does too and both are equal.

In regular operator systems composition is always feasible, and the semigroup operation is compatible with the operation of superposition of the maps $A\colon \boldsymbol{G}_A \to \boldsymbol{X}$.

The notion of operator system was first introduced by Słowikowski in [6] and then described in detail in [8], but it lurks in all the papers cited. We follow the terminology of Słowikowski, but we do not require here *a priori* that the semigroup $\boldsymbol{S}$ be commutative, and we do not assume *a priori* that the semigroup operation in $\boldsymbol{S}$ is identical with the superposition of its elements considered as maps $A\colon \boldsymbol{G}_A \to \boldsymbol{X}$, as it is done in [5]. Dropping these two requirements we avoid some inconvenient conditions on domains $\boldsymbol{G}_A$. An important example of a linear operator system is described in [1]. It is $((D), C)$, where C is the space of all real continuous functions $x(t)$ defined for real t, and (D) is the semigroup of formal differential operators generated by $D = d/dt$. The domain G_{D^k} is the set of functions with continuous k-th derivates. This operator system is not regular. The smallest regular operator system containing it is $((D), Y)$, where Y is the linear space of distributions of finite order. Likewise, in abstract theory of the operator systems, one can develop a theory of extensions to regular operator systems. This is the main point of all the papers cited. We shall state here one more extension theorem, which we claim is still more handy.

II. Let (S, X) be an operator system. An *ideal* (or a *regularizer*) of this operator system is a function $\mathfrak{J}$ which associates every element $A \in S$ with a subgroup $\mathfrak{J}(A) \subset X$, so that

(1) $$\mathfrak{J}(B) \subset \mathfrak{J}(AB),$$

(2) $$\mathfrak{J}(AB) \cap A^{-1}(X) = A^{-1}(\mathfrak{J}(B)),$$

(3) $$\mathfrak{J}(AB) = \mathfrak{J}(BA).$$

If in addition

(4) $$\mathfrak{J}(I) = \{0\},$$

then the ideal J is called an *extensor* (cf. [8], p. 254). In the special case of the operator system $((D), C)$ defined above, the function which associates D^k with the set on which D^k vanishes is an extensor.

We distinguish a special class of operator systems (S, X) which satisfy the following condition:

(*) *For every* $A, B \in S$ *and* $x \in X$ $\quad B^{-1}(A^{-1}x) \cap (AB)^{-1}x \cap (BA)^{-1}x \neq 0$.

This condition amounts to saying that for every $A, B \in S$ and $x \in X$ there exists $u \in G_{AB} \cap G_{BA} \cap G_A$ such that

$$x = (AB)u = (BA)u = B(Au).$$

LEMMA I. *If* (S, X) *is an operator system satisfying* (*) *and if* $\mathfrak{J}$ *is an extensor for* (S, X), *then*

$$A(Bu) - B(Av) = 0 \quad \textit{implies} \quad u - v \in \mathfrak{J}(AB).$$

Proof. Suppose that $A(Bu) = B(Av) = y$. It follows from (*) that there exist elements s, t, w such that

$$y = (AB)s = (BA)s,$$
$$y = (AB)t = A(Bt),$$
$$y = (BA)w = B(Aw).$$

We have

$$\begin{aligned} A(Bu) - A(Bt) = 0 &\quad \text{implies} \quad B(u-t) \in \mathfrak{J}(A) \text{ and } u - t \in \mathfrak{J}(A), \\ B(Aw) - B(Av) = 0 &\quad \text{implies} \quad A(w-v) \in \mathfrak{J}(B) \text{ and } w - v \in \mathfrak{J}(AB), \\ (AB)t - (AB)s = 0 &\quad \text{implies} \quad t - s \in \mathfrak{J}(AB), \\ (BA)s - (BA)w = 0 &\quad \text{implies} \quad s - w \in \mathfrak{J}(AB). \end{aligned}$$

Therefore $u - v = (u-t) + (t-s) + (s-w) + (w-v) \in \mathfrak{J}(AB)$, which is what we wanted to prove.

The main point of this paper is

THE EXTENSION THEOREM. *Let* (S, X) *be an operator system satisfying* (*), *and let* $\mathfrak{J}$ *be an extensor. Then there exists a unique regular operator system* (D, Y) *such that*

a. X *is a subgroup of* Y,

b. D *is a commutative semigroup of endomorphisms for which the domains are the entire group* Y,

c. *For each* $A \in S$ *there exists an* $\bar{A} \in D$ *such that the map* A: $G_A \to X$ *is contained in the map* $\bar{A}$: $Y \to Y$, *and the map*: $A \to \bar{A}$ *makes* D *a homomorphic image of* S,

d. *For every* $y \in Y$ *there exist* $A \in S$ *and* $x \in X$ *such that* $y = \bar{A}x$,

e. $\bar{A}^{-1}(0) \cap X = \mathfrak{J}(A)$.

Proof. We consider the Cartesian product $S \times X$ and a relation of equivalence in it:

$(A, x) \sim (B, y)$ iff there are $u, v \in X$ such that $x = Bu$, $y = Av$ and $u - v \in \mathfrak{J}(AB)$.

We shall show that this relation is really a relation of equivalence, i.e. that it is reflexive, symmetric, and transitive. The first two properties hold, it is trivial. We shall only prove that $\sim$ is transitive. Suppose that

$$(A, x) \sim (B, y), \quad (B, y) \sim (C, z).$$

There exist u, v such that $x = Bu$, $y = Av$, and $u - v \in \mathfrak{J}(AB)$, and there exist v', w such that $y = Cv'$, $z = Bw$, and $v' - w \in \mathfrak{J}(BC)$.

We set $x = Cs$ and $z = At$, and then we have

$$Cs - Bu = 0,$$
$$Av - Cv' = 0,$$
$$At - Dw = 0.$$

Again, setting

$$s = Bs^*, \quad u = Cu^*, \quad v' = Av'^*,$$
$$t = Bt^*, \quad v = Cv^*, \quad w = Aw^*,$$

we have

$$C(Bs^*)-B(Cu^*)=0\,,$$
$$A(Cv^*)-C(Av'^*)=0\,,$$
$$A(Bt^*)-B(Aw^*)=0\,,$$

and

$$C(u^*-v^*)\in\mathfrak{J}(AB),\qquad A(v'^*-w^*)\in\mathfrak{J}(BC)\,.$$

By Lemma I it follows that

$$s^*-u^*\in\mathfrak{J}(BC)\subset\mathfrak{J}(ABC)\,,$$
$$v^*-v'^*\in\mathfrak{J}(AC)\subset\mathfrak{J}(ABC)\,,$$
$$t^*-w^*\in\mathfrak{J}(AB)\subset\mathfrak{J}(ABC)\,,$$

and that

$$u^*-v^*\in\mathfrak{J}(ABC),\qquad v'^*-w^*\in\mathfrak{J}(ABC)\,.$$

We have therefore

$$t^*-s^*=(u^*-s^*)-(w^*-t^*)-(v^*-v'^*)-(u^*-v^*)-(v'^*-w^*)\in\mathfrak{J}(ABC),$$
$$t-s=B(t^*-s^*)\in\mathfrak{J}(AC);$$

this last equality means that $(A,x)\sim(C,z)$, which is what we wanted to prove.

We have also

1. $(A,x)\sim(I,Ax)$ provided Ax exists,
2. $(A,x)\sim(B,y)$ implies $(CA,x)\sim(CB,y)$,
3. $(AB,x)\sim(BA,x)$.

We shall prove, for instance, 2. Suppose that $(A,x)\sim(B,y)$. This means that there are $u,v\in\boldsymbol{X}$ such that

$$x=Bu,\qquad y=Av,\qquad\text{and}\qquad u-v\in\mathfrak{J}(AB)\,.$$

Moreover, it follows from (*) that there are $\tilde{u},\tilde{v}\in\boldsymbol{X}$ such that

$$y=(CA)\tilde{u}=(AC)\tilde{u}=A(C\tilde{u})\,,$$
$$x=(CB)\tilde{v}=(BC)\tilde{v}=B(C\tilde{v})\,.$$

Hence

$$A(C\tilde{v}-v)=0\qquad\text{and}\qquad B(C\tilde{u}-u)=0\,,$$

and hence

$$C\tilde{v}-v\in\mathfrak{J}(A)\subset\mathfrak{J}(AB)\,,$$
$$C\tilde{u}-u\in\mathfrak{J}(B)\subset\mathfrak{J}(AB)\,,$$

and hence

$$C(\tilde{u}-\tilde{v})=C\tilde{u}-C\tilde{v}=(C\tilde{u}-u)-(C\tilde{v}-v)+u-v\in\mathfrak{J}(AB)\,.$$

Therefore, $\tilde{u}-\tilde{v}\in\mathfrak{J}(ABC)\subset\mathfrak{J}(CACB)$, and thus $(CA,x)\sim(CB,y)$, which was to be proved.

We consider the family of equivalence classes. We observe that:

LEMMA II. *Any two classes $[(A,x)]$, $[(B,y)]$ can always be represented by pairs with the same first term AB:*

$$(A,x)\sim(AB,z)\qquad\textit{and}\qquad(B,y)\sim(AB,s)$$

for some $z,s\in\boldsymbol{X}$.

Proof. There exists an element u such that $x=(AB)u$. We have $(A,x)\sim(AB,Au)$. Indeed, the element u has the property that $(AB)u=x$, $Au=Au$, and $u-u\in\mathfrak{J}(A^2B)$.

Likewise, there exists an element v such that $y=(AB)v$, and, by the same argument, $(B,y)\sim(AB,Bv)$. The lemma is therefore proved.

We can provide the family of equivalence classes with a group structure setting

$$[(A,x)]+[(B,y)]=[(AB,Au)]+[(AB,Bv)]=[(AB,Au+Bv)]\,.$$

We denote this group by $\boldsymbol{Y}$. We have the natural embedding

$$\boldsymbol{X}\to x\to[(I,x)]\in\boldsymbol{Y}\,.$$

For every $Q\in\boldsymbol{S}$, the map $\bar{Q}$: $[(A,x)]\to[(AQ,x)]$ is an endomorphism of $\boldsymbol{Y}$. The group $\boldsymbol{D}$ of these endomorphisms is commutative, and it is a homomorphic image of $\boldsymbol{S}$.

III. J. S. e Silva formulated the following theorem (cf. [2], p. 177).

Let $\boldsymbol{S}$ be a semigroup of homomorphisms $A\in\boldsymbol{S}$ defined on subgroups $\boldsymbol{G}_A$ of a given group $\boldsymbol{X}$, and mapping onto the entire group, $A(\boldsymbol{G}_A)=\boldsymbol{X}$ for each A, and let $\boldsymbol{S}$ contain the identity map, $I\in\boldsymbol{S}$. For each A, there is given a subgroup $\mathfrak{N}(A)\subset\boldsymbol{X}$. In order that there exist a group $\boldsymbol{Y}\supset\boldsymbol{X}$ such that each $A\in\boldsymbol{S}$ can be prolonged to an endomorphism $\bar{A}$: $\boldsymbol{Y}\to\boldsymbol{Y}$ so that

(1) $\bar{A}(\boldsymbol{Y})=\boldsymbol{Y}$,
(2) $\bar{A}\bar{B}=\bar{B}\bar{A}=\overline{AB}$,
(3) every element $y\in\boldsymbol{Y}$ has the form $y=\bar{A}x$, $A\in\boldsymbol{S}$, $x\in\boldsymbol{X}$,
(4) $\bar{A}^{-1}(0)\cap\boldsymbol{X}=\mathfrak{N}(A)$,

it is necessary and sufficient that

(α') $\mathfrak{N}(AB)=\mathfrak{N}(BA)$,
(α'') $\mathfrak{N}(AB)\supset B^{-1}\mathfrak{N}(A)$,
(α''') $\mathfrak{N}(AB)\supset\mathfrak{N}(B)$,
(β) for every $x\in\boldsymbol{X}$, $(AB)^{-1}x-(BA)^{-1}x\subset\mathfrak{N}(AB)$,
(γ) $A^{-1}x\subset\mathfrak{N}(AB)$ implies $x\in\mathfrak{N}(B)$,
(δ) $\mathfrak{N}(I)=\{0\}$.

This extension is always unique.

One can easily prove that the function $\mathfrak{N}\colon A \to \mathfrak{N}(A)$ is an extensor for this special operator system. Indeed, the conditions 1, 2, 3 are equivalent to (α'), (α'') and (γ), and (α'''), respectively. The condition (β) can be formulated as follows

$$(**)\qquad A(Bu)-B(Av)=0 \quad \text{implies} \quad u-v \in \mathfrak{N}(AB).$$

This condition imposed on an operator system $(\boldsymbol{S}, \boldsymbol{X})$ is less convenient that (*), since it involves the extensor $\mathfrak{N}$. At first (**) seems to be a weaker condition than (*), but that is only apparent. We can prove the following

LEMMA III. *If $\boldsymbol{S}$ is a semigroup of homomorphisms, and if $(\boldsymbol{S}, \boldsymbol{X})$ is an operator system with an extensor $\mathfrak{N}$ satisfying (**), then setting*

$$\boldsymbol{G}_A^* = \boldsymbol{G}_A + \mathfrak{N}(A),$$

and extending every $A \in \boldsymbol{S}$ in the obvious way to the new domain $\boldsymbol{G}_A^*$ one gets an operator system that already satisfies (*).

Proof. We assume that an operator system $(\boldsymbol{S}, \boldsymbol{X})$ satisfies the condition of Silva (**), and we shall prove that the operator system with the expanded domains $\boldsymbol{G}_A$ satisfies condition (*).

Let x be an arbitrary element from $\boldsymbol{X}$, and $A, B \in \boldsymbol{S}$. Since A and B map their old domains onto $\boldsymbol{X}$, there exist $u \in \boldsymbol{G}_B$ and $v \in \boldsymbol{G}_A$ such that

$$x = A(Bu) = B(Av).$$

We have $u = v+(u-v)$. Let us calculate $B(Au)$ in the system with expanded domains:

$$B(Au) = B\big(A(v+(u-v))\big),$$

and, on the other hand, since $u-v \in \mathfrak{N}(AB)$,

$$x = B(Av) = B(Av)+(BA)(u-v) = (BA)v+(BA)u-v,$$

and since v, $u-v \in \boldsymbol{G}_{BA}^*$, we have $u \in \boldsymbol{G}_{BA}^*$ and

$$x = (BA)v+(BA)(u-v) = (BA)\big(v+(u-v)\big) = (BA)u.$$

Therefore $x = A(Bu) = (AB)u = (BA)u$, condition (*) is satisfied.

It follows immediately from this lemma that our extension theorem implies the theorem formulated by Silva.

We know from Lemma I that if $(\boldsymbol{S}, \boldsymbol{X})$ is an operator system satisfying (*), and if $\mathfrak{J}$ is an extensor for $(\boldsymbol{S}, \boldsymbol{X})$, then condition (**) of Silva is always satisfied. The theorem of Silva however does not imply our theorem. If we do not assume that $(\boldsymbol{S}, \boldsymbol{X})$ is operator system such that the semigroup operation in $\boldsymbol{S}$ is superposition of homomorphisms, then

condition (**) alone is not enough to prove the extension theorem. It is enough only if $\boldsymbol{S}$ is a semigroup of homomorphisms with the operation of superposition.

Our theorem also implies the Fundamental Theorem of Słowikowski (cf. [5], p. 5, and [8], p. 263). Słowikowski assumes *a priori* that he has an operator system with a commutative semigroup. In this respect the difference is that we put the commutativity condition into the extensor. However, he proves his Fundamental Theorem for operator systems $(\boldsymbol{S}, \boldsymbol{X})$ such that $\boldsymbol{S}$ is commutative and the domains satisfy the condition

$$({}^{\,*}_{**})\qquad \begin{array}{c} \boldsymbol{G}_{AB} \subset \boldsymbol{G}_B, \quad B(\boldsymbol{G}_{AB}) \subset \boldsymbol{G}_A, \quad \text{and} \\ (AB)x = A(Bx) \quad \text{for every} \quad x \in \boldsymbol{G}_{AB}. \end{array}$$

The advantage of (*) is obvious.

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