Topologies induced by groups of characters

by

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It is a consequence of the Pontryagin duality theorem that a dense subgroup H of a compact Abelian group G carries the topology induced by the continuous characters on H. By a theorem of Weil (see 1.1 and 1.2 below), a topological group is a dense subgroup of a compact group if and only if it is totally bounded. In section 1 we exploit these observations; that is, by studying groups of continuous characters we derive a number of results about totally bounded Abelian groups. Similarities between compact and totally bounded Abelian groups are to be expected; the differences recorded in 1.6 and 1.7 are striking.

Our point of view allows us to give in section 2 a simplified proof of a slight generalization of the following theorem, proved in [6] by Kertész and Szele. Every infinite Abelian group can be topologized in such a way that it is a first countable (equivalently, metrizable) topological group. In section 3, we use the techniques of section 1 to study and characterize the totally bounded group topologies on the integers.

For another investigation of non locally compact topological Abelian groups and their homomorphisms, we refer the reader to Hejcman [3].

1. Totally bounded groups. The topological groups $(G, \mathfrak{T})$ to be considered in this paper will be Abelian and Hausdorff. A subset B of G is said to be *bounded* if for each neighborhood V of the identity there is a finite subset F of G for which $B \subset \bigcup_{x \in F} xV$. The group $(G, \mathfrak{T})$ is said to be *locally bounded* provided that G contains a bounded nonvoid open set and *totally bounded* (¹) if G itself is bounded.

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(¹) The group $(G, \mathfrak{T})$, if not compact, necessarily admits various distinct compatible uniformities. We restrict our attention here to the so-called left uniformity, consisting of all sets $\{(x, y) \in G \times G : x^{-1}y \in U\}$ $(U \in \mathfrak{T})$. Expressions like "totally bounded" and "Cauchy net", when they occur in this paper, always refer to this uniformity (which, because G is Abelian, coincides with the right uniformity for G).

Evidently every subgroup of a compact (resp. locally compact) group is totally bounded (resp. locally bounded). That no other such groups exist is the content of the following theorem of A. Weil.

1.1. THEOREM (Weil [7]). *Every locally bounded group G is topologically isomorphic with a dense subgroup of a locally compact group $\bar{G}$, which is unique up to a topological isomorphism leaving G fixed pointwise. If G is totally bounded, then $\bar{G}$ is compact.*

It is a well-known theorem (see, for example, 22.17 of [4]) that the family of continuous characters on a locally compact Abelian group separates points. We note in passing that from Weil's theorem it follows that the same assertion is true for a locally bounded Abelian group. For an even more extensive class of topological groups with sufficiently many continuous characters, the reader is referred to [1].

Notation. The group of continuous characters on the topological group $(G, \mathfrak{T})$ will be denoted by the symbol $(G, \mathfrak{T})^\wedge$. The symbol G_d will denote the group G endowed with the discrete topology; thus $(G_d)^\wedge$ consists of *all* characters on G.

For any point-separating subgroup $\mathcal{H}$ of $(G_d)^\wedge$, we let $\mathfrak{T}_{\mathcal{H}}$ be the topology induced on G by $\mathcal{H}$. It is easy to see that $(G, \mathfrak{T}_{\mathcal{H}})$ is a topological group. A basis at the identity consists of all sets

$$U(\mathcal{F}, \varepsilon) = \{x \in G\colon\ |\chi(x)-1| < \varepsilon \text{ for all } \chi \in \mathcal{F}\},$$

where $\varepsilon > 0$ and $\mathcal{F}$ is a finite subset of $\mathcal{H}$.

1.2. THEOREM. *Let $(G, \mathfrak{T})$ be an Abelian topological group and let $\mathcal{H} = (G, \mathfrak{T})^\wedge$. Then the following assertions are equivalent:*

(a) *$(G, \mathfrak{T})$ is totally bounded;*

(b) *$(G, \mathfrak{T})$ can be embedded in a compact group $\bar{G}$;*

(c) *$\mathfrak{T} = \mathfrak{T}_{\mathcal{H}}$.*

Proof. (a) $\Rightarrow$ (b). This is 1.1.

(b) $\Rightarrow$ (c). Each element in $\mathcal{H}$ is the restriction to G of exactly one element of $(\bar{G})^\wedge$. By Pontryagin's duality theorem (24.3 in [4]), the topology on the compact group $\bar{G}$ is that induced by the family $(\bar{G})^\wedge$. Hence the topology $\mathfrak{T}$ on G is that induced by $\mathcal{H}$.

(c) $\Rightarrow$ (a). To show that $(G, \mathfrak{T}_{\mathcal{H}})$ is totally bounded, we choose a basic open set $U(\mathcal{F}, \varepsilon)$ in G. Let $\mathcal{F} = \{\chi_1, \ldots, \chi_m\}$ and find an integer N for which $|e^{2\pi i/N}-1| < \varepsilon/2$. For each m-tuple $a = (k_1, \ldots, k_m)$ of integers $(0 \leqslant k_j \leqslant N-1)$, consider the set

$$X_a = \{x \in G\colon\ |\chi_j(x) - e^{2\pi i k_j/N}| < \varepsilon/2 \text{ for } 1 \leqslant j \leqslant m\}.$$

Whenever X_a is nonvoid, we choose a point x_a in X_a. Then it is easy to see that

$$G \subset \bigcup_a x_a \cdot U(\mathcal{F}, \varepsilon).$$

In fact, for each y in G, there is an m-tuple $a = (k_1, \ldots, k_m)$ for which

$$|\chi_j(y) - e^{2\pi i k_j/N}| < \varepsilon/2 \qquad (1 \leqslant j \leqslant m).$$

The theorem above is a statement about point-separating groups of characters of the form $(G, \mathfrak{T})^\wedge$, where $(G, \mathfrak{T})$ is a totally bounded topological group. The following theorem shows that, in fact, every point-separating group of characters has this form.

1.3. THEOREM. *If G is an Abelian group and $\mathcal{H}$ is any point-separating group of characters on G, then $(G, \mathfrak{T}_{\mathcal{H}})^\wedge = \mathcal{H}$.*

Proof. It is clear that $\mathcal{H} \subset (G, \mathfrak{T}_{\mathcal{H}})^\wedge$.

Consider a character Ψ in $(G, \mathfrak{T}_{\mathcal{H}})^\wedge$. Let τ be the homomorphism of G into the torus $T^{\mathcal{H}}$ defined by $\big(\tau(x)\big)_\chi = \chi(x)$. Since $\mathcal{H}$ separates points, the function τ is one-to-one. Hence the function Θ on $\tau(G)$ defined by

$$\Theta\big(\tau(x)\big) = \Psi(x)$$

is well defined. A routine argument shows that Θ is a continuous character on $\tau(G)$. Being therefore uniformly continuous, the character Θ can be extended to a continuous character on $[\tau(G)]^-$. This character, in turn, can be extended to a continuous character on $T^{\mathcal{H}}$ (see 24.12 of [4]), which we again denote by Θ. By 23.21 of [4] there are elements $\chi_1, \ldots, \chi_n$ of $\mathcal{H}$ and continuous characters $\theta_1, \ldots, \theta_n$ on T such that

$$\Theta\big((t_\chi)\big) = \prod_{k=1}^{n} \theta_k(t_{\chi_k}) \qquad \text{whenever} \qquad (t_\chi) \in T^{\mathcal{H}}.$$

For each $k = 1, \ldots, n$ there is an integer m_k such that $\theta_k(t) = t^{m_k}$ for each t in T. Hence for each x in G we have

$$\Psi(x) = \Theta\big(\tau(x)\big) = \Theta\Big(\big(\chi(x)\big)\Big) = \prod_{k=1}^{n} \theta_k\big(\chi_k(x)\big) = \prod_{k=1}^{n} [\chi_k(x)]^{m_k}.$$

Thus $\Psi = \prod_{k=1}^{n} \chi_k^{m_k}$, so that Ψ belongs to $\mathcal{H}$.

1.4. COROLLARY. *Let G be an Abelian group and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be point-separating subgroups of $(G_d)^\wedge$. If $\mathfrak{T}_{\mathcal{H}_1} \subset \mathfrak{T}_{\mathcal{H}_2}$, then $\mathcal{H}_1 \subset \mathcal{H}_2$; if $\mathfrak{T}_{\mathcal{H}_1} = \mathfrak{T}_{\mathcal{H}_2}$, then $\mathcal{H}_1 = \mathcal{H}_2$.*

1.5. COROLLARY. *If $\mathcal{H}$ is a point-separating group of characters on an Abelian group G, then $(G, \mathfrak{T}_{\mathcal{H}})$ is totally bounded.*

Proof. Use 1.3 and the implication (c) $\Rightarrow$ (a) of 1.2.

The following theorem has an analogue valid for general uniform spaces. We treat the special case to which our techniques are readily adapted.

1.6. THEOREM. *Let $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ be a family of topologies on an Abelian group G, for each of which $(G, \mathcal{T}_\alpha)$ is a totally bounded topological group. If $\mathcal{T}$ is the smallest topology containing each $\mathcal{T}_\alpha$, then $(G, \mathcal{T})$ is a totally bounded topological group.*

Proof. The straightforward proof that $(G, \mathcal{T})$ is a topological group is omitted.

For each α in A, we set $\mathcal{H}_\alpha = (G, \mathcal{T}_\alpha)^\wedge$. Letting $\mathcal{H}$ be the subgroup of $(G_d)^\wedge$ generated by $\bigcup_\alpha \mathcal{H}_\alpha$, one may easily verify that $\mathcal{T} = \mathcal{T}_{\mathcal{H}}$. The conclusion now follows from 1.5.

1.7. THEOREM. *For any infinite Abelian group G, there is a unique (nondiscrete) totally bounded group topology $\mathcal{T}$ such that every homomorphism of G into a totally bounded group is continuous. In fact, $\mathcal{T} = \mathcal{T}_{\mathcal{H}}$ where $\mathcal{H} = (G_d)^\wedge$.*

Proof. Let $\mathcal{H} = (G_d)^\wedge$ and $\mathcal{T} = \mathcal{T}_{\mathcal{H}}$. We first show that a homomorphism f from $(G, \mathcal{T})$ into an arbitrary totally bounded group H is continuous. For every continuous character χ on H, it is obvious that the character $\chi \circ f$ is continuous on $(G, \mathcal{T})$. The continuity of f now follows from the fact that the topology on H is that induced by its continuous characters.

To establish the uniqueness of $\mathcal{T}$, we consider any totally bounded group topology $\mathcal{T}'$ on G different from $\mathcal{T}$. Then $\mathcal{T}' \subsetneq \mathcal{T}$, so the identity map is a discontinuous homomorphism from $(G, \mathcal{T}')$ onto $(G, \mathcal{T})$.

1.8. Discussion. If in 1.6 the words "totally bounded" are replaced throughout by the word "compact", then (except in the trivial case in which the $\mathcal{T}_\alpha$'s all coincide) the conclusion must fail. For otherwise the compact topology $\mathcal{T}$ would properly contain some Hausdorff topology $\mathcal{T}_\alpha$, which is impossible.

Theorem 1.7 also gives rise to a distinction between totally bounded and compact groups. Indeed, every infinite compact Abelian group $(G, \mathcal{T})$ admits a discontinuous homomorphism into the circle group; that is, $(G, \mathcal{T})^\wedge \subsetneq (G_d)^\wedge$. To see this, we recall from [5] Kakutani's identity $\operatorname{card}(G_d)^\wedge = 2^{\operatorname{card}(G)}$. Applying this identity also to the discrete group $(G, \mathcal{T})^\wedge$ we obtain the inequalities

$$\operatorname{card}(G, \mathcal{T})^\wedge < \operatorname{card}(G) < \operatorname{card}(G_d)^\wedge .$$

In corollary 1.4 we gave a one-to-one order-preserving correspondence between totally bounded group topologies for an arbitrary (Abelian) group G and point-separating subgroups of $(G_d)^\wedge$. We now characterize these subgroups of $(G_d)^\wedge$.

1.9. THEOREM. *Let G be an Abelian group and let $\mathcal{H}$ be a subgroup of $(G_d)^\wedge$. Then $\mathcal{H}$ is point-separating if and only if $\mathcal{H}$ is dense in the compact group $(G_d)^\wedge$.*

Proof. For a family $\mathcal{F}$ of characters, we set

$$A(\mathcal{F}) = \{x \in G\colon\ \Psi(x) = 1 \text{ for each } \Psi \text{ in } \mathcal{F}\} .$$

Theorem 24.10 of [4] states that

$$\mathcal{H}^- = \{\chi \in (G_d)^\wedge\colon\ \chi(A(\mathcal{H}^-)) = 1\} .$$

Since $A(\mathcal{H}^-) = A(\mathcal{H})$ and $(G_d)^\wedge$ separates points, we have $\mathcal{H}^- = (G_d)^\wedge$ if and only if $A(\mathcal{H})$ contains only the identity element of G, a condition equivalent to the condition that $\mathcal{H}$ separates points.

1.10. COROLLARY. *If the Abelian group G admits a finite point-separating group $\mathcal{H}$ of characters, then G is finite and is isomorphic with $\mathcal{H}$.*

Proof. By 1.9, we have $\mathcal{H} = \mathcal{H}^- = (G_d)^\wedge$. Since the character group of a finite group is isomorphic to the group itself, we infer that G is isomorphic with $\mathcal{H}$.

1.11. THEOREM. *Let G be an Abelian group and let $\mathcal{H}$ be a point-separating subgroup of $(G_d)^\wedge$. Then $(G, \mathcal{T}_{\mathcal{H}})$ is first countable if and only if $\mathcal{H}$ is countable.*

Proof. If $\mathcal{H}$ is countable, then the family of sets $U(\mathcal{F}, 1/n)$, where $\mathcal{F}$ is a finite subset of $\mathcal{H}$ and $n > 0$, is a countable base at the identity of G.

If there is a countable base at the identity, we may suppose that it has the form $\{U(\mathcal{F}_n, \varepsilon_n)\}_{n=1}^{\infty}$. Let $\mathcal{H}_0$ be the subgroup of $\mathcal{H}$ generated by the set $\bigcup_{n=1}^{\infty} \mathcal{F}_n$; clearly $\mathcal{H}_0$ is countable. Since each $U(\mathcal{F}_n, \varepsilon_n)$ is $\mathcal{T}_{\mathcal{H}_0}$-open, we have $\mathcal{T}_{\mathcal{H}} \subset \mathcal{T}_{\mathcal{H}_0}$. Thus 1.4 implies that $\mathcal{H} \subset \mathcal{H}_0$ and hence $\mathcal{H}$ is countable.

Weil's space $\overline{G}$ mentioned in 1.1 is the completion of G with respect to the uniformity referred to in footnote 1. It is obtained by the adjunction to G of enough points to ensure the convergence (to a point in $\overline{G}$) of each Cauchy net in G. The fact that the group operation $(x, y) \to xy^{-1}$ from $G \times G$ into G admits a continuous extension to $\overline{G} \times \overline{G}$ results from the fact that on $G \times G$ the function is uniformly continuous into the complete space $\overline{G}$.

Weil's construction of $\overline{G}$, couched entirely in the vocabulary of uniform spaces, is of necessity topological in nature; the group-theoretic properties of G are, so to speak, ignored as long as possible. Restricting our attention to the case in which G is totally bounded, we offer in theorem 1.12 below another characterization of $\overline{G}$.

We emphasize that our theorem does not replace or reprove Weil's result. Indeed, our proof depends directly upon the "existence" portion of Weil's theorem.

1.12. Theorem. *Let $(G, \mathcal{T})$ be a totally bounded Abelian group and let $\mathcal{H} = (G, \mathcal{T})^\wedge$. Let ν be the mapping from G into $(\mathcal{H}_d)^\wedge$ defined as follows:*

$$\nu(x)(\chi) = \chi(x) \quad \text{for} \quad \chi \text{ in } \mathcal{H}.$$

Then ν is a topological isomorphism of G onto a dense subgroup of $(\mathcal{H}_d)^\wedge$.

Proof. It is obvious that ν is an algebraic isomorphism. Since $\nu(G)$ is a point-separating group of characters on $\mathcal{H}$, it follows from 1.9 that $\nu(G)$ is dense in $(\mathcal{H}_d)^\wedge$. The topology $\mathcal{T}$ is by 1.2 the topology induced on G by $\mathcal{H}$. Since the topology on $(\mathcal{H}_d)^\wedge$ is also induced by $\mathcal{H}$, ν is a topological isomorphism.

2. On a theorem of Kertész and Szele. The generalization of the Kertész-Szele theorem promised in the introduction is 2.2 below.

2.1. Lemma. *If G is a topological group and H is a dense subgroup of G having a base at the identity of cardinality $\mathfrak{n}$, then G itself has a base at the identity of cardinality $\mathfrak{n}$. In particular, if H is metrizable, then G is metrizable.*

Proof. This is a special case of the following elementary result: If Y is a dense subspace of a regular topological space X, and if $\mathcal{B}$ is a local base in Y for the point $p \in Y$, then $\{\mathrm{int}_X \mathrm{cl}_X U\colon U \in \mathcal{B}\}$ is a local base in X for p.

2.2. Theorem. *Any infinite Abelian group G is algebraically isomorphic with a dense nondiscrete subgroup of a metrizable locally compact group $\bar{G}$.*

Proof. Let H be a countably infinite subgroup of G. Clearly there exists a countably infinite point-separating group $\mathcal{H}$ of characters on H. The nondiscrete totally bounded group $(H, \mathcal{T}_{\mathcal{H}})$ is first countable by 1.11. We now make G into a topological group $(G, \mathcal{T})$ by decreeing that $\mathcal{T}_{\mathcal{H}}$-open neighborhoods of the identity e in $\mathcal{H}$ are a base at e in G. Then H is $\mathcal{T}$-open in G and $(G, \mathcal{T})$ is a nondiscrete locally bounded metrizable group. The locally compact completion $\bar{G}$ of $(G, \mathcal{T})$ given by 1.1 is metrizable by 2.1.

We note that if G is countable, then the group H of the proof above may be taken as G itself, in which case $\bar{G}$ is compact.

3. Locally bounded topologies on the integers. We have already identified, at least in principle, all totally bounded group topologies on an arbitrary Abelian group H: They are the topologies of the form $\mathcal{T}_{\mathcal{H}}$, where $\mathcal{H}$ is a dense subgroup of $(H_d)^\wedge$. For each such $\mathcal{H}$, $\mathcal{T}_{\mathcal{H}}$ is the topology that H receives when the isomorphism ν given in 1.12 from H into $(\mathcal{H}_d)^\wedge$ is declared a homeomorphism. Another way to obtain all totally bounded group topologies on H is to determine all compact groups

containing a dense subgroup algebraically isomorphic with H. Sometimes such a compact group will contain many isomorphs of H that are not topologically isomorphic. The following theorem, an immediate consequence of 1.1, is an aid in determining to what extent this situation may hold.

3.1. Theorem. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be locally compact group topologies for the Abelian group G. If the subgroups H_1 and H_2 of G are $\mathcal{T}_1$-dense and $\mathcal{T}_2$-dense, respectively, and if τ' is a topological isomorphism of $(H_1, \mathcal{T}_1)$ onto $(H_2, \mathcal{T}_2)$, then there is a unique extension τ of τ' that is a topological isomorphism of $(G, \mathcal{T}_1)$ onto $(G, \mathcal{T}_2)$.*

We now show that the hypotheses of 3.1 may arise nontrivially. Consider the circle group T with its ordinary compact topology $\mathcal{T}_1$. Let σ be a discontinuous automorphism of T that leaves the group $H = \{e^{2\pi i r}\colon r$ is rational$\}$ pointwise fixed, and let $\mathcal{T}_2$ be the topology on T under which σ is a homeomorphism. Then H is $\mathcal{T}_1$-dense and $\mathcal{T}_2$-dense in G, and the identity mapping of $(H, \mathcal{T}_1)$ onto $(H, \mathcal{T}_2)$ is a topological isomorphism. Note that the extension τ given by 3.1 is not the identity mapping on G; in fact, $\tau = \sigma$.

The non-algebraic analogues of theorem 3.1 and the example above for uniform spaces are given in 15. O of [2].

3.2. Definitions. The element y of the topological group G will be called a *topological generator* of G if y generates a dense subgroup of G. A group possessing a topological generator is called a *monothetic* group.

We denote the group of integers by Z. If y is a topological generator of a group G, we write $\mathcal{T}_y$ for the topology on Z under which the mapping $n \to y^n$ is a topological isomorphism from Z into G.

3.3. Corollary (to 3.1). *Let y_1 and y_2 be topological generators of the compact monothetic group G. Then $\mathcal{T}_{y_1} = \mathcal{T}_{y_2}$ if and only if there is a topological automorphism τ of G mapping y_1 onto y_2.*

Proof. For $k = 1, 2$, let $H_k = \{y_k^n\colon n \in Z\}$ and apply 3.1.

3.4. Theorem. *Let y and z be elements of T having infinite order. Then $\mathcal{T}_z \subset \mathcal{T}_y$ if and only if $z = y^m$ for some integer m. Thus $\mathcal{T}_z = \mathcal{T}_y$ if and only if $y = z$ or $y = z^{-1}$.*

Proof. Let $\mathcal{H}_y$ and $\mathcal{H}_z$ be the subgroups of $(Z_d)^\wedge$ generated by the characters whose values at 1 are y and z, respectively. It is easy to see that $\mathcal{T}_y = \mathcal{T}_{\mathcal{H}_y}$ and $\mathcal{T}_z = \mathcal{T}_{\mathcal{H}_z}$. Since the inclusion $\mathcal{H}_z \subset \mathcal{H}_y$ holds if and only if $z = y^m$ for some integer m, the result follows from 1.4.

Theorem 3.4 above shows that Z receives many different totally bounded group topologies from T. The compact group Δ_p of p-adic integers (defined, for example, in 10.2 of [4]) is also monothetic, but Z can receive only one topology from it.

3.5. THEOREM. *If y and z are topological generators of Δ_p, then $\mathcal{C}_y = \mathcal{C}_z$.*

Proof. From 10.16.a and 26.18.e of [4] one readily sees that there is a topological automorphism of Δ_p mapping y onto z. Now apply 3.3.

The topologies that Z receives from T and Δ_p are clearly 0-dimensional. The following theorem shows that all totally bounded group topologies on Z are 0-dimensional.

3.6. THEOREM. *Every totally bounded group topology on an arbitrary countable Abelian group G is 0-dimensional.*

Proof. The topology in question has the form $\mathcal{C}_{\mathcal{K}}$ for some point-separating group $\mathcal{K}$ of characters on G. For χ in $\mathcal{K}$, it is obvious that $\chi(G)$ is a countable subgroup of T. Hence there is a sequence $\{t_n(\chi)\}_{n=1}^{\infty}$ in $T\setminus\chi(G)$ such that $|1-t_n(\chi)| < 1/n$ for $n = 1, 2, \ldots$ It is easy to see that each set $U(\{\chi\}, |1-t_n(\chi)|)$ is open and closed in $\mathcal{C}_{\mathcal{K}}$. Since

$$\bigcap_{\chi\in\mathcal{F}} U(\{\chi\}, |1-t_n(\chi)|) \subset U(\mathcal{F}, 1/n)$$

for any finite subset $\mathcal{F}$ of $\mathcal{K}$ and $n > 0$, the open and closed sets of the form $\bigcup_{\chi\in\mathcal{F}} U(\{\chi\}, |1-t_n(\chi)|)$ constitute a base at the identity of G for $\mathcal{C}_{\mathcal{K}}$.

Added June 19, 1964. Responding to a question posed in conversation, Adam Kleppner has pointed out that the existence of non locally bounded group topologies for Z has been known for many years. The following example was kindly communicated to us by Shizuo Kakutani. Let f be any boundet real-valued function on Z that is not almost periodic and yet for which the set of ε-translation numbers in unbounded for each $\varepsilon > 0$. The expression

$$d(m, n) = \sup_{k\in Z} |f(m+k) - f(n+k)|$$

defines a metrizable topology of this type. That there are such functions was schown by Harald Bohr.

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