that in general χ_i takes $(\overline{K_0 - K_{i+1}})$ onto $(\overline{Q - C_{t_{i+1}}})$, I_0 onto A, and $E_{p/2^q}$ onto $D'_{p/2^q}$ if $2^q \leqslant 2^i$. The limit of the χ_i's is the required extension taking Q onto itself.

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On a special metric and dimension *

by

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Once we have characterized [3] a metric space of covering dimension $\leqslant n$ by means of a special metric as follows:

A metric space R has dim $\leqslant n$ *if and only if we can introduce a metric ϱ in R which satisfies the following condition:*

For every $\varepsilon > 0$ and for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R satisfying

$$\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon, \qquad i = 1, \ldots, n+2\ (^1)$$

there is a pair of indices i, j such that

$$\varrho(y_i, y_j) < \varepsilon \qquad (i \neq j).$$

For separable metric spaces, this theorem was simplified by J. de Groot [2] as follows:

A separable metric space R has dim $\leqslant n$ *if and only if we can introduce a totally bounded metric ϱ in R which satisfies the following condition:*

For every $n+3$ points $x, y_1, \ldots, y_{n+2}$ in R, there is a triplet of indices i, j, k such that

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_k) \qquad (i \neq j).$$

The former theorem is not so smart though it is valid for every metric space. The problem of generalizing the latter theorem, omitting the condition of totally boundedness, to general metric spaces still remains unanswered. However, we can characterize the dimension of a general metric space by a metric satisfying a stronger condition as follows.

THEOREM. *A metric space R has* dim $\leqslant n$ *if and only if we can introduce a metric ϱ in R which satisfies the following condition:*

For every $n+3$ points $x, y_1, \ldots, y_{n+2}$ in R, there is a pair of indices i, j such that

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_j) \qquad (i \neq j).$$

* The content of this paper is a development in detail of our brief note *On a special metric characterizing a metric space of* dim $\leqslant n$, Proc. of Japan Acad. 39 (1963).

(1) $S_{\varepsilon/2}(x) = \{y \mid \varrho(x, y) < \varepsilon/2\}$.

Proof. Sufficiency. We shall prove that the following weaker condition is sufficient for R to have $\dim \leqslant n$.

We can introduce a metric ϱ in R such that for a definite number $\delta > 0$ and for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ in R with $\varrho(x, y_j) < \delta$, $j = 1, \ldots, n+2$, there is a pair of indices i, j such that

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_j) \quad (i \neq j).$$

For $n = 0$, the condition for ϱ implies that we can introduce a non-Archimedean metric in R. Hence, by de Groot's theorem [1], R has $\dim \leqslant 0$.

To prove our assertion by induction with respect to the number n we assume its validity and suppose ϱ is a metric satisfying the condition for δ and for every $n+4$ points $x, y_1, \ldots, y_{n+3}$ in R. Let F be a given closed set of R; then for an arbitrary positive number $\varepsilon < \delta$ we consider the open neighborhood $S_\varepsilon(F) = \{S_\varepsilon(p) \mid p \in F\}$ of F. To assert $\dim R \leqslant n+1$ it suffices to show

$$\dim \mathrm{B}S_\varepsilon(F) \leqslant n,$$

where $\mathrm{B}S_\varepsilon(F)$ denotes the boundary of $S_\varepsilon(F)$. For if we can prove the assertion, then for given disjoint closed sets F, G and for an m_0 with $1/m_0 < \delta$

$$S = \bigcup_{m=m_0}^{\infty} [S_{1/m}(F) - \overline{S_{1/m}(G)}]$$

is an open set such that

$$F \subset S \subset R - G,$$

$$B(S) \subset [\bigcup_{m=m_0}^{\infty} \mathrm{B}S_{1/m}(F)] \cup [\bigcup_{m=m_0}^{\infty} \mathrm{B}S_{1/m}(G)];$$

hence by the sum-theorem we obtain $\dim B(S) \leqslant n$ proving $\dim R \leqslant n+1$. If we denied the assertion, then by the inductive assumption there would be $n+3$ points $x, y_1, \ldots, y_{n+2}$ in $\mathrm{B}S_\varepsilon(F)$ such that

$$\varrho(x, y_j) < \varepsilon, \qquad \varrho(y_i, y_j) > \varrho(x, y_j)$$

for every pair i, j with $i \neq j$. We choose a small neighborhood $U(x)$ of x such that for every point x' of $U(x)$,

$$\varrho(x', y_j) < \varepsilon$$

and

$$\varrho(y_i, y_j) > \varrho(x', y_j), \quad i \neq j,$$

hold. Then there exists a point y_{n+3} of F satisfying

$$S_\varepsilon(y_{n+3}) \cap U(x) \neq \emptyset.$$

Take a point $x' \in S_\varepsilon(y_{n+3}) \cap U(x)$; then

$$\begin{aligned}
&\varrho(x', y_j) < \varepsilon < \delta, && j = 1, \ldots, n+3, \\
&\varrho(y_i, y_j) > \varrho(x', y_j), && i \neq j,\ 1 \leqslant i, j \leqslant n+2, \\
&\varrho(y_i, y_{n+3}) \geqslant \varepsilon > \varrho(x', y_{n+3}), && i = 1, \ldots, n+2, \\
&\varrho(y_{n+3}, y_j) \geqslant \varepsilon > \varrho(x', y_j), && j = 1, \ldots, n+2.
\end{aligned}$$

But this contradicts the property of ϱ. Therefore we can conclude that

$$\dim \mathrm{B}S_\varepsilon(F) \leqslant n$$

and accordingly

$$\dim R \leqslant n+1.$$

To carry out the proof of necessity we need the following terminology which is a slight modification of the concept 'rank' of a collection of sets established in [5] or [6].

Definition. Let $\mathfrak{S}$ be a collection of subsets of R. We call the Rank of $\mathfrak{S}$ not greater than n and denote it by

$$\operatorname{Rank} \mathfrak{S} \leqslant n$$

if $\mathfrak{S}$ has the following property:

If

$$U_1, \ldots, U_l \in \mathfrak{S}, \qquad \overline{U}_1 \cap \ldots \cap \overline{U}_l \neq \emptyset,$$

$$U_i \not\subset U_j \text{ for every pair } i, j \text{ with } i \neq j,$$

then $l \leqslant n$.

Incidentally, we call a collection $\{U_1, \ldots, U_l\}$ of subsets or the subsets themselves independent if $U_i \not\subset U_j$ for every pair i, j with $i \neq j$.

Proof of necessity Let R be a metric space of $\dim \leqslant n$; then we shall explain how to define a metric ϱ of R which satisfies the desired condition. By use of the decomposition theorem we decompose R as

$$R = \bigcup_{i=1}^{n+1} A_i$$

for 0-dimensional spaces A_i, $i = 1, \ldots, n+1$.

The point of the proof is to define a sequence

(1) $$\mathfrak{V}_1 > \mathfrak{V}_2^{**} > \mathfrak{V}_2 > \mathfrak{V}_3^{**} > \ldots \ (^2)$$

of locally finite open coverings such that

(2) $$\operatorname{mesh} \mathfrak{V}_m = \sup\{\delta(V) \mid V \in \mathfrak{V}_m\} < 1/m$$

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(2) As for terminologies and notations about coverings, see J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940.

and a locally finite open covering $\mathfrak{S}'_{m_1\ldots m_p} = \{S'_{m_1\ldots m_p}(V) | V \in \mathfrak{V}_{m_1}\}$ for each finite sequence $m_1, \ldots, m_p$ of integers with $1 \leqslant m_1 < m_2 < \ldots < m_p$ such that

(3) $$\mathfrak{S}'_m = \mathfrak{V}_m \quad (S'_m(V) = V \text{ for } V \in \mathfrak{V}_m),$$

(4) $$\text{if} \quad 2^{-m_1} + \ldots + 2^{-m_p} > 2^{-l_1} + \ldots + 2^{-l_q}, \quad \text{then} \quad \mathfrak{S}'_{m_1\ldots m_p} > \mathfrak{S}'_{l_1\ldots l_q},$$

(5) $$\operatorname{Rank} \bigcup \{\mathfrak{S}'_{m_1\ldots m_p} | 1 \leqslant m_1 < m_2 < \ldots < m_p\} \leqslant n+1.$$

We shall define, by induction on the number m, locally finite open coverings $\mathfrak{V}_1, \ldots, \mathfrak{V}_m$ and $\{\mathfrak{S}'_{m_1\ldots m_p} | 1 \leqslant m_1 < \ldots < m_p \leqslant m\}$ satisfying the following condition besides (1), (2), (3), (4). If we put, for brevity,

$$\{\mathfrak{S}'_{m_1\ldots m_p} | 1 \leqslant m_1 < \ldots < m_p \leqslant m\} = \{\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m)}\},$$

then

(6) $$U, U' \in \mathfrak{S}_i \text{ implies either } U \not\subset U' \text{ or } U = U',$$

(7) $$U, U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)} \text{ and } U \subsetneqq U' \text{ imply } \overline{U} \subset U',$$

(8) $$\operatorname{Rank} \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)} \leqslant n+1,$$

(9) $$\operatorname{ord}_p B(\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}) \leqslant i-1 \quad \text{for} \quad p \in A_i,$$

where for a collection $\mathfrak{S}$ of subsets and a point p of R, $B(\mathfrak{S})$ denotes the collection $\{B(U) | U \in \mathfrak{S}\}$ [3] and $\operatorname{ord}_p \mathfrak{S}$ denotes the number of the members of $\mathfrak{S}$ which contain p.

For $m = 1$ we construct a locally finite open covering $\mathfrak{V}'_1 = \{V'_\alpha | \alpha \in A'_1\}$ with

$$\operatorname{ord} \overline{\mathfrak{V}}'_1 \leqslant n+1, \quad \operatorname{mesh} \mathfrak{V}'_1 < 1,$$

where for a collection $\mathfrak{V}$ of subsets $\overline{\mathfrak{V}}$ denotes the collection $\{\overline{V} | V \in \mathfrak{V}\}$. Then there is an open covering $\mathfrak{V}''_1 = \{V''_\alpha | \alpha \in A'_1\}$ for which

$$\overline{V}''_\alpha \subset V'_\alpha.$$

Then, as we have seen in [4], Lemma 2.1, we can construct open sets V'''_α, $\alpha \in A'_1$, such that

$$\overline{V}''_\alpha \subset V'''_\alpha \subset V'_\alpha$$
$$\operatorname{ord}_p \{B(V'''_\alpha) | \alpha \in A'_1\} \leqslant i-1 \quad \text{for} \quad p \in A_i.$$

We choose from $\{V'''_\alpha | \alpha \in A'_1\}$ the members V'''_α for which $V'''_\alpha \subset V'''_\beta (\beta \in A'_1)$ implies $V'''_\alpha = V'''_\beta$ and make a collection $\mathfrak{V}_1$ out of them. Then it is easy to see that $\mathfrak{V}_1 = \mathfrak{S}'_1$ is a locally finite open covering satisfying all the required conditions. Now, let us assume that we have already defined $\mathfrak{V}_1, \ldots, \mathfrak{V}_m$ and $\{\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m)}\}$ to define $\mathfrak{V}_{m+1}$ and

$$\{\mathfrak{S}_{k(m)+1}, \ldots, \mathfrak{S}_{k(m+1)}\} = \{\mathfrak{S}'_{m_1\ldots m_p} | 1 \leqslant m_1 < \ldots < m_p = m+1\}.$$

[3] We often call a collection of subsets merely a *collection*. $B(U)$ denotes the boundary of U.

First, we construct a locally finite open covering $\mathfrak{V}$ with

$$\operatorname{mesh} \mathfrak{V} < 1/(m+1), \quad \mathfrak{V}^{**} < \mathfrak{V}_m$$

such that

(10) $$\text{if } U_1, \ldots, U_l \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)} \text{ and } \overline{U}_1 \cap \ldots \cap \overline{U}_l = \emptyset, \text{ then}$$
$$S^3(U_1, \mathfrak{V}) \cap \ldots \cap S^3(U_l, \mathfrak{V}) = \emptyset,$$

(11) for each $p \in R$, $S^3(p, \mathfrak{V})$ meets only finitely many members of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$,

(12) $$\text{if } U, U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)} \text{ and } \overline{U} \subset U', \text{ then } \overline{S^2(U, \mathfrak{V})} \subset U',$$

(13) $$\text{if } U, U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)} \quad \text{and} \quad \overline{U} \not\subset U', \quad \text{then}$$
$$S^2(U, \mathfrak{V}) \not\subset U'.$$

Since $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$ is locally finite, we can choose such a $\mathfrak{V}$ as follows.

By use of the local finiteness of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$, we can easily choose a sufficiently refined $\mathfrak{V}$ to satisfy (11) and (12) besides $\operatorname{mesh} \mathfrak{V} < 1/(m+1)$ and $\mathfrak{V}^{**} < \mathfrak{V}_m$.

We shall show how to define $\mathfrak{V}$ to satisfy (10) and (13), too. For each point p of R, we define a set $S(p)$ by

$$S(p) = \bigcap \{R - \overline{U} | p \notin \overline{U}, \ U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}\};$$

then

$$\mathfrak{S} = \{S(p) | p \in R\}$$

is an open covering since $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$ is locally finite. If we choose $\mathfrak{V}$ such that $\mathfrak{V}^{**} < \mathfrak{S}$, then $\mathfrak{V}$ clearly satisfies (10).

For each pair $U, U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$ satisfying $\overline{U} \not\subset U'$, we assign a point

$$p(U, U') \in U' - \overline{U}.$$

For a definite member U of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$,

$$N(\overline{U}) = R - \bigcup \{p(U, U') | \overline{U} \not\subset U'\}$$

is an open neighborhood of $\overline{U}$ by virtue of the local finiteness of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$. We choose an open neighborhood $N'(\overline{U})$ of $\overline{U}$ such that

$$N'(\overline{U}) \subset N(\overline{U}),$$
$$\mathfrak{N} = \{N'(\overline{U}) | U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}\} \text{ is locally finite.}$$

Now, we can choose $\mathfrak{V}$ such that

$$S^2(\overline{U}, \mathfrak{V}) = S^2(U, \mathfrak{V}) \subset N'(\overline{U}) \quad \text{for every } U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)};$$

then $\mathfrak{V}$ clearly satisfies (13). We note that if $\mathfrak{V}$ satisfies one of (10), (11), (12) or (13), then every refinement of $\mathfrak{V}$ also satisfies the same condition. Thus we can construct $\mathfrak{V}$ satisfying all the desired conditions.

Let $\mathfrak{V} = \{V_\alpha | \alpha \in A\}$; then we can construct an open covering $\mathfrak{W}' = \{W'_\alpha | \alpha \in A\}$ satisfying
$$\overline{W}'_\alpha \subset V_\alpha .$$
Since by (11) each $S(V_\alpha, \mathfrak{V})$ meets at most finitely many of $U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$, for each of those U we can define an open set $V_\alpha(U)$ such that

(14) $$\overline{W}'_\alpha \subset V_\alpha(U) \subset \overline{V_\alpha(U)} \subset V_\alpha ,$$

(15) if $U \neq U'$, then either $\overline{V_\alpha(U)} \subset V_\alpha(U')$ or $\overline{V_\alpha(U')} \subset V_\alpha(U)$,

(16) if $U \in \mathfrak{S}'_{m_1 \ldots m_p}$, $U' \in \mathfrak{S}'_{l_1 \ldots l_q}$, $2^{-m_1} + \ldots + 2^{-m_p} < 2^{-l_1} + \ldots + 2^{-l_q}$, then $\overline{V_\alpha(U)} \subset V_\alpha(U')$.

By virtue of (9) we can choose $V_\alpha(U)$ satisfying the following condition, too,

(17) $$\operatorname{ord}_p B(\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)} \cup \mathfrak{V}') \leqslant i-1 \quad \text{for} \quad p \in A_i ,$$

where
$$\mathfrak{V}' = \{V_\alpha(U) | \alpha \in A, \; U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}\} \text{ [4]}.$$
Suppose that $S'_{m_1 \ldots m_p}(V) = U$ $(V \in \mathfrak{V}_{m_1})$ is a member of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$; then we put
$$S'_{m_1 \ldots m_p m+1}(V) = \bigcup \{V_\alpha(U) | \alpha \in A, \; S(V_\alpha, \mathfrak{V}) \cap U \neq 0\},$$
$$\mathfrak{S}'_{m_1 \ldots m_p m+1} = \{S'_{m \cdot \ldots m_p m+1}(V) | V \in \mathfrak{V}_{m_1}\}.$$
By (11), $\mathfrak{S}'_{m_1 \ldots m_p m+1}$ is a locally finite open covering.

(18) We choose only those members of $\mathfrak{S}'_{m_1 \ldots m_p m+1}$ which are not contained in any other member and denote the collection of those members also by $\mathfrak{S}'_{m_1 \ldots m_p m+1}$.

Adding those locally finite open coverings $\mathfrak{S}'_{m_1 \ldots m_p m+1}$, $1 \leqslant m_1 < \ldots < m_p \leqslant m$, to the collection
$$\Sigma = \{\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m)}\}$$
we obtain a new collection
$$\Sigma' = \{\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m)}, \mathfrak{S}_{k(m)+1}, \ldots, \mathfrak{S}_{k(m+1)-1}\}.$$
Then we can see that this collection Σ' of coverings satisfies the conditions (6), (7), (8), (9).

As for (6) we have just altered $\mathfrak{S}'_{m_1 \ldots m_p m+1}$ so that Σ' satisfies that condition. To see (7) let
$$U, U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)-1} \quad \text{and} \quad U \subsetneqq U' .$$
If
$$U \in \mathfrak{S}_{k(m)+1} \cup \ldots \cup \mathfrak{S}_{k(m+1)-1}, \qquad U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)},$$

[4] See [4], Lemma 2.1.

then
$$U = \bigcup \{V_\alpha(U_0) | \alpha \in A, \; S(V_\alpha, \mathfrak{V}) \cap U_0 \neq 0\}$$
for some $U_0 \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$. We shall denote such a set U by $S_0^2(U_0, \mathfrak{V})$ in the rest of the proof. Hence
$$\overline{U}_0 \subset U \subset U'$$
Therefore it follows from (12) that
$$\overline{U} \subset \overline{S^2(U_0, \mathfrak{V})} \subset U' .$$
If
$$U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}, \qquad U' \in \mathfrak{S}_{k(m)+1} \cup \ldots \cup \mathfrak{S}_{k(m+1)-1},$$
then
$$U' = S_0^2(U'_0, \mathfrak{V})$$
for some $U'_0 \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$. If we assume that $\overline{U}'_0 \not\supset U$, then by (13)
$$S^2(U'_0, \mathfrak{V}) \not\supset U ,$$
and hence $U' \not\supset U$, which is a contradiction. Thus we obtain $U \subset \overline{U}'_0$ which implies
$$\overline{U} \subset \overline{U}'_0 \subset S_0^2(U'_0, \mathfrak{V}) = U' .$$
If
$$U, U' \in \mathfrak{S}_{k(m)+1} \cup \ldots \cup \mathfrak{S}_{k(m+1)-1},$$
then
$$U = S_0^2(U_0, \mathfrak{V}), \qquad U' = S_0^2(U'_0, \mathfrak{V}),$$
for some $U_0, U'_0 \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m)}$ with $U_0 \neq U'_0$. Hence, by (13), we obtain $U_0 \subset \overline{U}'_0$. If U_0, U'_0 belong to the same covering $\mathfrak{S}_i$, then U and U' also belong to the same covering which is impossible since $U \subsetneqq U'$. Hence we suppose that U_0 and U'_0 belong to the distinct coverings $\mathfrak{S}'_{m_1 \ldots m_p}$ and $\mathfrak{S}'_{l_1 \ldots l_q}$, respectively. If
$$2^{-m_1} + \ldots + 2^{-m_p} < 2^{-l_1} + \ldots + 2^{-l_q},$$
then it follows from (16), the local finiteness of $\mathfrak{V}$ and $U_0 \subset \overline{U}'_0$ that
$$\overline{U} = \overline{S_0^2(U_0, \mathfrak{V})} = \bigcup \{\overline{V_\alpha(U_0)} | \alpha \in A, \; S(V_\alpha, \mathfrak{V}) \cap U_0 \neq 0\}$$
$$\subset \bigcup \{V_\alpha(U'_0) | \alpha \in A, \; S(V_\alpha, \mathfrak{V}) \cap U'_0 \neq 0\} = S_0^2(U'_0, \mathfrak{V}) = U' .$$
If
$$2^{-m_1} + \ldots + 2^{-m_p} > 2^{-l_1} + \ldots + 2^{-l_q},$$
then by (4)
$$U_0 \subset \overline{U}'_0 \subset \overline{U}_1 \quad \text{for some} \quad U_1 \in \mathfrak{S}'_{m_1 \ldots m_p}.$$
Hence
$$U = S_0^2(U_0, \mathfrak{V}) \subsetneqq U' = S_0^2(U'_0, \mathfrak{V}) \subset S_0^2(U_1, \mathfrak{V}),$$

but this contradicts the definition (18) of $\mathfrak{S}_{m_1 \dots m_p m+1}$. Thus in any case we can conclude $\overline{U} \subset U'$ proving (7) for Σ'.

Now, to prove (8) for Σ' we suppose

$$U_1, \dots, U_l \in \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m+1)-1}, \quad \overline{U}_1 \cap \dots \cap \overline{U}_l \neq \emptyset,$$
$$U_i \not\subset U_j \quad \text{if} \quad i \neq j.$$

If we assume

$$U_1, \dots, U_h \in \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m)},$$
$$U_{h+1}, \dots, U_l \in \mathfrak{S}_{k(m)+1} \cup \dots \cup \mathfrak{S}_{k(m+1)-1},$$

then

$$U_s = S_0^2(U_{0s}, \mathfrak{B}), \quad s = h+1, \dots, l$$

for some $U_{0s} \in \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m)}$. Since $\{U_1, \dots, U_l\}$ is independent, it follows from (7) for Σ and (12) that $U_1, \dots, U_h, U_{0h+1}, \dots, U_{0l}$ are also independent. If we assume that

$$\overline{U}_1 \cap \dots \cap \overline{U}_h \cap \overline{U}_{0h+1} \cap \dots \cap \overline{U}_{0l} = \emptyset,$$

then we obtain from (10)

$$\overline{U}_1 \cap \dots \cap \overline{U}_l \subset \overline{U}_1 \cap \dots \cap \overline{U}_h \cap \overline{S^2(U_{0h+1}, \mathfrak{B})} \cap \dots \cap \overline{S^2(U_{0l}, \mathfrak{B})}$$
$$\subset S^3(U_1, \mathfrak{B}) \cap \dots \cap S^3(U_h, \mathfrak{B}) \cap S^3(U_{0h+1}, \mathfrak{B}) \cap \dots \cap S^3(U_{0l}, \mathfrak{B}) = \emptyset,$$

which is a contradiction. We can, therefore, conclude

$$\overline{U}_1 \cap \dots \cap \overline{U}_h \cap \overline{U}_{0h+1} \cap \dots \cap \overline{U}_{0l} \neq \emptyset,$$

and hence by (8) for Σ, we obtain $l \leqslant n+1$, i.e.

$$\operatorname{Rank} \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m+1)-1} \leqslant n+1$$

proving (8) for Σ'.

Finally, to prove (9) for Σ', we suppose

$$p \in A_i,$$
$$p \in B(U_1) \cap \dots \cap B(U_l),$$
$$U_1, \dots, U_h \in \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m)},$$
$$U_{h+1}, \dots, U_l \in \mathfrak{S}_{k(m)+1} \cup \dots \cup \mathfrak{S}_{k(m+1)-1},$$
$$U_s = S_0^2(U_{0s}, \mathfrak{B}), \quad s = h+1, \dots, l,$$
$$U_{0s} \in \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m)}.$$

Since $\{V_\alpha(U_{0s}) |\ \alpha \in A\}$ is locally finite, we obtain

$$B(U_s) \subset \bigcup \{BV_\alpha(U_{0s}) |\ \alpha \in A,\ S(V_\alpha, \mathfrak{B}) \cap U_{0s} \neq \emptyset\}.$$

Hence

$$p \in B(U_1) \cap \dots \cap B(U_h) \cap BV_{\alpha(h+1)}(U_{0h+1}) \cap \dots \cap BV_{\alpha(l)}(U_{0l})$$

for some distinct members $V_{\alpha(h+1)}(U_{0h+1}), \dots, V_{\alpha(l)}(U_{0l})$ of $\mathfrak{B}'$ defined in (17). Therefore from (17) we obtain $l \leqslant i-1$, i.e.

$$\operatorname{ord}_p B(\mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m+1)-1}) \leqslant i-1.$$

Thus Σ' satisfies all of (6)-(9).

Finally we shall define $\mathfrak{B}_{m+1} = \mathfrak{S}'_{m+1} = \mathfrak{S}_{k(m+1)}$. For the preceding covering $\mathfrak{W}'$ defined just before (14) we construct a locally finite open covering $\mathfrak{W}$ such that $\mathfrak{W} < \mathfrak{W}'$ and

$$\text{(19)} \qquad \operatorname{Rank} \mathfrak{S}_1 \cup \dots \cup \mathfrak{S}_{k(m+1)-1} \cup \mathfrak{W} \leqslant n+1.$$

Since Σ' satisfies (8) and (9) such a covering $\mathfrak{W}$ can be constructed by a slight modification of the process used in [6], proof of Theorem 2.

In general, let $\mathfrak{W}'$ be an open covering and $\mathfrak{S}$ a locally finite open covering with $\operatorname{Rank} \mathfrak{S} \leqslant n+1$ and $\operatorname{ord}_p B(\mathfrak{S}) \leqslant i-1$ for $p \in A_i$; then we can assert that there exists a locally finite open covering $\mathfrak{W}$ satisfying

$$\mathfrak{W} < \mathfrak{W}', \quad \operatorname{Rank} \mathfrak{S} \cup \mathfrak{W} \leqslant n+1.$$

Let

$$B_k = \{p |\ \operatorname{ord}_p B(\mathfrak{S}) \geqslant k\}, \quad k = 0, \dots, n;$$

then

$$B_k \subset A_{k+1} \cup \dots \cup A_{n+1},$$

and hence we obtain

$$\dim B_k \leqslant n-k$$

by the decomposition theorem. Since $B(\mathfrak{S})$ is locally finite, each B_k is a closed set and satisfies $B_{k+1} \subset B_k$. For every point p of $B_k - B_{k+1}$, we can choose an open neighborhood $U(p)$ of p such that

$$\overline{U(p)} \cap \overline{S} \neq \emptyset, \quad U(p) \cap (R-S) \neq \emptyset$$

for exactly k members S of $\mathfrak{S}$ (5). Then

$$\mathfrak{U}_k = \{U(p) |\ p \in B_k - B_{k+1}\}$$

is a collection which covers $B_k - B_{k+1}$ and consists of open sets which do not intersect B_{k+1}.

Now we shall define $n+1$ locally finite collections $\mathfrak{P}_k$, $k = 0, \dots, n$ of open sets such that

$$\mathfrak{P}_{k-1} \subset \mathfrak{P}_k < \mathfrak{W}', \quad \operatorname{ord} \mathfrak{P}_k \leqslant k+1,$$
$$\mathfrak{P}_k - \mathfrak{P}_{k-1} < \mathfrak{U}_{n-k}, \quad \mathfrak{P}_k \text{ covers } B_{n-k},$$

where we put

$$\mathfrak{U}_n = \{U(p) |\ p \in B_n\}, \quad \mathfrak{P}_{-1} = \emptyset.$$

(5) In general, if two subsets U and S satisfy this condition, then we call U *overflows* S.

To this end we shall show by induction on the number m that for every m with $0 \leqslant m \leqslant n$ we can define $m+1$ locally finite open collections $\mathfrak{P}_0^m, \dots, \mathfrak{P}_m^m$ such that

$$\mathfrak{P}_{k-1}^m \subset \mathfrak{P}_k^m < \mathfrak{W}', \qquad \operatorname{ord} \mathfrak{P}_k^m \leqslant k+1,$$
$$\mathfrak{P}_k^m - \mathfrak{P}_{k-1}^m < \mathfrak{U}_{n-k}, \qquad \mathfrak{P}_k^m \text{ covers } B_{n-k}, \qquad k = 0, \dots, m.$$

For $m = 0$ we choose, by use of $\dim B_n \leqslant 0$, an open covering $\mathfrak{Q}$ of B_n with

$$\operatorname{ord} \mathfrak{Q} \leqslant 1, \qquad \mathfrak{Q} < \mathfrak{U}_n \wedge \mathfrak{W}'.$$

Since $\mathfrak{Q}$ is a locally finite closed collection of order $\leqslant 0$ in R, we can easily see that there exists a locally finite open collection $\mathfrak{P}_0^0$ in R such that

$$\operatorname{ord} \mathfrak{P}_0^0 \leqslant 1, \qquad \mathfrak{Q} < \mathfrak{P}_0^0 < \mathfrak{U}_n \wedge \mathfrak{W}'.$$

Then $\mathfrak{P}_0^0$ is the desired open collection for $m = 0$.

Now, let us suppose we have defined $\mathfrak{P}_0^m, \dots, \mathfrak{P}_m^m$ at desire. Let

$$\mathfrak{P}_k^m = \{P_\alpha|\ \alpha < \alpha_{k+1}\}, \qquad k = 0, 1, \dots, m.$$

Since $\dim B_{n-m-1} \leqslant m+1$, we can choose a locally finite open covering $\mathfrak{N}$ of B_{n-m-1} satisfying

$$\operatorname{ord} \overline{\mathfrak{N}} \leqslant m+2, \qquad \overline{\mathfrak{N}} < \mathfrak{P}_m^m \cup \mathfrak{U}_{n-m-1}, \qquad \overline{\mathfrak{N}} < \mathfrak{W}'.$$

Since B_{n-m-1} is closed, $\overline{\mathfrak{N}}$ is a locally finite closed collection in R of order $\leqslant m+2$. Hence we can easily see that there exists a locally finite open collection $\mathfrak{M}$ in R such that

$$\operatorname{ord} \mathfrak{M} \leqslant m+2, \qquad \overline{\mathfrak{N}} < \mathfrak{M} < \mathfrak{P}_m^m \cup \mathfrak{U}_{n-m-1}, \qquad \mathfrak{M} < \mathfrak{W}'.$$

Consequently $\mathfrak{M}$ covers B_{n-m-1}. Then, putting

$$P'_\alpha = \cup \{M|\ M \in \mathfrak{M},\ M \subset P_\alpha,\ M \not\subset P_\beta \text{ for every } \beta < \alpha\},$$
$$\mathfrak{P}_k^{m+1} = \{P'_\alpha|\ \alpha < \alpha_{k+1}\}, \qquad k = 0, 1, \dots, m,$$
$$\mathfrak{P}_{m+1}^{m+1} = \mathfrak{P}_m^{m+1} \cup \{M|\ M \not\subset P_\alpha \text{ for every } \alpha < \alpha_{m+1}\},$$

we assert that $\mathfrak{P}_0^{m+1}, \dots, \mathfrak{P}_{m+1}^{m+1}$ are the desired locally finite open collections for $n = m+1$. The only problem is to show that $\mathfrak{P}_k^{m+1}$ covers B_{n-k}. Since P_{m+1}^{m+1} clearly covers B_{n-m-1}, we may assume $k \leqslant m$. To see this we note that each element of $\mathfrak{P}_m^m - \mathfrak{P}_k^m$ does not meet B_{n-k}, because

$$\mathfrak{P}_m^m - \mathfrak{P}_k^m < \mathfrak{U}_{n-k-1} \cup \dots \cup \mathfrak{U}_{n-m},$$

and each member of $\mathfrak{U}_{n-k-1} \cup \dots \cup \mathfrak{U}_{n-m}$ does not meet B_{n-k}. Furthermore note that each member of $\mathfrak{U}_{n-m-1}$ does not meet B_{n-k} either. Let

p be a given point of B_{n-k}; then $p \in M$ for some $M \in \mathfrak{M}$. From the above note and $\mathfrak{M} < \mathfrak{P}_m^m \cup \mathfrak{U}_{n-m-1}$, it follows that

$$p \in M \subset P$$

for some $P \in \mathfrak{P}_k^m$ which means by the definition of $\mathfrak{P}_k^{m+1}$

$$P \in M \subset P'$$

for some $P' \in \mathfrak{P}_k^{m+1}$. Hence $\mathfrak{P}_k^{m+1}$ covers B_{n-k}. Thus we get desired collections $\mathfrak{P}_0^m, \dots, \mathfrak{P}_m^m$ for every integer m with $0 \leqslant m \leqslant n$.

Putting

$$\mathfrak{P}_k = \mathfrak{P}_k^n, \qquad k = 0, 1, \dots, n,$$

we get the initially desired $n+1$ collections. We put

$$\mathfrak{P}_k = \{P_\gamma|\ \gamma \in \Gamma_k\}, \qquad k = 0, \dots, n.$$

We note that $\mathfrak{P}_n$ is a locally finite open covering of R. Hence there exists an open covering

$$\mathfrak{W}_n = \{W_\gamma|\ \gamma \in \Gamma_n\}$$

of R such that $\overline{W}_\gamma \subset P_\gamma$.

If we put

$$\mathfrak{W}_k = \{W_\gamma|\ \gamma \in \Gamma_k\}, \qquad k = 0, \dots, n,$$

then we can easily see that

$$\mathfrak{W}_n < \mathfrak{W}', \qquad \operatorname{Rank} \mathfrak{W}_n \leqslant \operatorname{ord} \overline{\mathfrak{W}}_n \leqslant n+1,$$
$$\operatorname{ord} \overline{\mathfrak{W}}_k \leqslant k+1, \qquad \mathfrak{W}_k - \mathfrak{W}_{k-1} < \mathfrak{U}_{n-k}.$$

Put $\mathfrak{W} = \mathfrak{W}_n$; then let us show that

$$\operatorname{Rank} \mathfrak{S} \cup \mathfrak{W} \leqslant n+1.$$

Suppose

$$p \in \overline{U}_1 \cap \dots \cap \overline{U}_k \cap \overline{U}_{k+1} \cap \dots \cap \overline{U}_{n+2},$$
$$U_i \not\subset U_j \quad \text{if} \quad i \neq j,$$
$$U_1, \dots, U_k \in \mathfrak{S}, \qquad U_{k+1}, \dots, U_{n+2} \in \mathfrak{W}.$$

Since

$$\operatorname{Rank} \mathfrak{S} \leqslant n+1, \qquad \operatorname{Rank} \mathfrak{W} \leqslant n+1,$$

it must be

$$1 \leqslant k \leqslant n+1.$$

Since

$$\operatorname{ord} \overline{\mathfrak{W}}_{n-k} \leqslant n-k+1,$$

at least one of $U_{k+1}, \dots, U_{n+2}$ does not belong to $\mathfrak{W}_{n-k}$. For example, let

$$U_{k+1} \in \mathfrak{W}_{l+1} - \mathfrak{W}_l$$

for some $l \geqslant n-k$. Then, since

$$\mathfrak{W}_{l+1}-\mathfrak{W}_l < \mathfrak{U}_{n-l-1},$$

and each member of $\mathfrak{U}_{n-l-1}$ overflows exactly $n-l-1$ members of $\mathfrak{S}$, U_{k+1} overflows at most $n-l-1$ members of $\mathfrak{S}$. Since $n-l-1 \leqslant k-1$, U_{k+1} overflows at most $k-1$ members of $\mathfrak{S}$. On the other hand, since $U_1, \ldots, U_k, U_{k+1}$ satisfies

$$U_i \not\subset U_j \quad \text{if} \quad i \neq j.$$

and

$$p \in \overline{U}_1 \cap \ldots \cap \overline{U}_k \cap \overline{U}_{k+1},$$

U_{k+1} overflows k members $U_1, \ldots, U_k$ of $\mathfrak{S}$, which is a contradiction. Therefore we can conclude that

$$\operatorname{Rank} \mathfrak{S} \cup \mathfrak{W} \leqslant n+1.$$

Thus $\mathfrak{W}$ is the desired covering.

Let W be a given member of $\mathfrak{W}$. For every member U of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)-1}$ such that

$$U \not\supset W, \quad U \cap W \neq \emptyset,$$

we assign a point

$$q(W, U) \in W - U.$$

Then

$$F(W) = \bigcup \{q(W, U) |\ U \not\supset W,\ U \cap W \neq \emptyset,\ U \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)-1}\}$$

is a closed set contained in W, because W meets only finitely many members of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)-1}$ since $\mathfrak{W} < \mathfrak{W}' < \mathfrak{V}$. Hence, by use of (9) for Σ', we can construct an open set $V(W)$ for every $W \in \mathfrak{W}$ such that

$$F(W) \subset V(W) \subset \overline{V(W)} \subset W,$$

$$\operatorname{ord}_p B(\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)-1}) \cup \{BV(W) |\ W \in \mathfrak{W}\} \leqslant i-1 \quad \text{for} \quad p \in A_i\ ^{(6)}.$$

Put

$$\mathfrak{V}_{m+1} = \mathfrak{S}'_{n+1} = \mathfrak{S}_{k(m+1)} = \{V(W) |\ W \in \mathfrak{W},\ V(W) \subset V(W_0)\ (W_0 \in \mathfrak{W}) \text{ implies } V(W) = V(W_0)\}.$$

Then it is easy to see from (6), (7) for Σ' and (19) that

$$\Sigma'' = \{\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m+1)}\}$$

also satisfies (6), (7), (8), (9).

Since (6) and (9) are clearly satisfied by Σ'', we shall only concern (7) and (8). Let

$$U, U' \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)} \quad \text{and} \quad U \subsetneqq U'.$$

(6) See [4], Lemma 2.1.

If

$$U \in \mathfrak{S}_i \quad (i \neq k(m+1)), \quad U' \in \mathfrak{S}_{k(m+1)},$$

then $U' \subset U_0$ for some $U_0 \in \mathfrak{S}_i$, because

$$\mathfrak{S}_{k(m+1)} < \mathfrak{W} < \mathfrak{V} < \mathfrak{V}^{**} < \mathfrak{V}_m < \mathfrak{S}_i.$$

This implies $U \subsetneqq U' \subset U_0$, which contradicts (6). Therefore we assume

$$U \in \mathfrak{S}_{k(m+1)}, \quad U' \in \mathfrak{S}_i \quad (i \neq k(m+1)).$$

Let

$$U = V(W) \quad \text{for} \quad W \in \mathfrak{W}.$$

Since $U \subset U'$, in view of the process defining $V(W)$ we obtain $W \subset U'$ and hence

$$\overline{U} \subset W \subset U'$$

proving (7). To see (8) we suppose

$$\overline{U}_1 \cap \ldots \cap \overline{U}_l \neq \emptyset$$

for independent members $U_1, \ldots, U_l$ of $\mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)}$. Let

$$U_1, \ldots, U_h \in \mathfrak{S}_1 \cup \ldots \cup \mathfrak{S}_{k(m+1)-1}, \quad U_{h+1}, \ldots, U_l \in \mathfrak{S}_{k(m+1)};$$

then we may assume

$$U_s = V(W_s) \quad \text{for} \quad W_s \in \mathfrak{W},\ s = h+1, \ldots, l.$$

Since $\overline{U}_s \subset W_s$,

$$\overline{U}_1 \cap \ldots \cap \overline{U}_h \cap \overline{W}_{h+1} \cap \ldots \cap \overline{W}_l \neq \emptyset.$$

Besides, $U_1, \ldots, U_h, W_{h+1}, \ldots, W_l$ are obviously independent by virtue of (6) for Σ' and the fact that we can suppose, without loss of generality, $\mathfrak{W}$ also satisfies the condition (6). Therefore from (19) we obtain $l \leqslant n+1$ proving (8). It is also easily seen that $\mathfrak{V}_1, \ldots, \mathfrak{V}_{m+1}$ and $\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m+1)}$ satisfy (1), (2), (3). The validity of (4) is shown by an argument which is a slight modification of the proof of Theorem 5 of [3] and omitted here. Thus we have defined $\mathfrak{V}_m$, $m = 1, 2, \ldots$ and $\mathfrak{S}'_{m_1 \ldots m_p}$, $1 \leqslant m_1 < \ldots < m_p$ which satisfy (1)-(5).

We now introduce a metric ϱ in R by use of the coverings

$$\mathfrak{S}'_{m_1 \ldots m_p}, \quad 1 \leqslant m_1 < \ldots < m_p \quad \text{and} \quad \mathfrak{S}'_0 = \{R\}$$

as follows:

$$\varrho(x, y) = \inf\{2^{-m_1} + \ldots + 2^{-m_p} |\ y \in S(x, \mathfrak{S}'_{m_1 \ldots m_p})\}.$$

The proof that ϱ is a metric is a slight modification of the proof of Theorem 5 of [3]. For that proof, we need, besides the structure of $\mathfrak{S}'_{m_1 \ldots m_p}(V)$, (1)-(4) and (16). Here we shall only prove that the metric ϱ

satisfies the special condition desired in the theorem. Let $x, y_1, \ldots, y_{n+2}$ be given $n+3$ points of R. For every $\varepsilon > 0$ we obtain

$$m_1^j, \ldots, m_{p(j)}^j, \qquad j = 1, \ldots, n+2$$

such that

$$\varrho(x, y_j) \leqslant 2^{-m_1^j} + \ldots + 2^{-m_{p(j)}^j} < \varrho(x, y_j) + \varepsilon$$

and

$$U_j \in \mathfrak{S}'_{m_1^j \ldots m_{p(j)}^j}$$

such that $x, y_j \in U_j$. It follows from (5) that there exist U_i and U_j $(i \neq j)$ such that $U_i \subset U_j$. Therefore

$$\varrho(y_i, y_j) \leqslant 2^{-m_1^j} + \ldots + 2^{-m_{p(j)}^j} < \varrho(x, y_j) + \varepsilon .$$

We take a pair i, j satisfying

$$\varrho(y_i, y_j) < \varrho(x, y_j) + \varepsilon_m$$

for a sequence $\{\varepsilon_m\}$ of positive numbers converging to 0. Then

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$$

proving the necessity. Thus among the conditions (6), (7), (8), (9) for $\mathfrak{S}_1, \ldots, \mathfrak{S}_{k(m)}$ the condition (8) is essential. The other conditions are needed only to continue the inductive argument.

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