A continuous real-valued function on E^n almost everywhere 1-1

by

J. L. Denny * (Bloomington, Indiana)

The purpose of this note is to prove the following

THEOREM. *There is a uniformly continuous function* $f\colon E^n\to(0,1)$ *and a Borel set* $D\subset E^n$ *with the Lebesgue measure of* $E^n\sim D$ *equal to zero such that the restriction of* f *to* D *is* 1-1. *Each partial function of* f *of a real variable is nondecreasing.*

The proof is based on the fact that there is an uncountable disjoint family of Borel sets each of which is in $(0,1)$ and which is the image by a continuous 1-1 function on E^1 of a Borel set whose complement has measure zero.

Let $n\geqslant 2$ be fixed and let $y=(y_1,\ldots,y_{n-1})$, $y_i\in\big(0,1/(n-1)\big)$, $i=1,\ldots,n-1$. Let $\{X_j\colon j=1,2,\ldots\}$ be a sequence of independent, identically distributed random variables such that $P\{X_1=i\}=y_i$, $i=1,\ldots,n-1$, and $P\{X_1=0\}=y_0=1-\sum_{i=1}^{n-1}y_i$. With each sequence $\{b_j\colon j=1,2,\ldots\}$ of outcomes of $\{X_j\}$ associate the real number

$$\sum_{j=1}^{\infty}b_jn^{-j}. \tag{1}$$

Let Y be the random variable defined on $\{X_j\}$ whose value at particular $\{b_j\}$ is given by (1) and let $F_y(x)=P\{Y\leqslant x\}$.

LEMMA 1. $F_y(\cdot)$ *is a strictly increasing continuous function on* $[0,1]$ *with* $F_y(0)=0$ *and* $F_y(1)=1$.

Proof. Being a Lebesgue-Stieltjes distribution function, $F_y(\cdot)$ is nondecreasing and is defined to be continuous from the right. The independence of the X_j implies that $P\{Y=x\}=0$ for $x\in[0,1]$ and so $F_y(\cdot)$

* Research sponsored by the Office of Naval Research under contract No. Nonr. 222 (43). This example appeared in the author's dissertation submitted to the University of California, Berkeley.

is continuous from the left. To verify that $F_y(\cdot)$ is strictly increasing observe that if $x < x'$ and $x, x' \in [0, 1]$, then $F_y(x) = F_y(x')$ is equivalent to $P\{x < Y \leqslant x'\} = 0$. There exists

$$z = \sum_{j=1}^{J} b_j n^{-j} \quad \text{and} \quad z' = \sum_{j=1}^{J-1} b_j n^{-j} + (b_J + 1) n^{-J}$$

where $b_J \neq n-1$ such that $z, z' \in (x, x')$. Then

$$P\{x < Y \leqslant x'\} \geqslant P\{z < Y \leqslant z'\} = \prod_{j=1}^{J} y_{b_j} > 0,$$

which gives strict monotony. Continuity and $0 \leqslant Y \leqslant 1$ give the last assertion.

LEMMA 2. *For each $x \in (0,1)$ and $y_i \in \big(0, 1/(n-1)\big)$, $i = 1, \ldots, j-1, j+1, \ldots, n-1$, $F_y(x)$ is a continuous and non-increasing function of y_j.*

Proof. Let $x = \sum_{j=1}^{\infty} a_j n^{-j}$; if there are two expansions either (fixed) is suitable. Let

$$D_1 = \Big\{z\colon\ z = \sum_{j=1}^{\infty} b_j n^{-j},\ b_1 < a_1,\ \sum_{j=1}^{\infty} b_j n^{-j} \text{ is the finite } n\text{-ary expansion of } z \text{ when it exists}\Big\}$$

and

$$D_m = \Big\{z\colon\ z = \sum_{j=1}^{\infty} b_j n^{-j},\ b_j = a_j,\ j = 1, \ldots, m-1,\ b_m < a_m,\ \sum_{j=1}^{\infty} b_j n^{-j} \text{ is the finite } n\text{-ary expansion of } z \text{ when it exists}\Big\},$$

$m = 2, 3, \ldots$ By definition, D_i is empty if $a_i = 0$ and otherwise the D_i are disjoint intervals. Since

$$[0, x) = \sum_{m=1}^{\infty} D_m$$

and since

$$P\{Y \in D_1\} = P\{X_1 < a_1\} = \sum_{i=0}^{a_1-1} y_i,$$

$$P\{Y \in D_m\} = P\{X_j = a_j,\ j = 1, \ldots, m-1,\ X_m < a_m\} = y_{a_1} y_{a_2} \cdots y_{a_{m-1}} \sum_{i=0}^{a_m-1} y_i,$$

it follows that

$$P\{Y \leqslant x\} = \sum_{m=1}^{\infty} \Big\{ y_{a_1} y_{a_2} \cdots y_{a_{m-1}} \sum_{i=0}^{a_m-1} y_i \Big\}.$$

Let $\tau_i^1 \equiv 0$ and τ_i^m be the number, possibly zero, of a_j's that equal i in the first $m-1$ a_j's, $i = 0, \ldots, n-1$. Agreeing that $\sum_{i=0}^{-1} y_i = 0$, the above gives

$$F_y(x) = \sum_{m=1}^{\infty} \Big\{ \Big(\prod_{i=0}^{n-1} y_i^{\tau_i^m} \Big) \sum_{i=0}^{a_m-1} y_i \Big\}. \tag{2}$$

Since $\sum_{i=0}^{a_m-1} y_i < 1$ and the rest of each term of (2) is a polynomial in y_j, it follows by the Weierstrass M-test that $F_y(x)$ is continuous in y_j. To verify monotony, the identity,

$$F_y(mn^{-(N+1)}) = \sum_{i=0}^{k-1} y_i + y_k F_y\big((m - kn^N) n^{-N}\big), \tag{3}$$

for $m = 0, \ldots, n^{N+1}$ and $mn^{-(N+1)} \in (k/n, (k+1)/n]$, $k = 0, \ldots, n-1$, is obtained from the independence of the X_j. In turn, (3) implies by an induction on $N = 1, 2, \ldots$, that for $y'_j > y_j$, $y' = (y_1, \ldots, y'_j, \ldots, y_{n-1})$, and each mn^{-N}, $m = 0, \ldots, n^N$, $F_{y'}(mn^{-N}) \leqslant F_y(mn^{-N})$. By the continuity of each $F_y(\cdot)$, from Lemma 1 and the denseness of the numbers mn^{-N}, $F_y(x)$ is a nonincreasing function of y_j.

Remark. The continuity of $F_y(\cdot)$ and (3) give

$$F_y(x) = \sum_{i=0}^{k-1} y_i + y_k F_y(nx - k), \tag{4}$$

$x \in (k/n, (k+1)/n]$, $k = 0, \ldots, n-1$. The representations (2) and (4) are extensions of functions studied by Salem [3] and de Rham [1], respectively. When $n = 2$ and 3 the monotony is strict. However if $n = 4$ and $x = .a_1 a_2 \ldots$ where each a_i is either 2 or 3 then $F_y(x)$ is a constant function of y_1.

LEMMA 3. *The function*

$$g\colon\ \big(0, 1/(n-1)\big) \times \ldots \times \big(0, 1/(n-1)\big) \times (0, 1) \to (0, 1),$$

whose value at $(y_1, \ldots, y_{n-1}, x)$ is that $z \in (0,1)$ such that $F_y(z) = x$ is well defined. For $y_i \in \big(0, 1/(n-1)\big)$, $i = 1, \ldots, n-1$, the function $g(y_1, \ldots, y_{n-1}, \cdot)\colon (0,1) \to (0,1)$ is strictly increasing and continuous. For each $x \in (0,1)$ and $y_i \in \big(0, 1/(n-1)\big)$, $i = 1, \ldots, j-1, j+1, \ldots, n-1$, $g(y_1, \ldots, y_{j-1}, \cdot, y_{j+1}, \ldots, y_{n-1}, x)\colon \big(0, 1/(n-1)\big) \xrightarrow{\text{into}} (0,1)$ is a continuous and non-decreasing function of y_j.

Proof. By Lemma 1, $F_y(\cdot)$ is a homeomorphism on $(0,1)$ onto $(0,1)$ and this assures the first two assertions. The last part of Lemma 2 plus computations give the last part of Lemma 3.

LEMMA 4. *The function g is continuous.*

Proof. By Lemma 3, each partial function of g of a real variable is nondecreasing. This ensures that for each $(y_1, \dots, y_{n-1}, x)$ in the domain of g, an open rectangle R containing $(y_1, \dots, y_{n-1}, x)$ may be constructed so that $\sup\{g(z)\colon z \in R\} - \inf\{g(z)\colon z \in R\}$ is less than a given $\varepsilon > 0$. This fact gives the lemma.

LEMMA 5. *Let*

$$A_y = \Big\{x\colon\ x \in (0,1),\ x = \sum_{j=1}^{\infty} a_j n^{-j},\ \lim_{m\to\infty} \Big(\sum_{j=1}^{m} I_i(a_j)\Big)\Big/m = y_i,$$

$$i = 1, \dots, n-1,\ y = (y_1, \dots, y_{n-1});\ I_i(z) = 1 \text{ or } 0 \text{ as } z = i \text{ or } z \neq i\Big\}.$$

For each $(y_1, \dots, y_{n-1}) \in \big(0, 1/(n-1)\big) \cdot \times \cdots \times \big(0, 1/(n-1)\big)$, $F_y[A_y]$ is a Borel set of Lebesgue measure one.

Proof. Surely A_y is a Borel set and by Lemma 1 and Lusin's theorem ([4], p. 244), $F_y[A_y]$ is a Borel set. By Kolmogorov's strong law of large numbers, $P\{Y \in A_y\} = 1$. Since $F_y(Y)$ is a uniformly distributed random variable on $(0,1)$, the assertion follows.

Remark. When $y_i = 1/n$, $i = 1, \dots, n-1$, then $F_y(x) = x$. Otherwise, by continuity of $F_y(\cdot)$, Kolmogorov's theorem, and de la Vallee Poussin's decomposition theorem ([2], p. 127), the derivative of $F_y(\cdot)$ equals zero almost everywhere.

LEMMA 6. *Let*

$$B = \{(y_1, \dots, y_{n-1}, x)\colon\ y_i \in \big(0, 1/(n-1)\big),\ i = 1, \dots, n-1,$$
$$x \in F_y[A],\ y = (y_1, \dots, y_{n-1})\}.$$

The set $B \subset \big(0, 1/(n-1)\big) \times \cdots \times \big(0, 1/(n-1)\big) \times (0,1)$ is a Borel set of Lebesgue measure $(n-1)^{-(n-1)}$.

Proof. Let $z = \sum_{j=1}^{\infty} a_j n^{-j}$ be the n-ary expansion of $z \in (0,1)$, taking the finite expansion whenever possible. For $i = 1, \dots, n-1$, define the sequence of functions $\{g_m^i\colon m = 1, 2, \dots\}$ by $g_m^i(z) = \sum_{j=1}^{m} I_i(a_j)/m$ when z is defined as above. The g_m^i are surely Borel functions and thus $g^{*i} = \lim\sup_{m\to\infty} g_m^i$ and $g_*^i = \liminf g_m^i$ are Borel functions. Let, for $i = 1, \dots$ $\dots, n-1$, h_i be the function defined by $h_i(y_1, \dots, y_{n-1}, x) = y_i$, surely a Borel function. By Lemma 4, g is continuous and so $g^{*i} \circ g$ and $g_*^i \circ g$ are real-valued Borel functions on $\big(0, 1/(n-1)\big) \times \cdots \times \big(0, 1/(n-1)\big) \times (0,1)$. Let

$$C_i = \big\{(y_1, \dots, y_{n-1}, x)\colon\ g^{*i}\big(g(y_1, \dots, y_{n-1}, x)\big) = g_*^i\big(g(y_1, \dots, y_{n-1}, x)\big) = h_i(y_1, \dots, y_{n-1}, x)\big\}$$

and let $C = \bigcap_{i=1}^{n-1} C_i$. Since all function are Borel functions, it follows that the C_i and hence C are Borel sets. But $B = C$, for $(y_1, \dots, y_{n-1}, x) \in B$ if and only if $x \in F_y[A_y]$ if and only if $g(y_1, \dots, y_{n-1}, x) \in A_y$ if and only if $g^{*i}\big(g(y_1, \dots, y_{n-1}, x)\big) = g_*^i\big(g(y_1, \dots, y_{n-1}, x)\big) = h_i(y_1, \dots, y_{n-1}, x)$ for $i = 1, \dots, n-1$, if and only if $(y_1, \dots, y_{n-1}, x) \in C$. To verify the asserted Lebesgue measure of B simply use Lemma 5 and the Fubini theorem.

LEMMA 7. *The restriction of g to B is 1-1.*

Proof. Let $(y_1, \dots, y_{n-1}, x)$, $(y_1', \dots, y_{n-1}', x') \in B$ be distinct. If there is j such that $y_j \neq y_j'$ then since $g(y_1, \dots, y_{n-1}, x) \in A_y$ and $g(y_1', \dots, y_{n-1}', x') \in A_{y'}$ and since $A_y \cap A_{y'}$ is empty, $g(y_1, \dots, y_{n-1}, x) \neq g(y_1', \dots, y_{n-1}', x')$. If $y_i = y_{i'}$, $i = 1, \dots, n-1$ then $x \neq x'$ and the strict monotony of $g(y_1, \dots, y_{n-1}, \cdot)$ proved in Lemma 3 implies the assertion.

Proof of the theorem. It remains only to choose an increasing homeomorphism h from E^1 onto $(0,1)$ such that h^{-1} satisfies Lusin's condition N, for istance, $h(x) = 1/\sqrt{2\pi} \int_{-\infty}^{x} e^{-y^2/2}\, dy$. For then $f(x_1, \dots, x_n) = g\big(h(x_1)/(n-1), \dots, h(x_{n-1})/(n-1), h(x_n)\big)$ meets the requirements.

Remark. The function f is a sufficient statistic for the family of all probability distributions dominated by Lebesgue measure since, except for a set of measure zero, f establishes a 1-1 correspondence with the sample.

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Reçu par la Rédaction le 29. 4. 1963

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FUNDAMENTA MATHEMATICAE
Warszawa 10 (Pologne), Śniadeckich 8

PRINTED IN POLAND

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