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On a problem of Erdös and Tarski

by

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In the paper Erdös-Tarski [1], the following properties of a cardinal number α are formulated:

$\boldsymbol{P}_1$: *There exists a simply ordered set of power α which has no well ordered or inversely well ordered subset of power α.*

$\boldsymbol{P}_2$: *There exists a division of the complete graph on a set of power α into two subgraphs neither of which includes a complete graph on a subset of power α.*

$\boldsymbol{P}_3$: *In the set algebra of all subsets of a set of power α, every α-complete prime ideal is principal.*

$\boldsymbol{P}_4$: *There is an α-complete and α-distributive Boolean algebra which is not isomorphic to any α-complete set algebra.*

$\boldsymbol{Q}$: *There is a ramification system $\mathcal{R}$ of order α such that for any $\eta < \alpha$ there are fewer than α elements of order η and every well ordered subset of $\mathcal{R}$ has power less than α.*

Here α is assumed to be an infinite cardinal.

It is shown in [1] that $\boldsymbol{P}_1$ implies $\boldsymbol{P}_2$, $\boldsymbol{P}_2$ implies $\boldsymbol{P}_3$, and $\boldsymbol{P}_3$ implies $\boldsymbol{P}_4$. It is also shown that $\boldsymbol{Q}$ implies $\boldsymbol{P}_3$ and that, for every inaccessible cardinal α, $\boldsymbol{P}_2$ implies $\boldsymbol{Q}$. Erdös and Tarski raised the question whether any of these implications holds in the opposite direction as well. It should be mentioned that all the properties $\boldsymbol{P}_1$-$\boldsymbol{P}_4$ hold for all accessible cardinals but fail for the smallest inaccessible cardinal ω. The problem whether $\boldsymbol{Q}$ holds for all accessible cardinals is not completely settled; it is known, however, that under the assumption of the generalized continuum hypothesis, the solution of the problem is affirmative for every accessible cardinal α which is not the immediate successor of a singular cardinal. For proofs and references for these results, see [1].

In Keisler-Tarski [6], the following condition is discussed:

$\boldsymbol{S}$: *There is an α-complete set algebra which is α-generated by α elements and in which every α-complete prime ideal is principal.*

It is known that, for all inaccessible α, $\boldsymbol{S}$ is equivalent to $\boldsymbol{Q}$. Like $\boldsymbol{P}_1$-$\boldsymbol{P}_4$, $\boldsymbol{S}$ is known to hold for all accessible α's and to fail for ω. Hence

P_2 implies S and S implies P_3 for all α, and in discussing the problem of inverse implications raised in [1], it proves convenient to replace Q by S (1).

The main result of this paper (Theorem 5, below) is that S implies P_1 and that consequently the three properties P_1, P_2, and S are equivalent (2). On the other hand, from a result stated in Hanf-Scott [4], it follows that P_3 does not imply S unless both hold for all cardinals larger than ω. (For a proof and a stronger form of this result, see [6].) Thus, among the problems concerning the implications between the conditions P_1-P_4 and S, essentially only one remains open, namely, the problem whether P_4 implies P_3.

Tarski has shown that properties Q, P_3, and P_4 are related to metamathematical problems concerning languages with infinitely long expressions. In particular, he introduced the metamathematical notion of strong incompactness and has shown that Q, and hence also S, applies to an inaccessible cardinal α just in case α is strongly incompact. Using incompactness results for languages with infinitely long expressions obtained by the author of this paper, he has further succeeded in showing that properties Q, S, P_3, and P_4 apply to members of a very comprehensive class of inaccessible cardinals larger than α; if we arrange all inaccessible cardinals $(>\omega)$ in a strictly increasing sequence $\theta_0, \theta_1, \ldots, \theta_\xi, \ldots$, then these properties apply, for example, to all those θ_ξ for which $\xi < \theta_\xi$. Our main theorem allows these results to be extended to properties P_1 and P_2 as well. Thus we see that conditions P_1 and P_2 hold for an inaccessible cardinal α just in case α is strongly incompact. Since it is known that under the generalized continuum hypothesis all accessible cardinals are strongly incompact, we conclude that under this hypothesis, each of the conditions P_1, P_2, and S is necessary and sufficient for an arbitrary cardinal α to be strongly incompact. Finally, P_1 and P_2, like the other properties in [1] turn out to apply to a comprehensive class of inaccessible cardinals. For the definition of the class of strongly incompact cardinals and for results concerning its extent, the reader is referred to the papers Tarski [13], Hanf [2], Hanf-Scott [4], and Keisler-Tarski [6].

Property P_1 is closely related to certain properties of simply ordered sets which were studied in Hausdorff [5] and Mahlo [8]. The results of this paper can be extended to some of these properties, e.g. to the constructability of ordered sets with given species. The author is planning to discuss this topic in a later paper.

(1) This is one of several helpful suggestions made by Tarski during the preparation of this paper. The equivalence of S and Q for inaccessible α is shown in [6], Theorem 4.31 where references to the earlier results in [1] and [9] are given. Note that the class C_0 of [6] is the class of all cardinals satisfying S. Thus Corollary 6 of this paper gives additional necessary and sufficient conditions for a cardinal to belong to C_0.

(2) This result was first stated by the author in the abstract [3].

In order to establish the main result, we define the notion of a Boolean algebra with an ordered α-basis and prove a theorem (Theorem 1 below) to the effect that, for α inaccessible, every α-distributive Boolean algebra of power at most α has an ordered α-basis. The notion in question is a generalization of the notion of an ordered basis introduced in Mostowski-Tarski [11]; the theorem just mentioned generalizes the well known theorem by which every denumerable Boolean algebra has an ordered basis (3). In Theorems 2, 3, and 4, which are not needed for the main result, we investigate further the conditions under which an ordered α-basis exists and we show how to construct a Boolean algebra with a given ordered set as its ordered α-basis. Theorem 5 is proved simply by showing that the ordered α-basis of a Boolean algebra satisfying the conditions of S must be a simply ordered set satisfying the conditions required in P_1.

In this paper, we make use of the terminology of [1]; in particular, recall that α-complete means that sets of power less than α have least upper (and greatest lower) bounds and that an α-distributive Boolean algebra is required to be α-complete. A set B will be said to be an ordered α-basis for an α-complete Boolean algebra $\mathfrak{A} = \langle A, \leqslant\rangle$ if and only if B is an α-complete subset of A which is simply ordered by $\leqslant$ and, for every $a \in A$, there exists a set R of ordered pairs of elements of B such that R has power less than α and

$$a = \sum_{\langle x,y\rangle \in R} (y-x)\,.$$

It is clear that an ordered α-basis of a Boolean algebra $\mathfrak{A}$ α-generates $\mathfrak{A}$. In case $\alpha = \omega$ or more generally, if α is inaccessible and $\mathfrak{A}$ is α-distributive, the converse also holds; every α-complete simply ordered set of elements of $\mathfrak{A}$ which α-generates $\mathfrak{A}$ is an ordered α-basis for $\mathfrak{A}$. This follows from the fact that, by the distributive law, any element can be written as a sum of products of elements of B and their complements, but in view of the ordering and completeness of B, each of the products reduces to a difference $x-y$ of two elements of B (note that 0 and 1 are in B since they are the empty sum and product respectively).

THEOREM 1. *Suppose that α is inaccessible. Then every α-distributive Boolean algebra of power at most α has an ordered α-basis.*

Proof. Let $\mathfrak{A} = \langle A, \leqslant\rangle$ be an α-distributive Boolean algebra and let $a_0, \ldots, a_\xi, \ldots$, where ξ runs over all ordinals less than α, be a sequence containing all the elements of A. Using transfinite recursion, we define

(3) This result was included in a second paper on Boolean algebras with ordered bases prepared by Mostowski and Tarski. This paper was set in type in 1939 but was destroyed during the war. The paper has never been reconstructed. The result mentioned, however, can be derived from a result in Mostowski [10] concerning the Stone space of a denumerable Boolean algebra.

sets B_ξ for each $\xi < \alpha$ as follows: $B_{\eta+1}$ consists of the elements of B_η together with all elements of the form $x + a_\eta(x'-x)$, where x and x' are any two successive elements of B_η (that is, $x, x' \in B_\eta$ and for no $y \in B_1$ is $x < y < x'$). For η a limit ordinal (or zero), B_η is the completion of $\bigcup_{\xi<\eta} B_\xi$, that is, the set of all elements of A which are expressible as sums or products of elements of $\bigcup_{\xi<\eta} B_\xi$. We now show by induction on η that the following conditions hold: (i) $B_\xi \subset B_\eta$ for all $\xi < \eta$. (ii) B_η is of power less than α. (iii) B_η is α-complete and simply ordered by $\leqslant$. (iv) a_η is a sum of differences of elements of $B_{\eta+1}$. Assume therefore that (i)-(iv) hold for all $\eta' < \eta$. (i) then follows immediately from the construction and (ii) is clear in the case that $\eta = \eta'+1$. In case η is a limit ordinal, B_η has power at most 2^β where β is the power of $\bigcup_{\xi<\eta} B_\xi$. By hypothesis of induction, each B_ξ has power less than α and so (ii) follows from the fact that α is inaccessible. For η a limit ordinal, (iii) is obvious since the union of an increasing family of simply ordered sets is simply ordered. For $\eta = \eta'+1$, B_η is simply ordered since each new element $x + a_{\eta'}(x'-x)$ lies between two successive elements of $B_{\eta'}$. B_η is α-complete since every subset of B_η either has a largest element or has the same least upper bound as the subset of $B_{\eta'}$ obtained by replacing each element $x + a_{\eta'}(x'-x)$ by the larger element x'. Thus (iii) holds. By the distributive law,

$$1 = \prod_{y \in B_\eta}(y+\bar{y}) = \sum_{Y \subseteq B_\eta}\Big(\prod_{y\in Y} y \cdot \prod_{y \in B_\eta - Y} \bar{y}\Big)$$

Now for each $Y \subseteq B_\eta$, if some element $y \in Y$ is smaller than some $z \in B_\eta - Y$, then $y \cdot \bar{z} = 0$ and hence

$$\prod_{y\in Y} y \cdot \prod_{y\in B_\eta - Y} \bar{y} = 0 .$$

Thus we need only consider the case when each element $y \in Y$ is larger than any $z \in B_\eta - Y$. Taking x to be the sum of all the elements of $B_\eta - Y$ and taking x' to be the product of the elements of Y, we see that either $x' = x$ or x' is the successor of x in B_η. In either case we have

$$\prod_{y\in Y} y \cdot \prod_{y\in B_\eta - Y} \bar{y} = x' - x$$

where x' denotes the successor of x in B_η if there is one and denotes x otherwise. Thus

$$a_\eta = a_\eta \cdot 1 = \sum_{x \in B_\eta} a_\eta \cdot (x'-x) .$$

But $a_\eta \cdot (x'-x) = [x + a_\eta \cdot (x'-x)] - x$ which is a difference of two elements of $B_{\eta+1}$ and so a_η is a sum of differences of elements of $B_{\eta+1}$. Therefore (iv) is established and the induction is complete. Let B be the union of all B_η for $\eta < \alpha$. By (iii) and (i), B is simply ordered by $\leqslant$. By (iv) and (ii), every element of A is a sum of fewer than α differences of elements of B. Hence B is an ordered α-basis of $\mathfrak{A}$.

THEOREM 2. *A necessary and sufficient condition for every α-distributive Boolean algebra of power at most α to have an ordered α-basis is that α satisfy one of the following three conditions:*

(a) *α is inaccessible,*

(b) *α is singular,*

(c) *for some β, $\beta < \alpha < 2^\beta$* [4].

Proof. The sufficiency of (a) is given by Theorem 1. In case α satisfies (b) or (c), it turns out that any α-distributive Boolean algebra of power at most α is atomistic and has fewer than α atoms. For α singular, this is easily seen by writing $1 = \prod_{a\in A}(a+\bar{a})$ as a product of fewer than α products each of fewer than α terms $a+\bar{a}$ and then applying the distributive law to each of these products of two-termed sums. Since $\mathfrak{A}$ is α^+-complete and of power at most α, it does not contain α disjoint elements, and so the resulting sums must each have fewer than α non-zero terms. Hence we can apply the distributive law a second time and we see that every non-zero term of the final sum must be an atom. In the case $\beta < \alpha < 2^\beta$, the atomicity of $\mathfrak{A}$ follows immediately from the proof that (iii) implies (i) in Theorem 2.14 of Smith-Tarski [12]. (This proof remains valid if the formula $\delta(A) < \beta$ in (iii) is replaced by the condition that the Boolean algebra does not contain a set of β disjoint elements.) It is clear that any α-complete atomistic Boolean algebra with fewer than α atoms has an ordered basis: Let a_ξ for $\xi < \beta$ be all the atoms of $\mathfrak{A}$; then the set of all elements $b_\eta = \sum_{\xi<\eta} a_\xi$ forms an ordered α-basis for $\mathfrak{A}$.

To show necessity, suppose that α does not satisfy (a), (b), or (c). Then α is regular and, for some $\beta < \alpha$, $2^\gamma = \alpha$ for all γ such that $\beta \leqslant \gamma < \alpha$. Let $\mathfrak{A}$ be the free α-distributive Boolean algebra on α generators. This Boolean algebra has power α since the set of all elements which are expressible as a sum or a product of generators has power $\sum_{\gamma<\alpha}\alpha^\gamma = \alpha$, then the set of all sums and products of these elements again has power α, and so on, by transfinite induction up to α. Suppose B is an ordered α-basis for $\mathfrak{A}$, and that G is a set of β of the free generators of $\mathfrak{A}$. Since each generator $g \in G$ can be expressed as a sum of fewer than α differences of elements of B, we see that all the elements of G can be expressed in terms of elements of some subset B' of B of power less than α. On the other hand, each element of B' can be expressed in terms of fewer than α of the free generators; hence there is some set G' of fewer than α generators

[4] Tarski has pointed out that conditions (b) and (c) can be replaced by the single condition: (d) For some β, $\beta < \alpha < \alpha^\beta$. Furthermore, he pointed out that conditions (a) and (b) can be replaced by the single condition that α is a strong limit number. Thus a necessary and sufficient condition for every α-distributive Boolean algebra of power at most α to have an ordered α-basis is that either for every $\beta < \alpha$, $2^\beta < \alpha$ or that for some $\beta < \alpha$, $2^\beta > \alpha$.

such that B' is included in the subalgebra α-generated by G'. Let B^* be the completion of B'; B^* is then also included in the subalgebra α-generated by G'. We wish now to show that there is a set of α disjoint elements, each of which is a difference of elements of B^*.

For each subset C of G, consider the element

$$c = \prod_{g \in C} g \cdot \prod_{g \in G-C} \bar{g} .$$

Since c is non-zero, its representation as a sum of differences of elements of B contains some non-zero difference $b-a$, where $a, b \in B$. It can be shown that $b-a$ is contained in some larger difference $e-d$ with $d, e \in B^*$ which is also contained in c. (If we use the irreducible representation of c as defined following this theorem, then a and b will already be elements of B^*.) By choosing such a difference $e-d$ corresponding to each subset C of G, we obtain the desired set D of disjoint elements each of which is a difference of elements of B^*. Since G has power β, D has power $2^\beta = \alpha$.

Consider now a generator g which is not in G'. No element of D can be included in or disjoint from g, for otherwise we would obtain an identity among the generators which does not hold in an arbitrary α-distributive Boolean algebra. But by the simple ordering of B, any difference of elements of B which does not contain any element of D can intersect at most two elements of D. Thus g cannot be the sum of fewer than α differences of elements of B and we are forced therefore to conclude that $\mathfrak{A}$ does not have an ordered α-basis.

Assuming the generalized continuum hypothesis, an immediate consequence of Theorem 2 is that a necessary and sufficient condition for every α-distributive Boolean algebra of power at most α to have an ordered α-basis in that α be a limit cardinal. Tarski has pointed out that the necessity of this condition actually implies the generalized continuum hypothesis. (Hence the necessity of the condition that α is a limit cardinal implies the sufficiency of the same condition.) Thus the generalized continuum hypothesis is equivalent, for example, to the statement that for every non-limit cardinal α, there exists an α-distributive Boolean algebra of power α which has no ordered α-basis.

We turn now to the problem of how the structure of an α-complete Boolean algebra $\mathfrak{A}$ is determined by the order type of its ordered α-basis B. In the following, R and S are assumed to range over subsets of $B \times B$ of power less than α. R is a *representation* of $a \in A$ if and only if

$$a = \sum_{\langle x, y\rangle \in R} (y-x) .$$

We now formulate (solely in terms of the ordering on B) a condition on a representation R which will insure that no subset of R with two or more elements can be replaced by a single element without changing the sum. R is *irreducible* if and only if the following conditions hold:

(i) *For any* $\langle x, y\rangle \in R$, $x < y$.

(ii) *If* $\langle x, y\rangle, \langle z, w\rangle \in R$ *and* $\langle x, y\rangle$ *overlaps* $\langle z, w\rangle$ (*i.e.* $x < w$ *and* $y > z$), *then* $\langle x, y\rangle = \langle z, w\rangle$.

(iii) *Suppose that* $x, y \in B$, $x < y$, *and, for all* $s, t \in B$, $x \leqslant s < t \leqslant y$ *implies that* $\langle s, t\rangle$ *overlaps some* $\langle u, v\rangle \in R$. *Then, for some* $\langle w, z\rangle \in R$, $w \leqslant x < y \leqslant z$ (5).

We write $S \leqslant R$ if, for every $\langle x, y\rangle \in S$, there is some $\langle w, z\rangle \in R$ such that $w \leqslant x < y \leqslant z$.

THEOREM 3. *Suppose that B is an ordered α-basis of an α-complete Boolean algebra $\mathfrak{A}$. Then*

(a) *If S and R are irreducible representations of a and b respectively, then $S \leqslant R$ holds if and only if $a \leqslant b$.*

(b) *Every element $a \in A$ has a unique irreducible representation.*

Proof. If $S \leqslant R$, then clearly $a \leqslant b$. If $a \leqslant b$, then $S \leqslant R$ follows from condition (iii). Thus (a) holds and hence every element has at most one irreducible representation. To show existence, suppose $a \in A$. Let S be any representation of a and let X be the range of the relation S. Two elements $x, y \in X$ will be said to be *equivalent* if $x-y$ and $y-x$ are both included in a. Taking R to be the set of all pairs $\langle x, y\rangle \in B \times B$ such that $x < y$ and x is the product and y the sum of some equivalence class of X, we easily check that R is an irreducible representation of a.

THEOREM 4. *Suppose that α is regular and B is an α-complete simply ordered set. If either* (i) *α is a strong limit number ($\beta < \alpha$ implies $2^\beta < \alpha$) or* (ii) *B has no family of α disjoint intervals, then there exists an α-complete Boolean algebra $\mathfrak{A}$ (unique up to isomorphism) having B as an ordered α-basis.*

A neccessary and sufficient condition for $\mathfrak{A}$ to be α-distributive is: (iii) *the completion of any subset of B of power less than α contains a gap.*

Proof. Let $\mathfrak{A} = \langle \mathcal{R}, \leqslant\rangle$, where $\mathcal{R}$ is the family of all irreducible subsets of $B \times B$ of power less than α. To show that $\mathcal{R}$ is closed to sums of fewer than α elements, let $\mathcal{R}'$ be a subset of $\mathcal{R}$ of power less than α.

(5) In case B is a densly ordered set, these conditions have a simple topological meaning, namely, that the set R is made up of pairs of endpoints of the open intervals of some regular open set. Kelley used this construction of the Boolean algebra of regular open sets in [7], p. 1172 to investigate Souslin's problem. The method of Kelley's proof of Theorem 12 is used in Theorem 4 below. Although it is not explicitly stated by Kelley, we can conclude that the Boolean algebra of regular open sets of a linear continuum which satisfies the countable interval condition is countably distributive just in case the linear continuum has no interval of real type.

Then the set X of all endpoints of elements of $\mathcal{R}'$ is of power less than a. Two elements x, y of X are said to be equivalent unless there exist elements $z, w \in B$ such that $x \leqslant z < w \leqslant y$ (or $y \leqslant z < w \leqslant x$) and for every $R \in \mathcal{R}$ and $\langle s, t\rangle \in R$, $\langle s, t\rangle$ does not overlap $\langle z, w\rangle$. Taking least upper and greatest lower bounds of the equivalence classes of X, we obtain an irreducible set which is easily shown to be the least upper bound of $\mathcal{R}'$. To show that $\mathcal{R}$ is closed to complementation, let R be an element of $\mathcal{R}$ and let X be the set of endpoints of R. Taking R' to be the set of intervals $\langle x, y\rangle$ such that x and y are successive elements of the completion of X and $\langle x, y\rangle \in R$, we see that either assumption (i) or (ii) implies that R' is of power less than a. It is then easy to check that R' is irreducible and is in fact the complement of R. Identifying each element $b \in B$ $(b \neq 0)$ with the irreducible set $\{\langle 0, b\rangle\}$ (and identifying 0, the smallest element of B, with the empty set), we see that B is an ordered a-basis of $\mathfrak{A}$.

To show the second part of the theorem, suppose first that $\mathfrak{A}$ is a-distributive and X is a subset of B of power less than a. Writing 1 as the product of terms $x+\overline{x}$ for $x \in X$ we see by the distributive law that there must be a subset Y of X such that the term

$$\prod_{x\in Y} x \cdot \prod_{x\in X-Y} \overline{x}$$

is different from 0. This gives two successive elements

$$\sum_{x\in X-Y} x \quad \text{and} \quad \prod_{x\in Y} \overline{x}$$

of the completion of X. Hence condition (iii) follows from the a-distributivity of $\mathfrak{A}$. Suppose now that (iii) holds and we have a double indexed family $a_{i,j}$ $(i \in I,\ j \in J)$ of elements of $\mathfrak{A}$ where I and J are of power less than a. It is clear that

$$a = \prod_{i\in I} \sum_{j\in J} a_{i,j}$$

is an upper bound of all the products

$$\prod_{i\in I} a_{i,f(i)}$$

for $f \in J^I$. Suppose that there were a smaller upper bound b. Let $\langle x, y\rangle$ be some interval contained in $a-b$ and let X be the set of all end points of the elements $a_{i,j}$ which lie between x and y. By (iii) the completion of X must contain a gap. Thus there exist elements s and t such that $x \leqslant s < t \leqslant y$ and no element of X lies between s and t. Hence the interval $\langle s, t\rangle$ is either included in or disjoint from each $a_{i,j}$ and we can conclude from the fact that $\langle s, t\rangle$ is included in a that, for each $i \in I$ there must be a $j \in J$ such that $\langle s, t\rangle$ is included in $a_{i,j}$. Chosing $f(i)$ to be such a j we see that $\langle s, t\rangle$ is included in

$$\prod_{i\in I} a_{i,f(i)}$$

and hence in b. This is impossible since $s < t$ and $\langle s, t\rangle$ is included in $a-b$. Hence a is the least upper bound of the products and we conclude that $\mathfrak{A}$ is distributive.

THEOREM 5. *If a cardinal a has property* $\mathbf{S}$, *then it has property* $\mathbf{P}_1$.

Proof. Since $\mathbf{P}_1$ holds for all accessible cardinals, we need only consider the case when a is inaccessible. Let $\mathfrak{A}$ be an a-complete set algebra which is a-generated by a elements and in which every a-complete prime ideal is principal. $\mathfrak{A}$ clearly has power a and since it is a set algebra, it is a-distributive. Hence by Theorem 1, $\mathfrak{A}$ has an ordered a-basis B and B must clearly be of power a. Thus it remains to show that B has no well ordered or inversely well ordered subset of power a. Suppose, on the contrary, that $x_0, \ldots, x_\eta, \ldots$ is an a-termed increasing sequence of elements of B. (The case of an a-termed decreasing sequence will be seen to be quite analogous.) We wish now to construct an a-complete non-principal prime ideal in $\mathfrak{A}$.

Let I_1 be the set of all $a \in A$ such that $a \leqslant x_\eta$ for some $\eta < a$. Let I_2 be the set of all $a \in A$ such that $a < \overline{x}_\eta$ for all $\eta < a$. Let $I = I_1 + I_2$. I_1 and I_2 and hence I are easily seen to be a-complete ideals of $\mathfrak{A}$. I is proper since if $a_1 \in I_1$ and $a_2 \in I_2$, then $a_1 \leqslant x_\eta$ for some $\eta < a$ and $a_2 < \overline{x}_{\eta+1}$ and so $a_1 + a_2 < 1$. To show that I is a prime ideal, consider any element $a \in A$. For some $R \subseteq B \times B$, R is of power less than a and

$$a = \sum_{\langle y,z\rangle \in R} (z-y)\,.$$

We distinguish two cases:

Case 1. For all $\langle y, z\rangle \in R$, $z \in I_1$ or $\overline{y} \in I_2$. Then, for all $\langle y, z\rangle \in R$, $z-y \in I$. Hence $a \in I$.

Case 2. For some $\langle y, z\rangle \in R$, $z \notin I_1$ and $\overline{y} \notin I_2$. From the definition of I_1 and I_2, it is seen that, for any $w \in B$, $w \in I_1$ or $\overline{w} \in I_2$. Hence $\overline{z} \in I_2$ and $y \in I_1$. Since $z-y \leqslant a$, we conclude that $\overline{a} \leqslant \overline{z} + y \in I_1 + I_2$. Hence $\overline{a} \in I$.

Finally, I is non-principal, for if $a \in A$ is an atom, then $a = z-y$, where y and z are two successive elements of B. But either $y \leqslant x_\eta$ for some $\eta < a$ in which case $z \leqslant x_{\eta+1}$ and so $a \in I_1$ or $y > x_\eta$ for all $\eta < a$ in which case $\overline{y}$ and hence a are in I_2. Thus I is the required a-complete non-principal prime ideal of $\mathfrak{A}$.

COROLLARY 6. *Properties* $\mathbf{P}_1$, $\mathbf{P}_2$, *and* $\mathbf{S}$ *are equivalent.*

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Erdös and Tarski in [3], Theorem 4.3, establish a connection between a representation problem for certain Boolean algebras and a problem about ramification systems (the exact problem depends on a given cardinal number). In this note we obtain a result (Theorem 2.1) which yields the converse of the implication proved in [3]. Actually we show that the ramification problem is equivalent to a compactness problem involving some special topological spaces. The definition of these spaces is given in Section 1, where the compactness problem is related to a prime ideal problem studied by Keisler and Tarski in [4]. The proof of equivalence of the representation problem of [3] and the prime ideal problem may be found in [4], Theorem 4.16.

In Section 3 the compactness problem is reformulated in simpler set-theoretical terms which make no reference to topological spaces.

§ 1. α-products of topological spaces. Throughout this note α, β, and γ will denote infinite cardinal numbers. Cardinals are considered as special kinds of ordinal numbers (initial numbers), and each ordinal coincides with the set of all smaller ordinals. The ξ-th infinite cardinal is denoted by ω_ξ. If $\alpha = \omega_\xi$, then $\alpha^+ = \omega_{\xi+1}$. The cardinal number of a set A is denoted by $|A|$. The set of all subsets of a set A is denoted by $S(A)$, and further

$$S_\alpha(A) = \{B \in S(A)\colon\ |B| < \alpha\}.$$

A topological space X is α-*complete* if the intersection of a family of power smaller than α of open sets is again open. Every space is of course ω_0-complete. Note that if α is a singular cardinal, then a space X is α-complete if and only if X is α^+-complete. The notion of an α-complete space is a natural generalization of the ordinary notion of a topological space, and many of the usual topological concepts may be appropriately modified for this class (see, e.g., [8]). We shall be concerned with two of these concepts, namely *compactness* and the formation of the *product topology*.