On monotonous mappings of complete lattices

by

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1. DEFINITION 1. Let S_1 and S_2 be (not necessarily disjoint) complete lattices and let f be a (one-valued) mapping of S_1 into S_2. f will be called *monotonous* if it is order-preserving, i.e. if for any pair s_1', s_1'' of elements of S_1 it holds

(1) $$s_1' \leqslant s_1'' \Rightarrow fs_1' \leqslant fs_1''.$$

Any homomorphism f of S_1 into S_2 is monotonous. For, if $s_1' \leqslant s_1''$ then $s_1' = s_1' \wedge s_1''$ and $fs_1' = f(s_1' \wedge s_1'') = fs_1' \wedge fs_1''$, hence $fs_1' \leqslant fs_1''$.

Let Z_1 be a subset of S_1. For any monotonous f and any $z_1 \in Z_1$ it will be $f \bigwedge_{s_1 \in Z_1} s_1 \leqslant fz_1$, so

(2a, b) $$f \bigwedge_{s_1 \in Z_1} s_1 \leqslant \bigwedge_{s_1 \in Z_1} fs_1 \quad \text{and dually} \quad \bigvee_{s_1 \in Z_1} fs_1 \leqslant f \bigvee_{s_1 \in Z_1} s_1 .$$

2. THEOREM 1. (Main Theorem). *Let $S_1, S_2, \ldots, S_n$ be (not necessarily disjoint) complete lattices, f_i monotonous mappings of S_i into S_{i+1}* (¹), *$i = 1, \ldots, n$, and $\sigma^\circ = \langle s_1^\circ, \ldots, s_n^\circ \rangle$ an n-tuple of elements $s_i^\circ \in S_i$.*

Then, there exists an n-tuple $\sigma' = \langle s_1', \ldots, s_n' \rangle$ of elements $s_i' \in S_i$ such that

(3_i) $$s_{i+1}' = s_{i+1}^\circ \vee f_i s_i', \qquad i = 1, \ldots, n,$$

and that, if $\zeta' = \langle z_1', \ldots, z_n' \rangle$ is any n-tuple of elements $z_i' \in S_i$ satisfying the relations

(4_i) $$s_{i+1}^\circ \vee f_i z_i' \leqslant z_{i+1}', \qquad i = 1, \ldots, n,$$

then

(5_i) $$s_i' \leqslant z_i', \qquad i = 1, \ldots, n.$$

Dually, there exists an n-tuple $\sigma'' = \langle s_1'', \ldots, s_n'' \rangle$ such that

($3_i'$) $$s_{i+1}'' = s_{i+1}^\circ \wedge f_i s_i'', \qquad i = 1, \ldots, n$$

and that, if $\zeta'' = \langle z_1'', \ldots, z_n'' \rangle$ is any n-tuple satisfying the relations

($4_i'$) $$z_{i+1}'' \leqslant s_{i+1}^\circ \wedge f_i z_i'', \qquad i = 1, \ldots, n,$$

then

($5_i'$) $$z_i'' \leqslant s_i''.$$

(¹) Throughout this paper an index j should always read "$j (\bmod n)$".

Proof. Let Σ be the set of all n-tuples $\sigma = \langle s_1, \dots, s_n\rangle$ of elements $s_i \in S_i$ with the property that for any of these n-tuples σ all the relations

(6$_i$) $$s_{i+1}^\circ \vee f_i s_i \leqslant s_{i+1}, \quad i = 1, \dots, n,$$

hold. We define s_i' as the g.l.b. of all s_i such that s_i is the ith component of some element σ of Σ:

(7$_i$) $$s_i' = \bigwedge_{\sigma\in\Sigma} s_i .$$

Then, by (2a), (6$_i$) and (7$_{i+1}$)

(8$_i$) $$f_i s_i' \leqslant \bigwedge_{\sigma\in\Sigma} f_i s_i \leqslant \bigwedge_{\sigma\in\Sigma} s_{i+1} = s'_{i+1} .$$

By (7$_{i+1}$) and (6$_i$) it is also

(9$_i$) $$s_{i+1}^\circ \leqslant s_{i+1}' ,$$

so

(10$_i$) $$s_{i+1}^\circ \vee f_i s_i' \leqslant s_{i+1}',$$

(11$_i$) $$f_{i+1}(s_{i+1}^\circ \vee f_i s_i') \leqslant f_{i+1} s_{i+1}',$$

(12$_i$) $$s_{i+2}^\circ \vee f_{i+1}(s_{i+1}^\circ \vee f_i s_i') \leqslant s_{i+2}^\circ \vee f_{i+1} s_{i+1}' .$$

This means by (6$_{i+1}$) that $\langle s_1^\circ \vee f_n s_n', s_2^\circ \vee f_1 s_1', \dots, s_n^\circ \vee f_{n-1} s_{n-1}'\rangle$ is an element of Σ, whence by (7$_{i+1}$)

(13$_i$) $$s_{i+1}' \leqslant s_{i+1}^\circ \vee f_i s_i' .$$

Together with (10$_i$) this yields

(14$_i$) $$s_{i+1}' = s_{i+1}^\circ \vee f_i s_i' .$$

On the other hand, let $\zeta' = \langle z_1', \dots, z_n'\rangle$ satisfy (4$_i$). Then by (6$_i$) ζ' is an element of Σ, whence, by (7$_i$), $s_i' \leqslant z_i'$. This completes the proof.

Remark. Because of (3$_i$) we see a posteriori that the definition (7$_i$) under the conditions (6$_i$) is equivalent to the same definition under the stronger conditions

(6$_i'$) $$s_{i+1}^\circ \vee f_i s_i = s_{i+1} .$$

However, if we would have taken this as the definition of the set Σ, we could (in the same way as before) infer only (10$_i$), but not the opposite relations (13$_i$).

3. We state some corollaries of the theorem. For $n = 2$, $n = 1$ we have:

COROLLARY 1. *Let S_1, S_2 be complete lattices and f_1, f_2 monotonous mappings of S_1 into S_2, S_2 into S_1, respectively. Then, for any pair of elements $\langle s_1^\circ, s_2^\circ\rangle$, $s_1 \in S_1$, $s_2 \in S_2$, there exists a pair $\langle s_1', s_2'\rangle$ [alternatively: $\langle s_1'', s_2''\rangle$] such that*

(15 a, b) $$s_2' = s_2^\circ \vee f_1 s_1', \quad s_1' = s_1^\circ \vee f_2 s_2', \quad [s_2'' = s_2^\circ \wedge f_1 s_1'', \quad s_1'' = s_1^\circ \wedge f_2 s_2'']$$

and that, if $\langle z_1, z_2\rangle$ is any pair satisfying

(16 a, b) $$s_2^\circ \vee f_1 z_1 \leqslant z_2, \quad s_1^\circ \vee f_2 z_2 \leqslant z_1, \quad [z_2 \leqslant s_2^\circ \wedge f_1 z_1, \quad z_1 \leqslant s_1^\circ \wedge f_2 z_2],$$

then

(17 a, b) $$s_1' \leqslant z_1, \quad s_2' \leqslant z_2, \quad [z_1 \leqslant s_1'', \quad z_2 \leqslant s_2''] .$$

COROLLARY 2. *Let f be a monotonous mapping of a complete lattice S into itself, and let s° be an element of S. Then there exist elements s', s'' such that*

(18 a, b) $$s' = s^\circ \vee f s', \quad s'' = s^\circ \wedge f s''$$

and that, if z is any element of S such that

(19 a, b) $$s^\circ \vee f z \leqslant z \quad [z \leqslant s^\circ \wedge f z],$$

then $s' \leqslant z$ $[z \leqslant s'']$.

For $\sigma^0 = \langle 0_1, \dots, 0_n\rangle$ $[\sigma^0 = [1_1, \dots, 1_n\rangle]$ we get from the theorem

COROLLARY 3. *Let $S_1, \dots, S_n$ be complete lattices and f_i monotonous mappings of S_i into S_{i+1}. Then there exist n-tuples $\sigma' = \langle s_1', \dots, s_n'\rangle$ $[\sigma'' = \langle s_1'', \dots, s_n''\rangle]$ satisfying*

(20$_i$) $$s_{i+1}' = f_i s_i', \quad s_{i+1}'' = f_i s_i''$$

and such that if $\zeta = \langle z_1, \dots, z_n\rangle$ is any n-tuple satisfying

(21 a, b) $$f_i z_i \leqslant z_{i+1} \quad [z_i \leqslant f_i z_i],$$

then

(22$_i$) $$s_i' \leqslant z_i \quad [z_{i+1} \leqslant s_i''] .$$

In particular, if ζ is any "closed f-chain" (as are σ' and σ''), i.e. if

(21$_i'$) $$z_{i+1} = f_i z_i ,$$

then

(22$_i'$) $$s_i' \leqslant z_i \leqslant s_i'' .$$

For $n = 2$ and $n = 1$ this yields:

COROLLARY 4. *Let S_1, S_2 be complete lattices and f_1, f_2 monotonous mappings of S_1 into S_2, S_2 into S_1, respectively. Then there exist pairs $\langle s_1', s_2'\rangle$, $[\langle s_1'', s_2''\rangle]$ such that*

(23 a, b) $$s_2' = f_1 s_1', \quad s_1' = f_2 s_2' \quad [s_2'' = f_1 s_1'', \; s_1'' = f_2 s_2''],$$

and that, if $\langle z_1, z_2\rangle$ is any pair such that

(24 a, b) $$f_1 z_1 \leqslant z_2, \quad f_2 z_2 \leqslant z_1 \quad [z_2 \leqslant f_1 z_1, \; z_1 \leqslant f_2 z_2],$$

then

(25 a, b) $$s_1' \leqslant z_1, \; s_2' \leqslant z_2 \quad [z_1 \leqslant s_1'', \; z_2 \leqslant s_2''] .$$

Corollary 5. *Let f be a monotonous mapping of a complete lattice S into itself. Then there exist fix-points s', s'' of S for f such that if z is any element of S satisfying*

(26a, b) $$fz \leqslant z \quad [z \leqslant fz],$$

then $s' \leqslant z$ $[z \leqslant s'']$; in particular such that any fix-point z of S for f satisfies $s' \leqslant z \leqslant s''$ ([2]).

4. Theorem 1 and Corollaries 1-5 were derived without use of the axiom of choice and the (completed) set of natural numbers.

By Theorem 1, to each n-tuple $\sigma^\circ = \langle s_1^\circ, \ldots, s_n^\circ \rangle$ were associated uniquely determined n-tuples $\varphi\sigma^\circ = \sigma' = \langle s_1', \ldots, s_n' \rangle$, $\psi\sigma^\circ = \sigma'' = \langle s_1'', \ldots, s_n'' \rangle$ satisfying (3_i) resp. $(3_i')$ and the extremal properties (5_i) resp. $(5_i')$.

With the use of transfinite ordinals we can give another definition of $\varphi\sigma$, $\psi\sigma$, alternative to (7_i) and its dual.

Let γ be the smallest ordinal with the property that the set of all preceding ordinals has a cardinality greater than the greatest of the cardinals $kS_1, \ldots, kS_n$ of the sets $S_1, \ldots, S_n$. We define a transfinite sequence of length $\gamma+1$ of n-tuples $\sigma_a = \langle s_{1a}, \ldots, s_{na} \rangle$ by transfinite recursion in the following way:

(27_i) $$s_{i0} = s_i^\circ, \quad \text{i.e.} \quad \sigma_0 = \sigma^\circ;$$

if $a > 0$ is an ordinal which is not a limit number, then

(28_i) $$s_{ia} = s_{i,a-1} \vee f_{i-1} s_{i-1,a-1};$$

if $a > 0$ is a limit-number, then

(29_i) $$s_{ia} = \bigvee_{\beta < a} s_{i\beta}.$$

Then obviously

(30_i) $$a_1 \leqslant a_2 \Rightarrow s_i^\circ \leqslant s_{ia_1} \leqslant s_{ia_2}.$$

First we prove that there is an ordinal δ, $\delta < \gamma$, such that $\sigma_\delta = \sigma_\gamma$.

For each i, $i = 1, \ldots, n$, there is an ordinal $\delta_i < \gamma$ such that $s_{i\delta_i} = s_{i,\delta_i+1}$. For, if for some i and all ordinals a $(a < \gamma)$

(31_i) $$s_{ia} < s_{i,a+1}$$

would hold, then, because of $\{s_{ia} | a < \gamma\} \subset S_i$, this would contradict the assumption $kS_i < k\{a | a < \gamma\}$. (With (31_i) also $s_{i\beta} \neq s_{ia}$, $a < \beta$ for limit-numbers β; otherwise $s_{ia} = s_{i,a+1}$, contrary to (31_i).) The greatest of all δ_i can be taken as δ; then $\sigma_\delta = \sigma_{\delta+1}$.

([2]) The special case of this Corollary when S is the power-set of a given set with $\subset$ as $\leqslant$ is contained in the author's short note *On monotone mappings of the power-set*, Portugaliae Mathematica 21, 2 (1962), pp. 111-112.

Theorem 2. *We have $\sigma_\delta = \sigma'$.*

Proof. From

(32_i) $$s_{i\delta} = s_{i,\delta+1}$$

we infer by (30_i) and (28_i), for $a = \delta+1$,

$$s_i^\circ \vee f_{i-1} s_{i-1,\delta} \leqslant s_{i\delta} \vee f_{i-1} s_{i-1,\delta} = s_{i,\delta+1} = s_{i\delta},$$

hence by Theorem 1, (4_{i-1}), (5_i) it holds

(33_i) $$s_i' \leqslant s_{i\delta}.$$

On the other hand by (27_i) and (3_{i-1})

(34_i) $$s_{i0} = s_i^\circ \leqslant s_i'$$

so the relations

(35_i) $$s_{ia} \leqslant s_i'$$

hold good for $a = 0$.

Let us suppose that they hold good for all a, $a < \beta$. In order to prove that they then hold good for β too, we distinguish two cases:

a) *β is a limit number.* Then, by (29_i), (35_i) is obviously satisfied for $a = \beta$.

b) *β is not a limit-number.* Then, by the induction hypothesis $((35_{i-1})$ for $a = \beta-1)$

(36_i) $$s_{i-1,\beta-1} \leqslant s_{i-1}',$$

(37_i) $$f_{i-1} s_{i-1,\beta-1} \leqslant f_{i-1} s_{i-1}',$$

(38_i) $$s_i^\circ \vee f_{i-1} s_{i-1,\beta-1} \leqslant s_i^\circ \vee f_{i-1} s_{i-1}' = s_i'.$$

Hence by (28_i) for $a = \beta$, (30_i) for $a_1 = 0$, $a_2 = \beta-1$, (38_i) and (36_{i+1})

(39_i) $$\begin{aligned} s_{i\beta} &= s_{i,\beta-1} \vee f_{i-1} s_{i-1,\beta-1} = (s_{i,\beta-1} \vee s_i^\circ) \vee f_{i-1} s_{i-1,\beta-1} \\ &= s_{i,\beta-1} \vee (s_i^\circ \vee f_{i-1} s_{i-1,\beta-1}) \leqslant s_{i,\beta-1} \vee s_i' = s_i'. \end{aligned}$$

This proves that (35_i) holds for any $a < \gamma$, in particular for $a = \delta$; this and (33_i) yields $\sigma_\delta = \sigma'$.

Dually we get a transfinite series with the limit $\psi\sigma^\circ$.

5. It is easy to see that the *mappings* $\varphi_i s_i^\circ = s_i'$ $[\psi_i s_i^\circ = s_i'']$ *are monotonous increasing* [*decreasing*], i.e.

$(40_i$a, b) $$s_i \leqslant \varphi_i s_i \quad [\psi_i s_i \leqslant s_i],$$

and idempotent mappings of S_i into itself.

Proof. The monotony is obvious from (6_i) and (7_i); (40a) follows from (6_i).

From (40a) and the monotony of φ_i we infer

(41) $$\varphi_i s_i \leqslant \varphi_i^2 s_i\,.$$

On the other hand, for $s_{i+1}^\circ = s_{i+1}$ (3_i) yields

(42) $$\varphi_{i+1}s_{i+1} = s_{i+1}\vee f_i\varphi_i s_i\,,$$

so

(43) $$f_i\varphi_i s_i \leqslant \varphi_{i+1}s_{i+1}$$

and also

(44) $$\varphi_{i+1}s_{i+1}\vee f_i\varphi_i s_i \leqslant \varphi_{i+1}s_{i+1}\,.$$

But (44) means that $\zeta' = \langle\varphi_1 s_1, \ldots, \varphi_n s_n\rangle$ satisfies (4_i) for $s_{i+1}^\circ = \varphi_{i+1}s_{i+1}$, hence by (5_i)

(45) $$\varphi_i^2 s_i \leqslant \varphi_i s_i\,.$$

(41) and (45) prove the idempotency of φ_i.

Let now $\varphi[\psi]$ have the meanings as in 4. Then, *σ is a fix-point for $\varphi[\psi]$ if and only if all the relations*

(46_i) $$f_i s_i \leqslant s_{i+1} \quad [s_{i+1} \leqslant f_i s_i]\,, \qquad i = 1, \ldots, n\,,$$

hold.

Proof. Let $\varphi\sigma = \sigma$. Then by (3_i)

(47_i) $$s_{i+1} = \varphi_{i+1}s_{i+1} = s_{i+1}\vee f_i\varphi_i s_i = s_{i+1}\vee f_i s_i\,,$$

so (46_i) is satisfied.

On the other hand, from (46_i) we infer

(48_i) $$s_{i+1}\vee f_i s_i = s_{i+1}\,,$$

so (4_i) is satisfied with $\zeta' = \sigma$ and therefore by (5_i)

(49_i) $$\varphi_i s_i \leqslant s_i\,.$$

$(40_i a)$ and (49_i) prove that now $\varphi\sigma = \sigma$.

6. As an illustration for applications of the Theorem 1 we prove the theorem of Cantor-Bernstein.

THEOREM OF CANTOR-BERNSTEIN. *Let S_1, S_2 be sets and f_1, f_2 (1-1)-mappings of S_1 into S_2, S_2 into S_1, respectively. Then, there exists a (1-1)-mapping of S_1 onto S_2.*

Proof. The power-sets $\mathbb{S}S_1$, $\mathbb{S}S_2$ of S_1, S_2 are complete lattices if $\leqslant$ is defined as the set-theoretical inclusion $\subset$. We extend f_1, f_2 to mappings of $\mathbb{S}S_1$ into $\mathbb{S}S_2$, $\mathbb{S}S_2$ into $\mathbb{S}S_1$ respectively by

(50a, b) $$f_1 Z_1 = \bigcup_{z_1\in Z_1}\{f_1 z_1\}\,, \qquad f_2 Z_2 = \bigcup_{z_2\in Z_2}\{f_2 z_2\}$$

for all $Z_1 \subset S_1$, $Z_2 \subset S_2$. Obviously, f_1, f_2 are monotonous.

By Corollary 1, for $s_1^\circ = S_1 - f_2S_2$, $s_2^\circ = O$; $s_1^\circ = \emptyset$, $s_2^\circ = S_2 - f_1S_1$ there exist sets U_1, U_2; V_1, V_2 such that (cf. (15a), (16a), (17a))

(51a, b) $$U_2 = f_1U_1\,, \qquad U_1 = (S_1 - f_2S_2)\cup f_2U_2\,;$$

(52a, b) $$V_2 = (S_2 - f_1S_1)\cup f_1V_1\,, \qquad V_1 = f_2V_2$$

and that for any pair U_1', U_2' satisfying (51a, b) [V_1', V_2' satisfying (52a, b)] it is

(52′a, b) $$U_1 \subset U_1'\,, \qquad U_2 \subset U_2'\,; \qquad [V_1 \subset V_1',\ V_2 \subset V_2']\,.$$

By Corollary 4 there exist sets W_1, W_2 such that (cf. (23b), (24b), (25b))

(53a, b) $$W_2 = f_1W_1\,, \qquad W_1 = f_2W_2$$

and that for any pair W_1', W_2' satisfying (53a, b) with "$\subset$" instead of "$=$" it is

(54) $$W_1' \subset W_1\,, \qquad W_2' \subset W\,.$$

We shall prove that U_1, V_1, W_1 and U_2, V_2, W_2 are disjoint and that $U_1\cup V_1\cup W_1 = S_1$, $U_2\cup V_2\cup W_2 = S_2$.

First ($A\backslash B$ is the set of elements of A which are not elements of B),

$$U_2\backslash W_2 = f_1U_1\backslash f_1W_1 = f_1(U_1\backslash W_1)\,;$$

$$\begin{aligned}U_1\backslash W_1 = [(S_1-f_2S_2)\cup f_2U_2]\backslash f_2W_2 &= (\text{because of } f_2W_2\subset f_2S_2)(S_1-f_2S_2)\cup\\ &\cup (f_2U_2\backslash f_2W_2) = (S_1-f_2S_2)\cup f_2(U_2\backslash W_2)\,,\end{aligned}$$

so by (51a) $U_1 \subset U_1\backslash W_1$, $U_2 \subset U_2\backslash W_2$, i.e. $U_1\cap W_1 = \emptyset$, $U_2\cap W_2 = \emptyset$. Analogously $V_1\cap W_1 = \emptyset$, $V_2\cap W_2 = \emptyset$.

Secondly,

$$\begin{aligned}U_2\backslash V_2 = f_1U_1\backslash[(S_2-f_1S_1)\cup f_1V_1] &= (\text{because of } f_1S_1\supset f_1U_1)\ f_1U_1\backslash f_1V_1\\ &= f_1(U_1\backslash V_1)\,;\end{aligned}$$

$$\begin{aligned}U_1\backslash V_1 &= [(S_1-f_2S_2)\cup f_2U_2]\backslash f_2V_2\\ &= (\text{because of } f_2V_2\subset f_2S_2)\,(S_1-f_2S_2)\cup(f_2U_2\backslash f_2V_2)\\ &= (S_1-f_2S_2)\cup f_2(U_2\backslash V_2)\,,\end{aligned}$$

so by (51a) $U_1\subset U_1\backslash V_1$, $U_2\subset U_2\backslash V_2$, i.e. $U_1\cap V_1 = \emptyset$, $U_2\cap V_2 = \emptyset$.

Thirdly, because of $A\backslash B\supset (A\cup C)\backslash(B\cup C)$,

$$\begin{aligned}f_1[S_1-(U_1\cup V_1\cup W_1)] = f_1S_1-(f_1U_1\cup f_1V_1\cup f_1W_1)\supset[(S_2-f_1S_1)\cup\\ \cup f_1S_1]\backslash[(S_2-f_1S_1)\cup f_1U_1\cup f_1V_1\cup W_2] = S_2-(U_2\cup V_2\cup W_2)\,,\end{aligned}$$

and analogously

$$f_2[S_2-(U_2\cup V_2\cup W_2)]\supset S_1-(U_1\cup V_1\cup W_1)\,,$$

so by (54) $S_1-(U_1\cup V_1\cup W_1)\subset W_1$, i.e. $S_1-(U_1\cup V_1\cup W_1) = \emptyset$ or $U_1\cup V_1\cup W_1 = S_1$ and analogously $U_2\cup V_2\cup W_2 = S_2$.

Hence the mapping f of S_1 into S_2 defined by

$$fs_1 = \begin{cases} f_1 s_1 & \text{for} \quad s_1 \in U_1 \cup W_1, \\ f_2^{-1} s_1 & \text{for} \quad s_1 \in V_1 \end{cases}$$

is (1-1) and onto.

A new proof of the well-ordering theorem, as suggested by the proof of our main theorem, will appear in Colloquium Mathematicum.

Addendum. The author wants to express his gratitude to the Editors of Fundamenta Mathematicae for calling his attention to the following: Our principal theorem is a generalization of a theorem by A. Tarski (*A lattice-theoretical fix-point theorem and its application*, Pacific J. Math. 5 (1955), pp. 285-309). Some results concerning the same topic are included also in the paper of E. S. Wolk, Canad. J. Math. 9 (1957), pp. 400-405 and in the paper of A. C. Davis, *A characterization of complete lattices*, Pacific J. Math. 5 (1955), pp. 311-319. Concerning the theorem of Cantor-Bernstein, a proof analogous to ours is included in a paper of R. Sikorski, *On a generalization of theorems of Banach and Cantor-Bernstein*, Colloquium Math. 1 (1948), p. 140-144, and also in the book of A. Tarski, *Cardinal Algebras*.

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Mehrfach wohlgeordnete Mengen und eine Verschärfung eines Satzes von Lindenbaum

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§ 1. Einleitung. Hat man in einer unendlichen Menge M eine Menge T von Wohlordnungen, so nennen wir eine Teilmenge $K \subset M$ einen *bezüglich T ordnungsgleichen Kern*, wenn in K alle Wohlordnungen aus T übereinstimmen. Nennen wir ferner eine Teilmenge $M' \subset M$, die $\overline{\overline{M'}} = \overline{\overline{M}}$ erfüllt, einen *Vollteil von M*, so besagt der Satz von Lindenbaum aus [2]:

Hat man eine unendliche Menge M und in M endlich viele Wohlordnungen, so gibt es zu diesen einen ordnungsgleichen Kern $K \subset M$ der Mächtigkeit $\overline{\overline{K}} = \overline{\overline{M}}$; $-K$ ist also Vollteil von M.

Ziel dieser Arbeit ist es, diesen Satz auch noch auf mehr als endlich viele Wohlordnungen auszudehnen. Es wird sich ergeben:

Ist $\overline{\overline{M}} = \aleph_{\nu+1}$, c die kleinste Kardinalzahl, für die $\aleph_\nu^c > \aleph_\nu$ ist, und T eine Menge von Wohlordnungen von M mit $\overline{\overline{T}} < c$, so gibt es einen ordnungsgleichen Kern bezüglich T, der Vollteil von M ist.

Für Limeskardinalzahlen wird ein analoger Satz hergeleitet werden.

Der Satz von Lindenbaum läßt sich beweisen, indem man ihn für den Fall zweier Wohlordnungen beweist, woraus sofort seine Gültigkeit für den Fall endlich vieler Wohlordnungen durch Schluß von n auf $n+1$ folgt. Für unseren Fall läßt sich diese Methode nicht mehr anwenden. Wichtigstes Hilfsmittel für unseren Beweis wird ein Satz graphentheoretischer Art (Satz 1) sein.

§ 2. Definitionen. Wir brauchen im Folgenden einige Begriffe, die etwas allgemeiner als die analogen Begriffe in [1] definiert sind.

Die zu einer Kardinalzahl k gehörige Anfangszahl bezeichnen wir wieder mit $\omega(k)$, und $W(\alpha)$ bezeichne die Menge aller Ordinalzahlen, die $<\alpha$ sind.

Unter einer *Folge innerhalb* $W(\omega_\mu)$ verstehen wir jede (transfinite) Folge f: $\alpha_0, \alpha_1, \ldots, \alpha_\varkappa, \ldots$, $\varkappa < \lambda$, von Ordinalzahlen aus $W(\omega_\mu)$; dabei heiße λ die *Länge der Folge*, sie sei auch mit $l(f)$ bezeichnet. Statt α_ν schreiben wir auch $f(\nu)$. Ist $\varkappa \leqslant \lambda$, so nennen wir die Folge der α_σ, $\sigma < \varkappa$, den