Zitate

[1] E. Harzheim, *Dualzerlegungen in totalgeordneten Mengen*, this volume, S. 81-91.

[2] A. Lindenbaum, *Z teorii uporządkowania wielokrotnego* (*Sur la théorie de l'ordre multiple*), Wiadomości Matematyczne 37 (1934), S. 1-35.

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Rotation groups under monotone transformations

by

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Lucille Whyburn [1] has several theorems concerning rotation groups under topological transformations. We generalize her results by relaxing the condition that the transformation be topological.

Unless otherwise specified, S is to mean a compact Hausdorff space (M is to mean a plane Peano continuum), T will be a monotone transformation such that $T(S) = S$ $(T(M) = M)$, and if T allows a set K of fixed points then $T^{-1}(K) = K$. The proofs of most of the theorems follow in a straightforward manner along the lines of Whyburn's proofs and are not included.

THEOREM 1. *If C_1 is a component of $S-K$, then $T(C_1)$ and $T^{-1}(C_1)$ are components of $S-K$.*

THEOREM 2. *Let C_0 be a component of $S-K$, $C_n = T^n(C_0)$ for each n, and let G be the collection C_n. Then G forms a commutative group where the group operation is defined for any two elements C_i and C_j as $C_iC_j = T^j(C_i)$.*

THEOREM 3. *Every component of $S-K$ lies in one and only one rotation group.*

LEMMA 1. *If $T(x_0) = x_0$ and $\{x_i\}$ is a sequence of distinct points converging to x_0, then $\{T^{-1}(x_i)\}$ is a sequence of sets converging to x_0.*

Proof. Letting $L = \bigcup_{i=0}^{\infty} x_i$, it follows from the continuity of T that $T^{-1}(L)$ is closed and hence compact. Suppose first that $x_0 \notin \overline{\lim_i}\, T^{-1}(x_i)$. Then there is an open set U_0 about x_0 such that $U_0 \cap T^{-1}(L) = x_0$. Let U_i for $i > 0$ be an open set about x_i such that $U_i \cap (\bigcup_{j\neq i, j=0}^{\infty} x_j) = \emptyset$. It follows that $U_i' = T^{-1}(U_i)$ is an open set containing $T^{-1}(x_i)$ such that $U_i' \cap (\bigcup_{j\neq i, j=0}^{\infty} T^{-1}(x_i)) = \emptyset$. Hence the collection $\{U_i'\}$ including $i = 0$ is an open covering of $T^{-1}(L)$ which contains no finite subcovering. This contradicts the fact that $T^{-1}(L)$ is compact. Suppose now that there exists a point y in $\overline{\lim}\, T^{-1}(x_i)$ such that $y \neq x_0$. Then by the definition of the limit superior, given any open set U about y, U intersects infinitely many of the sets $T^{-1}(x_i)$ and hence $T(U)$ contains an infinity of the points x_i. Therefore, $T(y)$ is a limit point of the sequence $\{x_i\}$. It is known that $T(y) \neq x_0$. This contradicts the fact that the sequence $\{x_i\}$ converges to x_0. Let $\{T^{-1}(x_{i_k})\}$ be any subsequence of $\{T^{-1}(x_i)\}$. It follows, from an argument similar to the one above, that $\underline{\lim}_{i_k} T^{-1}(x_{i_k}) = x_0$. Hence it follows that $\lim_i T^{-1}(x_i) = x_0$.

THEOREM 4. *If C_i and C_j are two elements of a rotation group, then $\bar{C}_i - C_i = \bar{C}_j - C_j$.*

Proof. It is now shown that $\bar{C}_i - C_i \subset \bar{C}_j - C_j$. Note that $C_i = T^i(C_0)$ and $C_j = T^j(C_0)$. Hence $C_j = T^{j-i}(C_i)$. First consider the case where $j-i = 0$. It then follows that $C_i = C_j$ and the theorem is proved. Next suppose that $j-i = k > 0$, then T^k is a continous mapping of C_i onto C_j. If $x \in \bar{C}_i - C_i$, then there is a sequence of points $\{x_i\}$ in C_i which converge to x. By the continuity of T^k it follows that the sequence $\{T^k(x_i)\}$ converges to $T(x) = x$. Therefore, x is in $\bar{C}_j - C_j$. Finally suppose that $j-i = -k < 0$, in which case $T^{-k}(C_i) = C_j$, or $T^k(C_j) = C_i$. Note that $(T^k)^{-1}$ is T^{-k}. Again let $\{x_i\}$ be a sequence of distinct points from C_i converging to x in $\bar{C}_i - C_i$. Since T^k satisfies the conditions in Lemma 1, it follows that $\lim T^{-1}(x_i) = x$. Therefore x is in $\bar{C}_j - C_j$ since $T^{-k}(x_i) \subset C_j$ for each $i > 1$. In a similar manner it can be shown that $\bar{C}_j - C_j \subset \bar{C}_i - C_i$. Therefore it follows that $\bar{C}_i - C_i = \bar{C}_j - C_j$.

COROLLARY. *If S is locally connected, then for any rotation group $F(\bigcup_i C) = \bigcup_i F(C_i) = F(C_k)$ for C_k a fixed element of the rotation group (where $F(A)$ denotes the boundary of A relative to S).*

COROLLARY. *If p is in $F(C_0)$ and accessible from C_0, then for any C_n in G, $n > -1$, p is accessible from C_n.*

THEOREM 5. *If C is an element of a rotation group of M under T of order greater than 1, then C has property S.*

From a result of G. T. Whyburn [2], it follows that under the hypothesis of Theorem 5, every point of $F(C)$ is accessible from C.

THEOREM 6. *If C is an element of a rotation group G of M with order greater then 1, then $F(C)$ is contained in some simple closed curve. If $F(C)$ contains more than two points, then the order of G is less than or equal two. If G is infinite, then $F(C)$ reduces to one point, and for any preassigned positive number ε, the diameter of C_i is less than ε for all but a finite number of the subscripts i.*

THEOREM 7. *Let M be a two-dimentional sphere. If there is a rotation group under T of order greater than 1, then K is a simple closed curve. Hence, there is only one rotation group and it has order 2.*

References

[1] L. Whyburn, *Rotation groups about a set of fixed points*, Fund. Math. 28 (1937), pp. 124-130.

[2] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Coll. Publ. 28 (1942).

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