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A note on v^*-algebras

by

W. Narkiewicz (Wrocław)

A v^*-algebra is an abstract algebra $\mathfrak{A} = (\mathcal{A}, F)$ satisfying the following conditions:

(i) *If $a \in \mathcal{A}$ and a is not an algebraic constant, then the set $\{a\}$ is a set of independent elements.*

(ii) *If $\{a_1, \ldots, a_n\}$ is a set of independent elements and $\{a_1, \ldots, a_{n+1}\}$ is not a set of independent elements, then a_{n+1} belongs to the subalgebra generated by $\{a_1, \ldots, a_n\}$.* (Independence is to be understood in the sense of E. Marczewski. See [1], [2].)

Some properties of v^*-algebras have been developed in [3]. Here we shall prove a strengthening of theorem II of [3].

Let $\mathfrak{A}$ be any algebra. By $A^{(n)}$ we denote the set of all algebraic functions of n variables, and by $A^{(n,k)}$ we denote the set of all functions of $A^{(n)}$ depending on at most k variables.

THEOREM. *If $\mathfrak{A} = (\mathcal{A}, F)$ is an n-dimensional v^*-algebra, and $A^{(3)} = A^{(3,1)}$, then there exist a group G of transformations of the set $\mathcal{A}$ and a subset $\mathcal{A}_0 \subset \mathcal{A}$ containing all fixed points of the transformations from G such that $G(\mathcal{A}_0) \subset \mathcal{A}_0$, and moreover every algebraic function of n variables is of the form:*

$$f(x_1, \ldots, x_n) = g(x_i) \quad \text{for} \quad g \in G \text{ and } 1 \leqslant i \leqslant n,$$

or

$$f(x_1, \ldots, x_n) = a \quad \text{for} \quad a \in \mathcal{A}_0.$$

In view of theorem II of [3] it suffices to prove that $A^{(n)} = A^{(n,1)}$, whence the theorem results at once from the following

LEMMA. *If $\mathfrak{A}$ is an v^*-algebra, $k \geqslant 3$, $A^{(k)} = A^{(k,1)}$, and $\dim \mathfrak{A} \geqslant k+1$, then $A^{(k+1)} = A^{(k+1,1)}$.*

Proof. Suppose that the set $A^{(k+1)} \setminus A^{(k+1,k)}$ is non-void, and let $f \in A^{(k+1)} \setminus A^{(k+1,k)}$. Hence the set $\{f(x_1, \ldots, x_{k+1}), x_2, \ldots, x_{k+1}\}$ in the algebra $A^{(k+1)}$ is independent, and thus this set generates the whole algebra $A^{(k+1)}$. There exists an $F \in A^{(k+1)}$ such that

$$x_1 = F\big(f(x_1, \ldots, x_{k+1}), x_2, \ldots, x_{k+1}\big).$$

By putting here $x_1 = X$, $x_2 = \ldots = x_{k+1} = Y$ we obtain

$$X = F\big(f(X, Y, \ldots, Y), Y, \ldots, Y\big).$$

From the hypothesis, $f(X, Y, \ldots, Y) \in A^{(2,1)}$. If we had $f(X, Y, \ldots, Y) = h(Y)$ with some h, then $X = F\big(h(Y), Y, \ldots, Y\big)$—a contradiction; hence $f(X, Y, \ldots, Y) = h(X)$ with a suitable $h \in A^{(1)}$.

The set $\{f(x_1, \ldots, x_{k+1}), x_1, x_3, x_4, \ldots, x_{k+1}\}$ is independent in the algebra $A^{(k+1)}$; hence, as before, we obtain

$$x_2 = G\big(f(x_1, x_2, \ldots, x_{k+1}), x_1, x_3, \ldots, x_{k+1}\big)$$

with a suitable $G \in A^{(k+1)}$.

By putting here $x_1 = X, x_2 = Y, x_3 = \ldots = x_{k+1} = Z$ we obtain

$$Y = G\big(f(X, Y, Z, \ldots, Z), X, Z, \ldots, Z\big) \tag{1}$$

From the hypothesis, $f(X, Y, Z, \ldots, Z) \in A^{(3,1)}$. If we had $f(X, Y, Z, \ldots, Z) = H(X)$, or $f(X, Y, Z, \ldots, Z) = H(Z)$ with some H, then $Y = G\big(H(X), X, Z, \ldots, Z\big)$, or $Y = G\big(H(Z), X, Z, \ldots, Z\big)$ and this is impossible. Hence $f(X, Y, Z, \ldots, Z) = H(Y)$ with a suitable $H \in A^{(1)}$. By setting here $Y = Z$, we obtain $f(X, Y, \ldots, Y) = H(Y)$; but we obtained above $f(X, Y, \ldots, Y) = h(X)$, whence H must be a constant function, $H(Y) = c$ for all Y; but now from (1) we have $Y = G\big(H(Y), X, Z, \ldots, Z\big) = G(c, X, Z, \ldots, Z)$—a contradiction, because the right side of the last equation does not depend on Y. Hence every function of $A^{(k+1)}$ depends on at most k variables, and from the hypothesis it follows that every function of $A^{(k+1)}$ depends on at most one variable, q.e.d.

The theorem just proved and theorem III of [3] imply the following

COROLLARY. *If* $\mathfrak{A}$ *is an* v^**-algebra,* $\dim \mathfrak{A} \geqslant 3$, *then* $\mathfrak{A}$ *is decomposable if and only if* $A^{(3)} = A^{(3,1)}$.

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A representation theorem for v^*-algebras

by

K. Urbanik (Wrocław)

By an *algebra* $\mathfrak{A}$ we mean a pair $(A, \boldsymbol{F})$ where A is a non-empty set and $\boldsymbol{F}$ is a family of A-valued functions of finitely many variables running over A. $\boldsymbol{F}$ is called the class of *fundamental* operations. The class $\boldsymbol{A}$ of *algebraic* operations is, by definition, the smallest class closed with respect to composition, containing all fundamental operations and all trivial operations defined by the formula

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots).$$

Two algebras $(A, \boldsymbol{F}_1)$ and $(A, \boldsymbol{F}_2)$ having the same class of algebraic operations will be treated here as identical. In particular, we have the equation $(A, \boldsymbol{F}) = (A, \boldsymbol{A})$. Further, if the class of algebraic operations on $(A, \boldsymbol{F}_1)$ is contained in the class of algebraic operations on $(A, \boldsymbol{F}_2)$, then we say that $(A, \boldsymbol{F}_1)$ is a *subsystem* of $(A, \boldsymbol{F}_2)$.

The subclass of all algebraic operations of n variables will be denoted by $\boldsymbol{A}^{(n)}$ $(n \geqslant 1)$. Further, by $\boldsymbol{A}^{(0)}$ we shall denote the set of all values of constant algebraic operations. Elements belonging to $\boldsymbol{A}^{(0)}$ will be called *algebraic constants*. If $1 \leqslant k \leqslant n$, then $\boldsymbol{A}^{(n,k)}$ will denote the subclass of $\boldsymbol{A}^{(k)}$ consisting of all operations depending on at most k variables, i.e. $f \in \boldsymbol{A}^{(n,k)}$ if there is an operation $g \in \boldsymbol{A}^{(k)}$ such that $f(x_1, x_2, \ldots, x_n) = g(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ for a system of indices $i_1, i_2, \ldots, i_k$ and for every $x_1, x_2, \ldots, x_n \in A$. By $\boldsymbol{A}^{(n,0)}$ $(n \geqslant 1)$ we shall denote the subclass of $\boldsymbol{A}^{(n)}$ containing all constant operations. The above definitions are given in a more detailed form in [1], [2] and [4].

Following E. Marczewski [1], we say that elements of a set I $(I \subset A)$ are *independent* if for each system of n different elements $a_1, a_2, \ldots, a_n$ from I and for each pair of operations $f, g \in \boldsymbol{A}^{(n)}$ the equation

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$$

implies that f and g are identical in $\mathfrak{A}$. A set whose elements are not independent will be called a set of *dependent* elements. An element $a \in A$ is said to be *self-dependent* if the one-point set containing a is a set of dependent elements.