On faithful representations of free products of groups

by

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1. O_3 denotes the group of real orthogonal matrices, with determinant equal to 1, acting on $\mathcal{R}^3$. $\mathcal{M}$ denotes the group of substitutions $w = (az+b)/(cz+d)$ with real a, b, c, d and $ad-bc = 1$.

Our chief result is the following [1]:

THEOREM 1. *If G_t $(t \in T)$ is a system of groups such that each G_t is isomorphic to a subgroup of O_3 or each G_t is isomorphic to a subgroup of $\mathcal{M}$, $\overline{\overline{G_t}} < 2^{\aleph_0}$ for each $t \in T$ and $\overline{\overline{T}} \leqslant 2^{\aleph_0}$ then the free product $\prod\limits_{t\in T}^{*} G_t$ is isomorphic to a subgroup of O_3 or $\mathcal{M}$, respectively.*

This theorem (in the case of O_3) was conjectured by J. de Groot ([11]). It is a finalisation of a long list of results: F. Hausdorff ([12]) has proved that the free product $Z_2 * Z_3$ (Z_n denotes the cyclic group of order $n \leqslant \infty$) is isomorphic to a subgroup of O_3. Therefore the free group $\prod\limits_{n=1}^{\infty *} Z_\infty$ is isomorphic to a subgroup of O_3 (by the elementary fact that each group of the form $G * H$ contains a subgroup isomorphic to $\prod\limits_{n=1}^{\infty *} Z_\infty$, except in the case when G or H is the unity group, or $G \cong H \cong Z_2$). W. Sierpiński ([20] Lemme 1) has proved that the free group $\prod\limits_{t\in\mathcal{R}}^{*} Z_\infty$ of potency $2^{\aleph_0}$ is isomorphic to a subgroup of O_3. All these results were obtained for the purpose of the s.c. paradoxical decompositions. J. de Groot ([11]) has studied the problem for himself and has given a simple proof and some improvements of the result of Sierpiński. Th. J. Dekker ([9]) has proved that $\prod\limits_{n\leqslant\infty}^{*} \prod\limits_{t\in\mathcal{R}}^{*} Z_n$ is isomorphic to a subgroup of O_3 [2].

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[1] The main results of this paper were announced without proof in [1]. Theorem 1 was then known to us in a somewhat weaker form improved now by the application of a result of S. Świerczkowski (see footnote [7]). Another result announced in [1], section 3, concerning the representations of free products by permutation groups will not be studied here since it has been independently obtained and refined by N. G. de Bruijn ([4], [5], [6]).

[2] For other investigations related to our subject see [3], [7], [8], [18].

Concerning the group $\mathcal{M}$ it is known that $Z_2 * Z_3$ is isomorphic to the subgroup of substitutions in which a, b, c, d are integers [3].

All these results were obtained without the use of the axiom of choice. Our theorem which generalises of course all of them is obtained by a different method and applies the axiom of choice. It should be mentioned that the continuum hypothesis is not applied, due to a method developed in [2].

It seems plausible that Theorem 1 holds if O_3 or $\mathcal{M}$ is replaced by any simple connected Lie group $\mathcal{G}$.

In Section 2 we give some general results reducing this problem to the proof of a lemma $L(\mathcal{G})$ which we have been able to establish only for O_3 and $\mathcal{M}$. The proof of $L(O_3)$ and $L(\mathcal{M})$ follows in Section 3. In Section 4 we prove other results on O_3 solving another problem of J. de Groot ([11]).

2. G denotes any group (the group operations are written multiplicatively and the unity is denoted by 1).

For every set $K \subset G$ we denote by $[K]$ the subgroup of G generated by K.

For a system of groups G_t $(t \in T)$ [4] we denote by $\prod\limits_{t\in T}^{*} G_t$ the free product of this system, that is the group generated by the set of all ordered pairs $\langle g, t\rangle$ with $t \in T$, $g \in G_t$ determined by the relations

$$\langle g, t\rangle^{-1} = \langle g^{-1}, t\rangle,$$
$$\langle g_1, t\rangle\langle g_2, t\rangle = \langle g_1 g_2, t\rangle,$$
$$\langle 1, t_1\rangle = \langle 1, t_2\rangle.$$

If we have a system of homomorphisms $h_t\colon G_t \to G$, then the *natural homomorphism* $h\colon \prod\limits_{t\in T}^{*} G_t \to [\bigcup\limits_{t\in T} h_t(G_t)]$ is determined by $h(\langle g, t\rangle) = h_t(g)$. In particular, if $G_t \subset G$ and $x_t \in G$ $(t \in T)$ then the natural homomorphism $\prod\limits_{t\in T}^{*} G_t \to [\bigcup\limits_{t\in T} x_t G_t x_t^{-1}]$ is determined by $\langle g, t\rangle \to x_t g x_t^{-1}$.

Let us consider the following proposition

$L(G)$. *For any* $a_1, \ldots, a_m \in G\setminus(1)$ *and any integers* $k_1, \ldots, k_m$ *different from* 0 $(m = 1, 2, \ldots)$ *the function of the variable* $x \in G$

$$a_1 x^{k_1} a_2 x^{k_2} \ldots a_m x^{k_m} \tag{1}$$

is not identically equal to 1.

$\mathcal{G}$ denotes any connected Lie group.

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[3] A simple proof is given by K. A. Hirsch [13]. For other results and references see [8], [10], [11], [13], [19].

[4] We do not suppose that $t_1 \neq t_2$ implies $G_{t_1} \neq G_{t_2}$.

THEOREM 2. *If $L(\mathcal{G})$ holds and G is a subgroup of $\mathcal{G}$ with $\overline{\overline{G}} < 2^{\aleph_0}$, then there exists such an element $x \in \mathcal{G}$, which is of infinite order, and such that the natural homomorphism $G * [(x)] \to [G \cup (x)]$ is an isomorphism* [5].

Proof. Let us consider the class $\boldsymbol{M}$ of all subsets of $\mathcal{G}$ having one of the following forms

$$\{x\colon x^k = 1\}, \quad k = 1, 2, \ldots \tag{2}$$

$$\{x\colon a_1 x^{k_1} \ldots a_m x^{k_m} = 1\} \quad (a_i,\ k_i \text{ and } m \text{ as in (1)}). \tag{3}$$

It is clear that $\overline{\overline{\boldsymbol{M}}} \leqslant \overline{\overline{G}} + \aleph_0 < 2^{\aleph_0}$. Moreover, each set $A \in \boldsymbol{M}$ is an analytic surface in $\mathcal{G}$ (in the sense of [2]). In fact, for sets of the form (2) this is clear and for sets of the form (3) this follows from $L(\mathcal{G})$ [6]. Therefore, by a theorem of [2], the set $\bigcup\limits_{A\in \boldsymbol{M}} A$ is a border set in $\mathcal{G}$ (i.e. its complement is dense in $\mathcal{G}$). It follows that there exists an element $x \in \mathcal{G}\setminus \bigcup\limits_{A\in \boldsymbol{M}} A$. Clearly, x satisfies the conclusion of Theorem 2; q.e.d.

LEMMA 1. *If $L(\mathcal{G})$ holds and G and H are subgroups of $\mathcal{G}$ with $\overline{\overline{G}}, \overline{\overline{H}} < 2^{\aleph_0}$, then there exists an $x \in \mathcal{G}$ such that the natural homomorphism $G * H \to [G \cup xHx^{-1}]$ is an isomorphism.*

Proof. By Theorem 2 there exists an $x \in \mathcal{G}$ such that the natural homomorphism $[G \cup H] * [(x)] \to [G \cup H \cup (x)]$ is an isomorphism. It is clear that for such an x the conclusion of Lemma 1 holds; q.e.d.

THEOREM 3. *If $L(\mathcal{G})$, $\tau \in T$, and G_t $(t \in T)$ is a system of subgroups of $\mathcal{G}$ with $\overline{\overline{G_t}} < 2^{\aleph_0}$ for any $t \in T$ and $\overline{\overline{T}} \leqslant 2^{\aleph_0}$, then there exists a system $(x_t)_{t\in T} \subset \mathcal{G}$ with $x_\tau = 1$, such that the natural homomorphism $\prod\limits_{t\in T}^{*} G_t \to [\bigcup\limits_{t\in T} x_t G_t x_t^{-1}]$ is an isomorphism.*

Proof. The set T can be well ordered in a sequence $(t_\xi)_{\xi<\varkappa}$ such that $\tau = t_0$ and

$$\sum_{\xi\leqslant\alpha} \overline{\overline{G}}_{t_\xi} < 2^{\aleph_0} \quad \text{for any} \quad \alpha < \varkappa \text{ [7]}.$$

Then by means of a simple induction on the basis of Lemma 1 we find a sequence $(x_{t_\xi})_{\xi<\varkappa} \subset \mathcal{G}$ for which the conclusion of Theorem 3 holds; q.e.d.

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[5] $A * B$ denotes the free product of the groups A and B. This theorem was conjectured ([17]) for $\mathcal{G}$ locally compact connected and simple. Of course such a $\mathcal{G}$ is a Lie group (by the approximation theorem of H. Yamabe — see [14], p. 175). This problem remains open.

[6] Since the mapping $x \to a_1 x^{k_1} \ldots a_m x^{k_m}$ is analytic, which is well known — see e.g. [15], proof of Lemma 2.

[7] This follows from [16], Lemma 2 (a result of S. Świerczkowski). It is interesting that this result replaces in some sense the regularity of the cardinal $2^{\aleph_0}$, which belongs of course to the classical conjectures of set theory.

Remark 1. In Lemma 1 and Theorem 3, a supposition $L'(\mathcal{G})$ weaker than $L(\mathcal{G})$ would be sufficient. $L'(\mathcal{G})$ is obtained by replacing (1) in $L(\mathcal{G})$ by

$$a_1xb_1x^{-1}a_2xb_2x^{-1}\dots a_nxb_nx^{-1} \quad \text{with} \quad a_i, b_i \in \mathcal{G}\setminus(1) \text{ and } n \geqslant 1.$$

PROBLEM. *Does $L(\mathcal{G})$ or $L'(\mathcal{G})$ hold true for every connected simple Lie group $\mathcal{G}$?*

Proof of Theorem 1. The statements $L(O_3)$ and $L(\mathcal{M})$ will be proved in Section 3; therefore Theorem 1 follows from Theorem 3.

Remark 2. One can obtain results analogous to Theorem 2 and 3 for any connected locally compact group $\mathcal{G}$ supposing $L(\mathcal{G})$ holds if one uses the continuum hypothesis or if one supposes the stronger inequalities $\overline{\overline{G}} \leqslant \aleph_0$, $\overline{\overline{G}}_t \leqslant \aleph_0$ for any $t \in T$ and $\overline{\overline{T}} \leqslant \aleph_1$. The proofs of such theorems are analogous: one has to use the theorem of Baire on the sets of the first category instead of the theorem of [2], and the only point to be completed is to prove that all sets of the form (2) or (3) are nowhere dense in $\mathcal{G}$. But this clearly follows on account of $L(\mathcal{G})$ from [15], Theorem 1, and the approximation theorem (l.c. [5]).

3. The statement $L(O_3)$ visibly follows from the following

LEMMA 2. *If $k_1, \dots, k_n$ are integers different from 0, $a_1, \dots, a_n \in O_3$, and $x_\varphi \in O_3$ $(0 \leqslant \varphi < 2\pi)$ where φ is the rotation angle of x_φ and all x_φ have a common rotation axis L and $a_s(L) \neq L$ for $s = 2, \dots, n$ then the matrix*

$$a_1x_\varphi^{k_1}a_2x_\varphi^{k_2}\dots a_nx_\varphi^{k_n} \tag{4}$$

is a non constant function of φ.

Proof. We can suppose from now on without loss of generality that $a_1 = 1$ and that we have chosen in $\mathcal{R}^3$ a coordinate system, such that the third coordinate axis is the line L.

Now we need some auxiliary statements.

(A) *The following conditions are equivalent (for $a \in O_3$)*

(i) $a(L) \neq L$;

(ii) $L \neq$ *(axis of a) and, if the rotation angle of a is π, then also L is not perpendicular to the axis of a;*

(iii) $a = (a_{ij})_{i,j=1,2,3}$ *and*

$$(a_{11} - a_{22} \neq 0 \ \text{ or } \ a_{12} + a_{21} \neq 0) \ \text{ and } \ (a_{11} + a_{22} \neq 0 \ \text{ or } \ a_{12} - a_{21} \neq 0).$$

The equivalences (i)$\leftrightarrow$(ii) and (ii)$\leftrightarrow$(iii) are elementary.

(B) *For any linear transformation $a = (a_{ij})_{i,j=1,2}$ of the plane $\mathcal{R}^2$ we have*

$$a(z) = Az + B\bar{z} \ [8],$$

where

(i) $$A = \tfrac{1}{2}[(a_{11} + a_{22}) + i(a_{21} - a_{12})], \quad B = \tfrac{1}{2}[(a_{11} - a_{22}) + i(a_{12} + a_{21})],$$

and

(ii) $$a_{11} = \operatorname{Re}(A+B), \quad a_{12} = -\operatorname{Im}(A-B), \quad a_{21} = \operatorname{Im}(A+B), \quad a_{22} = \operatorname{Re}(A-B).$$

This is elementary.

(C) *All the functions $\cos^n\varphi$ and $\sin\varphi\cos^n\varphi$ $(n = 0, 1, 2, \dots)$ are independent, i.e. a finite linear form formed of them vanishes if and only if all the coefficients vanish.*

This follows by independence of $\cos^n\varphi$ $(n = 0, 1, 2, \dots)$ and the fact that $\cos^n\varphi$ are even functions and $\sin\varphi\cos^n\varphi$ are odd functions.

(D) $$x_\varphi^k = 2^{|k|-1}\cos^{|k|-1}\varphi \begin{pmatrix} \cos\varphi & -\sigma\sin\varphi & 0 \\ \sigma\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + b, \quad (k = \pm 1, \pm 2, \dots),$$

where $\sigma = \operatorname{sgn} k$ and b is a matrix with elements of the form

$$\sum_{r=0}^{|k|-2} \alpha_r \sin\varphi\cos^r\varphi + \sum_{r=0}^{|k|-1} \beta_r \cos\varphi^r.$$

This follows from the known formulae

$$\cos n\varphi = 2^{n-1}\cos^n\varphi + P_1, \quad \sin n\varphi = \sin\varphi(2^{n-1}\cos^{n-1}\varphi + P_2), \tag{5}$$

(where P_1 resp. P_2 are polynomials in $\cos\varphi$ of degree $< n$ resp. $n-1$) and from the equality

$$x_\varphi^k = \begin{pmatrix} \cos k\varphi & -\sin k\varphi & 0 \\ \sin k\varphi & \cos k\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any matrix $a = (a_{ij})_{i,j=1,2,3}$ we take the notation

$$a^* = (a_{ij})_{i,j=1,2}.$$

(E) (i) *The elements of the matrix (4) can be represented uniquely in the form*

$$\sum_{r=0}^{d-1} \alpha_r^{ij} \sin\varphi\cos^r\varphi + \sum_{r=0}^{d} \beta_r^{ij}\cos^r\varphi \quad (i, j = 1, 2, 3),$$

where $d = |k_1| + \dots + |k_n|$.

(ii) *The elements of the matrix*

$$c = (c_{ij})_{i,j=1,2} = \prod_{s=1}^{n} a_s^* \begin{pmatrix} \cos\varphi & -\sigma_s\sin\varphi \\ \sigma_s\sin\varphi & \cos\varphi \end{pmatrix},$$

[8] We identify the point $(\xi, \eta) \in \mathcal{R}^2$ with the complex number $z = \xi + i\eta$, and the point $(\xi, -\eta) \in \mathcal{R}^2$ with $\bar{z} = \xi - i\eta$.

where $\sigma_s = \operatorname{sgn} k_s$, *can be represented uniquely in the form*

$$c_{ij} = \sum_{r=0}^{n-1} \gamma_r^{ij} \sin\varphi \cos^r\varphi + \sum_{r=0}^{n} \delta_r^{ij} \cos^r\varphi \qquad (i, j = 1, 2).$$

(iii) $\alpha_{d-1}^{ij} = 2^{d-n}\gamma_{n-1}^{ij}$ *and* $\beta_d^{ij} = 2^{d-n}\delta_n^{ij}$ $(i, j = 1, 2)$.

The statements (i) and (ii) can be easily proved by induction on n if one applies (C), the formulae (5) and $\sin^2\varphi = 1 - \cos^2\varphi$. The statement (iii) follows in the same way from (C) and (D).

Now, by (C) and (E) (i), for proving Lemma 2 it is enough to show that one of the terms $\alpha_{d-1}^{ij}\sin\varphi\cos^{d-1}\varphi + \beta_d^{ij}\cos^d\varphi$ $(i, j = 1, 2, 3)$ does not vanish. By (E) (iii) it is enough to prove that one of the terms $\gamma_{n-1}^{ij}\sin\varphi\cos^{n-1}\varphi + \delta_n^{ij}\cos^n\varphi$ $(i, j = 1, 2)$ does not vanish.

We have

$$a_1^*(z) = z, \qquad x_\varphi^*(z) = e^{\varphi}z,$$

and we put

$$a_s^*(z) = A_s z + B_s \bar{z} \qquad (s = 2, 3, \ldots, n),$$
$$c(z) = Cz + D\bar{z}.$$

Then by (A), the supposition $a_s(L) \neq L$ of the Lemma 2 and (B) (i) we have

(6) $$A_s \neq 0 \neq B_s \qquad (s = 2, 3, \ldots, n).$$

By the definition ((E) (ii)) of the matrix c we have

(7) $$C = \sum C_{\varkappa_2,\ldots,\varkappa_n} e^{i(\sigma_1+\varkappa_2\sigma_2+\ldots+\varkappa_n\sigma_n)\varphi},$$
$$D = \sum D_{\varkappa_2,\ldots,\varkappa_n} e^{i(\sigma_1+\varkappa_2\sigma_2+\ldots+\varkappa_n\sigma_n)\varphi},$$

where the sums are running over all sequences $\varkappa_2, \ldots, \varkappa_n$ with $\varkappa_s = \pm 1$ and $C_{\varkappa_2,\ldots,\varkappa_n}$ and $D_{\varkappa_2,\ldots,\varkappa_n}$ are products of some of the numbers A_s and B_s and their conjugates. Clearly, by (6) all the numbers $C_{\varkappa_2,\ldots,\varkappa_n}$ and $D_{\varkappa_2,\ldots,\varkappa_n}$ are different from 0. In each of the sums (7) there is exactly one term in which $\sigma_1 + \varkappa_2\sigma_2 + \ldots + \varkappa_n\sigma_n = \sigma_1 n$ (since this holds if and only if $\varkappa_s = \sigma_1\sigma_s$). Hence C and D contain one term $Ke^{i\sigma_1 n\varphi}$ and $Le^{i\sigma_1 n\varphi}$ respectively with $K \neq 0 \neq L$. Therefore by (B) (ii) we have

$$c_{11} = \operatorname{Re}[e^{i\sigma_1 n\varphi}(K+L) + t_{11}], \qquad c_{12} = -\operatorname{Im}[e^{i\sigma_1 n\varphi}(K-L) + t_{12}],$$
$$c_{21} = \operatorname{Im}[e^{i\sigma_1 n\varphi}(K+L) + t_{21}], \qquad c_{22} = \operatorname{Re}[e^{i\sigma_1 n\varphi}(K-L) + t_{22}],$$

where t_{ij} are trigonometrical sums of degree $< n$.

Then one of the c_{ij}, when represented in the form (E) (ii), contains a non-vanishing term $\gamma_{n-1}^{ij}\sin\varphi\cos^{n-1}\varphi + \delta_n^{ij}\cos^n\varphi$, which concludes the proof.

LEMMA 3. *$L(\mathcal{M})$ holds.*

Proof. The mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d}$$

is a homomorphism of the unimodular group of matrices onto $\mathcal{M}$ corresponding to the identification of matrices of opposite sign. Keeping in mind this fact we can work with matrices.

We shall prove that the matrix (all matrices considered are unimodular)

$$\prod_{s=1}^{n} a_s x^{k_s}, \qquad \text{where} \qquad a_s = \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix} \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (s = 1, \ldots, n)$$

(k_s are integers different from 0, $n \geqslant 1$), depends on the matrix x.

We can suppose without loss of generality that $\gamma_s \neq 0$ for $s = 1, \ldots, n$ (using an inner automorphism of the group). Let us verify under this assumption that the matrix

$$(\pi_{ij}(t))_{i,j=1,2} = \prod_{s=1}^{n} a_s x_t^{k_s}, \qquad \text{where} \qquad x_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

depends on t. Clearly

$$a_s x_t^{k_s} = \begin{pmatrix} \alpha_s & \alpha_s k_s t + \beta_s \\ \gamma_s & \gamma_s k_s t + \delta_s \end{pmatrix}.$$

By an easy induction on n we verify that

$$\pi_{22}(t) = \gamma_1 \ldots \gamma_n k_1 \ldots k_n t^n + P,$$

where P is a polynomial in t of degree $< n$; q.e.d.

4. In this section we give some more special results on O_3 which are easy consequences of Lemma 2.

THEOREM 4. *Let G be a subgroup of O_3 and φ a fixed angle with $0 < \varphi < 2\pi$ and $\varphi \neq \pi$, and Q an axis (a plane) in $\mathcal{R}^3$ containing the origin o and such that $a(Q) \neq Q$ for every $a \in G \setminus (1)$. Then every rotation $r \in O_3$ with axis Q (with axis contained in Q and rotation angle φ), except a set of $\overline{\overline{G}} + \aleph_0$ such rotations, is such that the natural homomorphism $G * [(r)] \to [G \cup (r)]$ is an isomorphism.*

Proof. The first case clearly follows from the Lemma 2 since every function

$$a_1 r^{k_1} a_2 r^{k_2} \ldots a_n r^{k_n}$$

(k_i integers different from 0 and $a_i \in G \setminus (1)$, $n \geqslant 1$) is analytical and depends on the rotation angle of r the axis being Q.

In the second case we consider the product

$$a_1 x b^{k_1} x^{-1} a_2 x b^{k_2} x^{-1} \dots a_n x b^{k_n} x^{-1}, \tag{8}$$

where b is a fixed rotation with axis contained in Q and angle φ, and x is a rotation with axis perpendicular to Q and variable angle. Putting $r = xbx^{-1}$ we obtain rotations with axis in Q and angle φ. Then we obtain the conclusion since, by Lemma 2, (8) depends essentially and analytically on the rotation angle of x.

Remark 3. Theorem 4 is a refinement of Theorem 2 for $G = O_3$.

THEOREM 5. *If x_φ and y_φ are rotations with fixed different axes and variable common rotation angle φ, then with the exception of an at most denumerable set of φ-s x_φ and y_φ are free generators of a free group* (9).

Proof. It is visible that there exist an $a \in O_3$, such that $y_\varphi = a x_\varphi a^{-1}$ for all φ-s. Therefore, by Lemma 2, the function

$$x_\varphi^{k_1} y_\varphi^{l_1} x_\varphi^{k_2} y_\varphi^{l_2} \dots x_\varphi^{k_n} y_\varphi^{l_n} = x_\varphi^{k_1} a x_\varphi^{l_1} a^{-1} x_\varphi^{k_2} a x_\varphi^{l_2} a^{-1} \dots x_\varphi^{k_n} a x_\varphi^{l_n} a^{-1}$$

depend essentially and analytically on φ, and the Theorem 5 follows.

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(9) This was conjectured by J. de Groot ([11]). Th. J. Dekker has shown ([9]) more than this: x_φ and y_φ are free generators if $\cos\varphi$ is transcendental.

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