On plane dendroids and their end points in the classical sense

by

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§ 1. Dendroids. A continuum X (i.e. a compact connected metric space) is called a *dendroid* (1) if it is arcwise connected and hereditarily unicoherent, i.e. if every two distinct points of it are joined by an arc contained in X and every subcontinuum of X (as well as whole X) is unicoherent (see [4], p. 104).

As has recently been proved by J. Charatonik (in a paper which is now being prepared for publication) the condition for X to be a dendroid is equivalent, among others, to each of the following ones:

(i) *every two distinct points of X are joined by exactly one irreducible continuum contained in X, namely by an arc,*

(ii) *X is an arcwise connected, homologically acyclic and* 1*-dimensional continuum (or a single point).*

By (ii) every non-degenerate dendroid is a unicoherent 1-dimensional continuum; therefore (see [4], p. 338) *every locally connected dendroid is a dendrite.* This means that dendroids constitute a generalization of dendrites. In this character they are found in literature. For instance in 1954 Borsuk [2] proved that dendroids have the fixed point property. This was generalized in 1958 by Ward [8], who was considering more abstract spaces, namely without the requirement that X be a metrizable space. Let us mention also paper [9] of Ward, where some additional references are given.

I was encouraged to study dendroids by Professor B. Knaster. I express my gratitude to him.

§ 2. End points in the classical sense. I say that a point x of an arcwise connected continuum X is an *end point of X in the classical sense* if x is an end point of every arc contained in X and containing x. It is well-known (see [4], p. 203) that if X is a locally connected con-

(1) This term was proposed by B. Knaster and became usual in his Seminar in Wrocław.

tinuum, then the above definition of end point is equivalent to the usual definition adopted according to Menger in the ordinary topological curve theory, i.e. to the condition $\operatorname{ord}_x X = 1$ (see [4], p. 200). But, as can easily be seen on very simple examples, if X is *not* locally connected, these notions of end point are different.

I use the term "classical sense", because it seems to me that this definition of end point is nearest to the original intuitive notion of end point (see for instance [1], p. 584), which was later refined for the purposes of various theories, e.g. the theory of locally connected continua.

The set of all end points of X in the classical sense will be denoted by X^e, to distinguish it from the set $X^{[1]} = \{x\colon \operatorname{ord}_x X = 1\}$. It is known (see [4], p. 204) that the set $X^{[1]}$ is always a G_δ-set (in X). We shall show in § 10 that there exists an arcwise connected plane continuum, even a dendroid, D such that D^e is not of the 1-st Borel class (in D). Similarly the proposition that $X^{[1]}$ is always a 0-dimensional set (see [4], p. 217) is not true for X^e: we shall construct in § 9 a plane dendroid D such that D^e is a 1-dimensional set. This shows that the sets X^e and $X^{[1]}$ have essentially dissimilar structures and that one of X^e seems to be more complicated.

All end points considered in the present paper are taken in the classical sense. We shall thus omit the words "in the classical sense".

The following theorem is due to J. Charatonik:

2.1. *If D is a dendroid, then D^e does not contain a non-degenerate continuum.*

Proof. Suppose on the contrary that there exists a non-degenerate continuum C contained in D^e. Let $p, q \in C$ and $p \neq q$. There exists in C a continuum irreducible between p and q (see [4], p. 132) which is not an arc by the inclusion $C \subset D^e$, contrary to the condition (i) in § 1.

If A is an arc with end points p and q, then we write $A = \widehat{pq}$. The following statements are consequences of (i), § 1:

2.2. *If D is a dendroid, $p, q \in D$ and $p \neq q$, then the arc $\widehat{pq}$ is uniquely determined by its end points p and q.*

2.3. *If D is a dendroid, $C \subset D$ is a continuum, $p, q \in C$ and $p \neq q$, then $\widehat{pq} \subset C$.*

If p and q are points of the Euclidean space, then by $\overline{pq}$ we denote the straight line segment with end points p and q.

§ 3. Two lemmas.

The following lemmas of a more general topological character will be needed:

3.1. *If X is a hereditarily unicoherent continuum and $\mathbf{C}$ is an arbitrary family of subcontinua of X such that every two elements of $\mathbf{C}$ have a point in common, then all elements of $\mathbf{C}$ have a point in common.*

Proof. According to the theorem of F. Riesz (see [4], p. 5), it is sufficient to show that every finite number of elements of $\mathbf{C}$ have a point in common, i.e. that proposition 3.1 is true for an arbitrary family $\mathbf{C}$ having n elements ($n = 2, 3, \ldots$).

For $n = 2$ this is a tautology.

Suppose that $n \geqslant 2$ and 3.1 is true for $\mathbf{C}$ having n elements. Then we shall prove 3.1 for $\mathbf{C}$ having $n+1$ elements.

Indeed, let $\mathbf{C} = \{C_1, \ldots, C_{n+1}\}$. Since $C_i \cap C_{n+1} \neq 0$ for $i = 1, \ldots, n$, the sets $C_i \cup C_{n+1}$ are continua, and thus also unicoherent continua, X being hereditarily unicoherent. This implies that $K_i = C_i \cap C_{n+1}$ is a continuum for $i = 1, \ldots, n$.

Moreover, the set $C_i \cup C_j \cup C_{n+1}$ is also a continuum for every $i, j = 1, \ldots, n$, because it is a union of non-disjoint continua. Thus $C_i \cup C_j \cup C_{n+1}$ are unicoherent continua, whence (see [5], p. 511) we have $0 \neq C_i \cap C_j \cap C_{n+1} = K_i \cap K_j$ for every $i, j = 1, \ldots, n$. Therefore applying 3.1 for the family $\{K_1, \ldots, K_n\}$ we obtain $0 \neq K_1 \cap \ldots \cap K_n = C_1 \cap \ldots \cap C_{n+1}$.

3.2. *If $A \subset X$ and $f\colon A \to f(A)$ is a continuous mapping, then for every $B \subset f(A)$ we have* (2)

$$\operatorname{Fr} f^{-1}(B) \subset f^{-1}(\operatorname{Fr} B) \cup \operatorname{Fr} A\,.$$

Proof. Since $\bar{A} - A \subset X - A \subset \overline{X - A}$, we have

$$\begin{aligned}\operatorname{Fr} f^{-1}(B) &= \overline{f^{-1}(B)} \cap \overline{X - f^{-1}(B)} = \overline{f^{-1}(B)} \cap [\overline{X - A} \cup \overline{A - f^{-1}(B)}] \\ &= [\overline{f^{-1}(B)} \cap \overline{X - A}] \cup [\overline{f^{-1}(B)} \cap A \cap \overline{A - f^{-1}(B)}]\,,\end{aligned}$$

whence we obtain 3.2 by the following formulas (see [3], p. 17 and 74):

$$\overline{f^{-1}(B)} \cap \overline{X - A} \subset \bar{A} \cap \overline{X - A} = \operatorname{Fr} A\,,$$

$$\begin{aligned}\overline{f^{-1}(B)} \cap A \cap \overline{A - f^{-1}(B)} &\subset f^{-1}(\bar{B}) \cap A \cap \overline{f^{-1}f(A) - f^{-1}(B)} \\ &= f^{-1}(\bar{B}) \cap A \cap \overline{f^{-1}(f(A) - B)} \subset f^{-1}(\bar{B}) \cap f^{-1}\big(\overline{f(A) - B}\big) \\ &= f^{-1}\big(\bar{B} \cap \overline{f(A) - B}\big) = f^{-1}(\operatorname{Fr} B)\,.\end{aligned}$$

§ 4. Crossed arcs.

The arcs A_1 and A_2 lying in the plane are said to be *crossed* (see [6]) if, for every two simple closed curves C_1 and C_2 lying in the plane and such that $A_1 \subset C_1$ and $A_2 \subset C_2$, the interior of C_1 intersects that of C_2.

(2) If A is a subset of the space X, we denote by $\operatorname{Fr} A$ the boundary of A in X, i.e. the set $\operatorname{Fr} A = \bar{A} \cap \overline{X - A}$, where $\bar{A}$ is the closure of A in X.

The following proposition is a simple consequence of the definition:

4.1. *If the arcs A_1, A_2 lying in the plane E^2 are crossed, then A_1 is not the topological limit* [3] *of arcs contained in $E^2 - A_2$.*

The crossing being obviously a symmetric relation, 4.1 is true also for A_2 instead of A_1.

Now let us observe that

4.2. *If $A_1, \ldots, A_5$ are arcs lying in the plane such that the intersection $A_1 \cap \ldots \cap A_5$ contains at least two points q and r, the sum $A_1 \cup \ldots \cup A_5$ is a dendrite and $p_1, \ldots, p_5$, being end points of $A_1 \cup \ldots \cup A_5$, are distinct end points of $A_1, \ldots, A_5$ respectively, then there exist an index $j \leqslant 5$ and an arc $A \subset A_1 \cup \ldots \cup A_5$ such that A_j and A are crossed.*

Proof. Let $s \in \widehat{qr}$ and $q \neq s \neq r$. Therefore the arc $\widehat{sp_i}$ contains q or r for every $i = 1, \ldots, 5$. Hence at least one of points q, r is contained in the intersection of three of the arcs $\widehat{sp_i}$. We may assume that $q \in \widehat{sp_1} \cap \widehat{sp_2} \cap \widehat{sp_3}$ (see fig. 1). It is obvious that we can distinguish an end point, for example p_2, of the oriented dendrite $\widehat{sp_1} \cup \widehat{sp_2} \cup \widehat{sp_3}$,

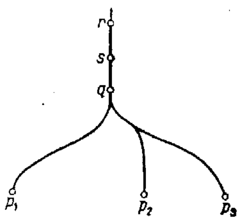

Fig. 1

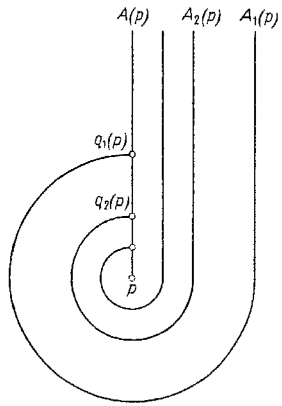

Fig. 2

such that p_2 lies between two other end points, p_1 and p_3 (compare fig. 1). Then A_2 and $\widehat{p_1p_3}$ are crossed arcs, and it is enough to put $j = 2$, $A = \widehat{p_1p_3}$.

§ 5. Singular end points. I say that the end point p of dendroid D is *singular* if (see fig. 2) there exist an arc $A(p) \subset D$ with end point p, a sequence of points $q_i(p)$ $(i = 1, 2, \ldots)$ of $A(p)$ and a sequence of arcs $A_i(p) \subset D$ with end points $q_i(p)$ respectively $(i = 1, 2, \ldots)$ such that

(3) By the topological limit $\operatorname*{Lim}_{i\to\infty} A_i$ of a sequence of sets $A_1, A_2, \ldots$ we understand the set $\operatorname*{Ls}_{i\to\infty} A_i$ provided that it coincides with $\operatorname*{Li}_{i\to\infty} A_i$. For the definitions of all limits Lim, Li and Ls see [3], p. 241-245.

(i) $$p = \lim_{i\to\infty} q_i(p)\,,$$

(ii) $$\{q_i(p)\} = A_i(p) \cap A(p) \quad \text{for} \quad i = 1, 2, \ldots,$$

(iii) $$A(p) = \operatorname*{Lim}_{i\to\infty} A_i(p)\,.$$

The set of all singular end points of dendroid D will be denoted by D_s^e.

It is not difficult to see that condition (iii) in above definition can be replaced by the following one [4]:

(iv) $$0 < \varepsilon < \delta[A_i(p)] \quad \text{for} \quad i = 1, 2, \ldots$$

It follows immediately from 4.1 and (i)-(iii) that

5.1. *If p is a singular end point of plane dendroid D and $A \subset D$ is an arc, then $A(p)$ and A are not crossed.*

We shall show that

5.2. *If $p_1, \ldots, p_n$ are distinct singular end points of plane dendroid D and the set $A(p_1) \cap \ldots \cap A(p_n)$ contains at least two points, then $n < 5$.*

Proof. Suppose $n \geqslant 5$. Then the set $A(p_1) \cup \ldots \cup A(p_5)$ is a dendrite (see § 1) having the points $p_1, \ldots, p_5$ as end points. Applying 4.2 for $A_i = A(p_i)$, where $i = 1, \ldots, 5$, we obtain an index $j \leqslant 5$ and an arc $A \subset A(p_1) \cup \ldots \cup A(p_5) \subset D$ such that $A(p_j)$ and A are crossed, which contradicts 5.1.

5.3. *If D is a plane dendroid, $p_0 \in P \subset D_s^e$ and $\aleph_0 < \overline{\overline{P}}$, then there exists $p' \in P$ such that $A(p_0) \cap A(p') = 0$.*

Proof. Suppose on the contrary that

(1) $$A(p_0) \cap A(p) \neq 0 \quad \text{for} \quad p \in P\,.$$

Evidently, among the points of P only p_0 and, perhaps, another end point q of $A(p_0)$ can belong to $A(p_0)$. Let $P' = P - \{p_0, q\}$. Hence $\aleph_0 < \overline{\overline{P'}}$ and $p \notin A(p_0)$ for $p \in P'$. Let $A'(p)$ be an arc with end point p $(p \in P')$ such that

(2) $$A'(p) \subset A(p) \quad \text{and} \quad A(p_0) \cap A'(p) = 0 \quad \text{for} \quad p \in P'.$$

We shall prove that

(3) among every five distinct points $p_1, \ldots, p_5$ belonging to P' there exists a pair p_i, p_j such that $A'(p_i) \cap A'(p_j) = 0$.

In fact, suppose on the contrary that every two of the sets $A'(p_1), \ldots, A'(p_5)$ have a point in common. Then we have

$$A'(p_1) \cap \ldots \cap A'(p_5) \neq 0$$

(4) For a subset A of the metric space X with the distance ϱ the symbol $\delta(A)$ denotes the *diameter* of A, i.e. $\delta(A) = \sup_{a_1, a_2 \in A} \varrho(a_1, a_2)$.

according to 3.1. Let $a \in A'(p_1) \cap \ldots \cap A'(p_5)$. Therefore

(4) $$a \in A(p_1) \cap \ldots \cap A(p_5) \quad \text{and} \quad a \notin A(p_0)$$

by virtue of (2).

But we infer from (1) and (4) that also every two of the sets $A(p_0), A(p_1), \ldots, A(p_5)$ have a point in common. Thus, by 3.1, all these sets have a point b in common. Hence $b \in A(p_0)$ and (4) implies that $a, b \in A(p_1) \cap \ldots \cap A(p_5)$ and $a \neq b$, contrary to 5.2. Therefore (3) is proved.

Applying the Dushnik-Miller theorem (see [7], for instance) for the relation R in the set P' defined as follows:

$$p_1 R p_2 \equiv [A'(p_1) \cap A'(p_2) = 0] \quad \text{for} \quad p_1, p_2 \in P',$$

we obtain from (3) that there exists a subset $P'' \subset P'$ such that $\aleph_0 < \overline{\overline{P''}}$ and $A'(p_1) \cap A'(p_2) = 0$ for every $p_1, p_2 \in P''$, $p_1 \neq p_2$.

It follows from (i) and (2) that for every $p \in P''$ at least one of the points $q_i(p)$, we denote it by $q_{i_p}(p)$, is an interior point of the arc $A'(p)$. Hence (ii) implies that the set $A'(p) \cup A_{i_p}(p)$ is a triod with the pith $A'(p)$ (see [6]). The sets $A'(p)$ being disjoint for $p \in P''$, we conclude that there exist two distinct points $p, q \in P''$ such that the arcs $A_{i_p}(p)$ and $A'(q)$ are crossed (see [6]). Thus the arcs $A_{i_p}(p)$ and $A(q)$ are also crossed by (2), and $A_{i_p}(p) \subset D$. This contradicts 5.1.

5.4. THEOREM. *If D is a plane dendroid, then D_s^e is at most countable.*

Proof. Let us suppose the contrary: $\aleph_0 < \overline{\overline{D_s^e}}$, i.e. $\aleph_1 \leqslant \overline{\overline{D_s^e}}$. Let R be a symmetric relation in the set D_s^e defined as follows:

$$s_1 R s_2 \equiv [A(s_1) \cap A(s_2) = 0] \quad \text{for} \quad s_1, s_2 \in D_s^e.$$

We infer from 5.3 that for every $Y \subset D_s^e$ such that $\aleph_1 \leqslant \overline{\overline{Y}}$ and for every $y \in Y$ the set of all elements p of Y satisfying $y \text{ non } Rp$ is at most countable, i.e.

$$\overline{\overline{\{p\colon\ p \in Y,\ y \text{ non } Rp\}}} < \aleph_1.$$

Therefore there exists (see [7]) a set $Z \subset D_s^e$ such that $\aleph_1 \leqslant \overline{\overline{Z}}$ and the conditions $z_1 \neq z_2$, $z_1, z_2 \in Z$ imply $z_1 R z_2$, i.e. the family of sets $A(z)$, where $z \in Z$, is a family of disjoint sets.

It follows from (i) that for every $z \in Z$ at least one of the points $q_i(z)$, we denote it by $q_{i_z}(z)$, is an interior point of the arc $A(z)$. Hence (ii) implies that $\{A(z) \cup A_{i_z}(z)\}_{z \in Z}$ is a family of triods lying in the plane and having disjoint piths $A(z)$ (see [6]). Thus there exist at least two distinct points $z, z' \in Z$ such that the arcs $A_{i_z}(z)$ and $A(z')$ are crossed (see [6]). This contradicts 5.1.

§ 6. A valuation of the Borel class of D^e, when D is a plane dendroid.

It is given by the following

THEOREM. *If D is a plane dendroid, then D^e is a $G_{\delta\sigma\delta}$-set.*

Proof. Let for $i, j, k = 1, 2, \ldots$ G_{ijk} be the set of points p of D such that there exists a subset $U \subset D$ satisfying the following four conditions ([5]):

(5) $$p \in U,$$

(6) $$\overline{U} \subset Q(p, 1/i),$$

(7) $$\operatorname{Fr} U \subset Q(p, 1/i) - \overline{Q(p, 1/j)},$$

(8) $$\delta(\operatorname{Fr} U) < 1/k.$$

By (5) and (7) the point p belongs to the interior of U (in D). Consequently, G_{ijk} is an open set (in D) for $i, j, k = 1, 2, \ldots$ Hence, by 5.4, the following formula:

(9) $$D^e = D_s^e \cup \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} G_{ijk}$$

gives the desired valuation.

To prove (9), let

(10) $$p \in D^e - D_s^e$$

and let i be an arbitrary natural number. We shall show that there exists a natural number j such that for every $k = 1, 2, \ldots$ there exists a set U satisfying (5)-(8).

If $D \subset Q(p, 1/i)$, then we put $U = D$, and (5)-(8) hold for every natural j and k, since $\operatorname{Fr} D = 0$. Thus we may assume that there exists a point $q \in D$ such that $\varrho(p, q) = 1/2i$ and $\widehat{pq} \subset Q(p, 1/i)$. Let $h\colon [0, 1] \to \widehat{pq}$ be a homeomorphism such that $h(0) = p$ and $h(1) = q$. Let T be the set of $t \in [0, 1]$ such that there exists a point $q_t \in D$ satisfying the conditions

(11) $$\widehat{q_t h(t)} \cap \widehat{pq} = \{h(t)\} \quad \text{and} \quad \varrho(p, q_t) = 1/i.$$

If there are no such t, we put $T = \{1\}$. If we had $\inf_{t \in T} t = 0$, points $t_n \in T$ would be chosen such that $\lim_{n \to \infty} t_n = 0$. Whence, putting $q_n(p) = h(t_n)$, $A(p) = \widehat{pq}$ and $A_n(p) = \widehat{q_{t_n} h(t_n)}$, we should obtain $\lim_{n \to \infty} q_n(p) = h(0) = p$ and $\{q_n(p)\} = A_n(p) \cap A(p)$ for $n = 1, 2, \ldots$ according to (11).

([5]) If $p \in D$ and $\varepsilon > 0$, we denote by $Q(p, \varepsilon)$ the open ball with centre p and radius ε, i.e. $Q(p, \varepsilon) = \{x\colon x \in D,\ \varrho(p, x) < \varepsilon\}$.

This would give conditions (i) and (ii) from § 5. Furthermore, we should find a natural number l such that $\varrho(p, h(t_n)) < 1/2i$ for $n > l$, whence

$$\varrho(q_{t_n}, h(t_n)) \geqslant \varrho(q_{t_n}, p) - \varrho(p, h(t_n)) > 1/i - 1/2i = 1/2i$$

for $n > l$, by (11). Thus we should have

$$0 < 1/2i < \delta[A_n(p)]$$

for $n > l$, and condition (iv) from § 5 would also be satisfied. Hence we should have $p \in D_s^e$, contrary to (10). Therefore $0 < \inf_{t \in T} t$.

Setting $r = h(\inf_{t \in T} t)$, we obtain $p \neq r \in Q(p, 1/i)$ and

(12) if $x \in D$ and $1/i \leqslant \varrho(p, x)$, then $r \in \widehat{px}$.

We take a natural number j such that $1/j < \varrho(p, r)/2$. It follows that $r \in Q(p, 1/i) - \overline{Q(p, 1/j)}$.

Now let k be an arbitrary natural number. Choosing a number $\varepsilon > 0$ such that (⁶)

$$\varepsilon < \min\{1/2k, \varrho[r, \operatorname{Fr}(Q(p, 1/i) - \overline{Q(p, 1/j)})]\},$$

we have

(13) $$\overline{Q(r, \varepsilon)} \subset Q(p, 1/i) - \overline{Q(p, 1/j)}$$

and

(14) $$\delta[Q(r, \varepsilon)] = 2\varepsilon < 1/k.$$

Let $A = D - Q(r, \varepsilon)$. Since A is a compact set, there exists (see [4], p. 122) a continuous function $f\colon A \to \mathcal{C}$ into the Cantor 0-dimensional set $\mathcal{C}$, such that the sets $f^{-1}(y)$, where $y \in f(A)$, coincide with the components of A. Let A_0 be the sum of all components of A which are contained in $Q(p, 1/i)$. Therefore

(15) $$f^{-1}f(A_0) = A_0 \subset Q(p, 1/i).$$

Since $Q(p, 1/i)$ is an open set and f^{-1} is an upper semicontinuous function (see [4], p. 42), $f(A_0)$ is an open set in $f(A)$ (see [4], p. 35).

It is obvious that $p \in A \cap Q(p, 1/i)$. Let C be the component of p in A. Suppose that $C - Q(p, 1/i) \neq 0$ and choose $x \in C - Q(p, 1/i)$. Thus we have $1/i \leqslant \varrho(p, x)$ and $\widehat{px} \subset C$ according to 2.3. Hence $\widehat{px} \subset A \subset D - \{r\}$, contrary to (12). Therefore $C \subset Q(p, 1/i)$, i.e. $C \subset A_0$. It follows that $p \in A_0$, whence $f(p) \in f(A_0)$.

(⁶) $\varrho(A, B) = \inf_{a \in A, b \in B} \varrho(a, b)$; $\varrho(x, A) = \varrho(\{x\}, A)$.

But since $f(A) \subset \mathcal{C}$ is a 0-dimensional set and $f(A_0)$ is an open neighbourhood of $f(p)$ in $f(A)$, there exists a B such that B is an open set in $f(A)$, $f(p) \in B$, $\bar{B} \subset f(A_0)$ and $\operatorname{Fr} B = 0$, where $\bar{B}$ and $\operatorname{Fr} B$ are the closure and the boundary of B in $f(A)$ respectively. Putting $X = D$, we infer from 3.2 that $\operatorname{Fr} f^{-1}(B) \subset \operatorname{Fr} A = \operatorname{Fr} Q(r, \varepsilon)$.

We set $U = f^{-1}(B)$. Thus $p \in U$, $\overline{U} \subset f^{-1}(\bar{B}) \subset f^{-1}f(A_0) \subset Q(p, 1/i)$ by (15), $\operatorname{Fr} U \subset \operatorname{Fr} Q(r, \varepsilon) \subset \overline{Q(r, \varepsilon)} \subset Q(p, 1/i) - \overline{Q(p, 1/j)}$ by (13), and $\delta(\operatorname{Fr} U) \leqslant \delta[\overline{Q(r, \varepsilon)}] = \delta[Q(r, \varepsilon)] < 1/k$ by (14). Hence all conditions (5)-(8) hold.

Therefore (10) implies $p \in \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} G_{ijk}$, whence

$$D^e \subset D_s^e \cup \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} G_{ijk}.$$

To finish the proof of (9), we must only show that if $p \in D - D^e$, then

$$p \in D - \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} G_{ijk}.$$

In fact, for such p there exist two points $q, r \in D$ such that $q \neq p \neq r$ and $p \in \widehat{qr}$. Let i be a natural number such that

$$1/i < \min\{\varrho(p, q), \varrho(p, r)\}.$$

Then for every $j = 1, 2, \ldots$ conditions (5), (6) and (7) imply

$$[\widehat{pq} - Q(p, 1/j)] \cap \operatorname{Fr} U \neq 0 \neq [\widehat{pr} - Q(p, 1/j)] \cap \operatorname{Fr} U,$$

whence (⁶)

$$0 < \varrho[\widehat{pq} - Q(p, 1/j), \widehat{pr} - Q(p, 1/j)] \leqslant \delta(\operatorname{Fr} U).$$

It follows that inequality (8) cannot be satisfied for every $k = 1, 2, \ldots$

§ 7. Some conditions for D^e to be a G_δ-set. Let $\mathbf{A}$ be a family of arcs lying in the topological space X. I say that $\mathbf{A}$ is a *family with straightness* if a real-valued function $\sigma\colon \mathbf{A} \to E^1$ (called a *straightness*) exists such that $\sigma(A) > 0$ for every $A \in \mathbf{A}$ and the conditions:

$$p_i \in \widehat{x_i y_i} \in \mathbf{A} \quad \text{for} \quad i = 1, 2, \ldots,$$
$$\lim_{i\to\infty} p_i = p, \quad \lim_{i\to\infty} x_i = x, \quad \lim_{i\to\infty} y_i = y,$$
$$x \neq p \neq y \quad \text{and} \quad 0 < \varepsilon \leqslant \sigma(\widehat{x_i y_i}) \quad \text{for} \quad i = 1, 2, \ldots$$

imply that the point p is contained in the interior of an arc $\widehat{xy} \in \mathbf{A}$ satisfying the conditions:

$$\widehat{xy} \subset \operatorname{Ls}_{i\to\infty} \widehat{x_i y_i}, \quad \varepsilon \leqslant \sigma(\widehat{xy}).$$

An arc A contained in the n-dimensional Euclidean space E^n is said to be a *polygonal line* if it is the sum of a finite number of straight line segments. Let us denote by $A^{(n)}$ the family of all polygonal lines contained in E^n.

7.1. *$A^{(n)}$ is a family with straightness.*

Proof. Let $A \in A^{(n)}$. Then A is the sum of k straight line segments $L_1, \ldots, L_k$ such that the common part $L_i \cap L_{i+1}$ is an end point of L_i and that of L_{i+1} for $i = 1, \ldots, k-1$. Let a_i $(i = 1, \ldots, k-1)$ denote the angle, less than π, formed by L_i and L_{i+1}, and put $a_k = \pi$. It is not difficult to verify that a straightness $\sigma\colon A^{(n)} \to E^1$ can be defined as follows (compare footnotes (4) and (6)):

$$\sigma(A) = \min_{i=1,\ldots,k} \{1/k, a_i, \delta(L_i), 1/\delta(L_i), \varrho(L_i, \bigcup_{\substack{j=1,\ldots,k \\ |i-j|>1}} L_j)\}.$$

7.2. *If all arcs contained in the dendroid D form a family with straightness, then D^e is a G_δ-set.*

Proof. Let σ be a straightness. If $p \in D - D^e$, then p is an interior point of some arc $\widehat{xy} \subset D$. Let F_j $(j = 1, 2, \ldots)$ be a set of points $p \in D - D^e$ such that p is an interior point of an arc $\widehat{xy} \subset D$ satisfying the condition:

$$1/j \leqslant \min\{\varrho(p, x), \varrho(p, y), \sigma(\widehat{xy})\}.$$

Therefore $D - D^e = \bigcup_{j=1}^{\infty} F_j$ and it is easy to see that F_j is a closed subset of D for $j = 1, 2, \ldots$ Hence $D - D^e$ is a F_σ-set, i.e. D^e is a G_δ-set.

I say that the dendroid D is *straightenable* if there exist a natural number n and a homeomorphism $h\colon D \to E^n$ such that every two points of $h(D)$ are joined in $h(D)$ by a polygonal line, i.e. $\widehat{xy} \in A^{(n)}$ for every $x, y \in h(D)$, $x \neq y$. We shall show in § 8 that there exists a plane dendroid which is not straightenable.

We infer from 7.1 and 7.2 that

7.3. *If D is a straightenable dendroid, then D^e is a G_δ-set.*

Now, understanding by a *triod* any set composed of three arcs such that the common part of every two of them is the same single point, called a *vertex* of the triod, let us denote by X^t the set of vertices of all triods contained in X.

7.4. *If D is a dendroid and $D^e \cap \overline{D^t}$ is a G_δ-set, then D^e is a G_δ-set.*

Proof. Since $\overline{D^t}$ is a closed set, there exist for every point $p \in (D - D^e) - \overline{D^t}$ two points $x, y \in D$ such that p is an interior point of the arc $\widehat{xy}$ and $\widehat{xy} \cap \overline{D^t} = 0$. Therefore we obtain the decomposition:

$$(D - D^e) - \overline{D^t} = \bigcup_{j=1}^{\infty} F_j,$$

where F_j $(j = 1, 2, \ldots)$ is the set of all points $p \in (D - D^e) - \overline{D^t}$ for which there exist points $x, y \in D$ such that $p \in \widehat{xy}$ and

$$1/j \leqslant \min\{\varrho(x, \widehat{py}), \varrho(y, \widehat{px}), \varrho(\widehat{xy}, \overline{D^t})\}.$$

We shall show that every F_j is a closed set. Indeed, let $p = \lim_{i\to\infty} p_i$, $p_i \in F_j$ for $i = 1, 2, \ldots$, and let x_i, y_i be points corresponding to p_i. By the compactness of D we can assume that $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} y_i = y$. Since $\operatorname{Ls}_{i\to\infty} \widehat{p_i y_i}$ is a continuum (see [4], p. 111) and contains p and y, we infer by 2.3 that $\widehat{py} \subset \operatorname{Ls}_{i\to\infty} \widehat{p_i y_i}$. Thus the inequalities $1/j \leqslant \varrho(x_i, \widehat{p_i y_i})$ for $i = 1, 2, \ldots$ give $1/j \leqslant \varrho(x, \widehat{py})$. Similarly $1/j \leqslant \varrho(y, \widehat{px})$. Moreover, the inequalities $1/j \leqslant \varrho(\widehat{x_i y_i}, \overline{D^t})$ for $i = 1, 2, \ldots$ imply that $1/j \leqslant \varrho(\widehat{px} \cup \widehat{py}, \overline{D^t})$, whence $(\widehat{px} \cup \widehat{py}) \cap \overline{D^t} = 0$ and the union $\widehat{px} \cup \widehat{py}$ does not contain a triod. It follows that $\widehat{px} \cup \widehat{py} = \widehat{xy}$. Therefore $p \in F_j$, i.e. F_j is a closed set.

But since

$$D - D^e = (\overline{D^t} - D^e \cap \overline{D^t}) \cup \bigcup_{j=1}^{\infty} F_j$$

and $D^e \cap \overline{D^t}$ is a G_δ-set, $D - D^e$ is a F_σ-set and 7.4 is proved.

Evidently we always have $X^e \cap X^t = 0$. Hence 7.4 implies

7.5. *If D is a dendroid and D^t is a closed set, then D^e is a G_δ-set.*

I say that the continuum X is a *star* if it is the union of arcs such that the common part of every two of them is the same single point p. Obviously every star X is a dendroid and $X^t = \{p\}$. Thus 7.5 implies that *the set of end points of every star is a G_δ-set* (the last statement has been suggested by K. Borsuk).

§ 8. The plane dendroid D for which D^e is not a G_δ-set.

We proceed to the construction of the plane dendroid D having the following properties:

(i) *D is not straightenable,*

(ii) *D^e is dense in D,*

(iii) *D^e is a F_σ-set,*

(iv) *D^e is not a G_δ-set.*

Let us observe that (iv) implies (i) by 7.3. Hence we shall show only (ii), (iii) and (iv).

We denote by $\mathcal{J}$ the closed interval $0 \leqslant x \leqslant 1$ and by $\mathcal{C}$ the Cantor discontinuum in $\mathcal{J}$. Let $\eta_1, \eta_2, \ldots$ be the sequence of all rational numbers of the open interval $\mathcal{J} - \{0\} - \{1\}$ and let $\theta_1, \theta_2, \ldots$ be the sequence of all (left and right) end points of component intervals of the set $E^1-\mathcal{C}$. In particular $0=\theta_i$ and $1=\theta_j$ for some natural i,j.

For each continuous real-valued function $f\colon \mathcal{J}\to E^1$ and a number $\varepsilon>0$ we define the following sets:

$$\Gamma(f)=\{(x,y)\colon\ y=f(x)\}\,,$$

$$\Gamma_-(f,\varepsilon)=\Gamma(f)\cup\bigcup_{i=1}^{\infty}\{(\eta_i,y)\colon\ f(\eta_i)-\varepsilon/i\leqslant y\leqslant f(\eta_i)\}\,,$$

$$\Gamma_+(f,\varepsilon)=\Gamma(f)\cup\bigcup_{i=1}^{\infty}\{(\eta_i,y)\colon\ f(\eta_i)\leqslant y\leqslant f(\eta_i)+\varepsilon/i\}\,.$$

Therefore $\Gamma(f)$ is an arc and $\Gamma_-(f,\varepsilon)$, $\Gamma_+(f,\varepsilon)$ are dendrites; the sets of their end points are countable and their closures contain $\Gamma(f)$. Furthermore we evidently have [7]

$$d[\Gamma(f),\Gamma_-(f,\varepsilon)]\leqslant\varepsilon\,,\qquad d[\Gamma(f),\Gamma_+(f,\varepsilon)]\leqslant\varepsilon\,. \tag{16}$$

It is not difficult to see that by successive construction we can find a sequence $f_1,f_2,\ldots$ of continuous real-valued functions $f_i\colon\mathcal{J}\to E^1$ and a sequence of positive numbers $\varepsilon_1,\varepsilon_2,\ldots$ such that:

$$\lim_{i\to\infty}\varepsilon_i=0\,, \tag{17}$$

$$f_i(0)=0\,,\qquad f_i(1)=\theta_i\qquad\text{for}\qquad i=1,2,\ldots\,, \tag{18}$$

$$\lim_{n\to\infty}\varrho(f_{i_n},f_{j_n})=0\ ^{(8)}\qquad\text{provided that}\qquad \lim_{n\to\infty}\theta_{i_n}=\lim_{n\to\infty}\theta_{j_n}\ \text{and} \tag{19}$$

$$\lim_{n\to\infty}\theta_{i_n}\in\mathcal{C}-\{\theta_1,\theta_2,\ldots\}\,,$$

and the sets

$$F_i=\begin{cases}\Gamma_-(f_i,\varepsilon_i) & \text{if } \theta_i \text{ is a left end point,}\\ \Gamma_+(f_i,\varepsilon_i) & \text{if } \theta_i \text{ is a right end point,}\end{cases} \tag{20}$$

where $i=1,2,\ldots$, satisfy the conditions:

$$F_i\cap F_j=\{(0,0)\}\qquad\text{for}\qquad i,j=1,2,\ldots;\ i\neq j\,, \tag{21}$$

$$\operatorname*{Lim}_{n\to\infty}\Gamma(f_{i_n})=F_k\qquad\text{provided that}\qquad \lim_{n\to\infty}\theta_{i_n}=\theta_k\qquad\text{and}\qquad\lim_{n\to\infty}i_n=\infty \tag{22}$$

(see fig. 3).

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(7) $d(A,B)$ denotes the Hausdorff distance from A to B (see [3], p. 106).

(8) $\varrho(f,g)=\max\limits_{x\in\mathcal{J}}|f(x)-g(x)|$.

Set

$$D=\overline{\bigcup_{i=1}^{\infty}F_i}\,. \tag{23}$$

We shall prove that D is a dendroid satisfying (ii), (iii) and (iv). Since F_i are closed sets, the condition

$$p\in\overline{\bigcup_{i=1}^{\infty}F_i}-\bigcup_{i=1}^{\infty}F_i \tag{24}$$

implies the existence of a sequence $p_1,p_2,\ldots$ of points such that $p=\lim\limits_{n\to\infty}p_n$, $p_n\in F_{i_n}$ and $\lim\limits_{n\to\infty}i_n=\infty$. We have $\lim\limits_{n\to\infty}\varepsilon_{i_n}=0$ by (17), whence

$$\operatorname*{Lim}_{n\to\infty}F_{i_n}=\operatorname*{Lim}_{n\to\infty}\Gamma(f_{i_n}) \tag{25}$$

by (16) and (20). According to the compactness of $\mathcal{C}$ we may assume that the sequence $\theta_{i_1},\theta_{i_2},\ldots$ is convergent to some point $\zeta\in\mathcal{C}$. If we had $\zeta=\theta_k$ for some natural k, the point p would belong to F_k according to (22) and (25), which contradicts (24). Hence $\zeta\in\mathcal{C}-\{\theta_1,\theta_2,\ldots\}$. It follows that for every sequence $\theta_{j_1},\theta_{j_2},\ldots$ such that $\lim\limits_{n\to\infty}\theta_{j_n}=\zeta$ we obtain from conditions (18), (19) and (21) that the sequences of functions f_{i_n} and f_{j_n} are uniformly convergent to the same continuous function $f_\zeta\colon\mathcal{J}\to E^1$. So $\Gamma(f_\zeta)=\operatorname*{Lim}\limits_{n\to\infty}\Gamma(f_{i_n})$, whence $p\in\Gamma(f_\zeta)$

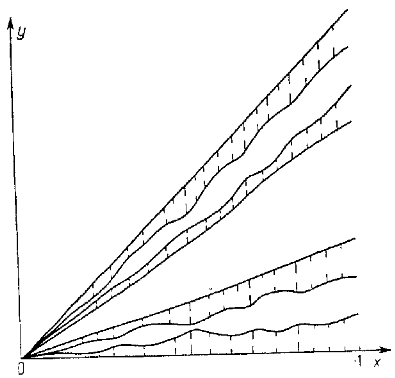

Fig. 3

by (25), and the function f_ζ is uniquely determined by the point ζ. Furthermore we obtain from (18)

$$f_\zeta(0)=0\qquad\text{and}\qquad f_\zeta(1)=\zeta \tag{26}$$

for every $\zeta\in\mathcal{C}-\{\theta_1,\theta_2,\ldots\}$.

Hence (23) gives the decomposition:

$$D=\bigcup_{i=1}^{\infty}F_i\cup\bigcup_{\zeta}\Gamma(f_\zeta)\,, \tag{27}$$

where $\zeta\in\mathcal{C}-\{\theta_1,\theta_2,\ldots\}$, i.e. D is the union of a family of continua each of which is a dendrite or an arc. We infer from (21) and from the definition of functions f_ζ that the common part of every two distinct elements of this family is the single point $(0,0)$. This implies immediately that D is a dendroid.

For the same reason each end point of the dendrite F_i $(i = 1, 2, ...)$ is an end point of the dendroid D. Hence $\Gamma(f_i) \subset \overline{D^e}$ for $i = 1, 2, ...$ by (20), and (ii) follows according to (22) and (23).

The set of the end points of F_i $(i = 1, 2, ...)$ is countable and contains the point $(1, \theta_i)$ by (18) and (20). Since, furthermore, $\Gamma(f_\zeta)$ is an arc from $(0, 0)$ to $(1, \zeta)$ for each $\zeta \in \mathcal{C} - \{\theta_1, \theta_2, ...\}$ according to (26), we infer from (27) that D^e is the union of a countable set $\{e_1, e_2, ...\}$ and the compact set $U = \{(1, y)\colon y \in \mathcal{C}\}$. Hence (iii) follows.

The decomposition $D^e = U \cup \{e_1\} \cup \{e_2\} \cup ...$ being a decomposition into compact frontier sets (in D^e) by (ii), the theorem of Baire is not valid for the space D^e. Hence (iv) follows (see [3], p. 316 and 320).

§ 9. The plane dendroid D for which D^e is a 1-dimensional set. This dendroid, which we are going to construct, has the following properties:

(i) $\dim D^e = 1$,

(ii) *D^e is dense in D,*

(iii) *D^e is a G_δ-set,*

(iv) *D^e is not a F_σ-set,*

(v) *there exists a point $c \in D - D^e$ such that $D^e \cup \{c\}$ is a biconnected set* [9],

(vi) *$\widehat{pc}$ is a straight line segment for every $p \in D - \{c\}$, i.e. $\widehat{pc} = \overline{pc}$; so D is a straightenable dendroid.*

Let us observe that (v) implies (i), and (vi) implies (iii) by 7.3.

Furthermore, (i) implies (iv). In fact, suppose on the contrary that $D^e = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed subsets of D for $i = 1, 2, ...$ Then no F_i° contains a non-degenerate continuum by 2.1. Thus F_i are 0-dimensional sets (see [4], p. 130), whence D^e is also a 0-dimensional set (see [3], p. 171). This contradicts (i).

Hence we shall show only (ii), (v) and (vi).

To define D, the following geometrical construction is needed:

By an *oriented* triangle T we mean a triangle (i.e. a 2-cell) in which an ordering $\preccurlyeq$ of vertices is distinguished. If a, b, c are vertices of T and this ordering is just $a \preccurlyeq b \preccurlyeq c$, then we write: $T = T(abc)$.

Let $T(abc)$ be a fixed oriented triangle and let a_i', b_i' be points of the side $\overline{ab}$ such that

$$(28)\qquad \varrho(a, a_i') = \varrho(a, b)/2i \quad \text{and} \quad \varrho(a, b_i') = \varrho(a, b)/(2i-1)$$

[9] According to Knaster and Kuratowski a set is said to be *biconnected* (see [4], p. 85) if it is connected and is not a sum of any two disjoint non-degenerate connected subsets.

for $i = 1, 2, ...$ Let ϑ be the distance from the point c to the straight line L_{ab} containing the points a, b, and let $\eta_1, \eta_2, ...$ be the sequence of all rational numbers of the closed interval $[0, \vartheta]$.

Finally, let a_i and b_i be points of the segments $\overline{ca_i'}$ and $\overline{cb_i'}$, respectively, such that

$$\varrho(a_i, L_{ab}) = \varrho(b_i, L_{ab}) = \eta_i \quad \text{for} \quad i = 1, 2, ...$$

Hence

$$(29)\qquad \operatorname*{Ls}_{i\to\infty} \{a_i\} = \overline{ac}$$

and

$$(30)\qquad \varrho(a_i, b_i) \leqslant \varrho(a, b)/2 \quad \text{for} \quad i = 1, 2, ...$$

according to (28).

We denote by $\boldsymbol{T}(abc)$ the sequence $\{T(a_ib_ic)\}_{i=1,2,...}$ of oriented triangles. Therefore

$$T_1 \cap T_2 = \{c\} \quad \text{for} \quad T_1, T_2 \in \boldsymbol{T}(abc),\ T_1 \neq T_2 .$$

Now we shall define for every $i = 1, 2, ...$ the countable family $\boldsymbol{S}_i$ of straight line segments and the countable family $\boldsymbol{T}_i$ of oriented triangles such that c is an end point of every segment belonging to $\boldsymbol{S}_i$ and the last vertex of every triangle belonging to $\boldsymbol{T}_i$ (i.e. every $T \in \boldsymbol{T}_i$ is of the form $T = T(a'b'c)$), and that [10]:

$$(31)\qquad \boldsymbol{S}_{i+1}^* \cup \boldsymbol{T}_{i+1}^* \subset \boldsymbol{S}_i^* \cup \boldsymbol{T}_i^* \quad \text{for} \quad i = 1, 2, ...,$$

$$(32)\qquad S_1 \cap S_2 = T_1 \cap T_2 = T_1 \cap S_1 = \{c\}$$

for $S_1, S_2 \in \boldsymbol{S}_i$, $T_1, T_2 \in \boldsymbol{T}_i$, $S_1 \neq S_2$, $T_1 \neq T_2$, $i = 1, 2, ...$

Namely, we put

$$\boldsymbol{S}_1 = \{\overline{ac}\}, \qquad \boldsymbol{T}_1 = \boldsymbol{T}(abc)$$

and

$$(33)\qquad \begin{aligned} \boldsymbol{S}_{i+1} &= \boldsymbol{S}_i \cup \{\overline{pc}\colon T(pqc) \in \boldsymbol{T}_i\}, \\ \boldsymbol{T}_{i+1} &= \bigcup \boldsymbol{T}(pqc), \quad \text{where} \quad T(pqc) \in \boldsymbol{T}_i, \end{aligned}$$

$i = 1, 2, ...$ (see fig. 4).

The dendroid D is defined by the formula

$$D = \bigcap_{i=1}^{\infty} (\boldsymbol{S}_i^* \cup \boldsymbol{T}_i^*) .$$

Evidently $p \in D - \{c\}$ implies that $\overline{pc} \subset \boldsymbol{S}_i^* \cup \boldsymbol{T}_i^*$ for every $i = 1, 2, ...$, whence $\widehat{pc} = \overline{pc}$ and (vi) holds.

We have $\boldsymbol{S}_i^* \subset \boldsymbol{S}_j^*$ for $i \leqslant j$ according to (33), and $\boldsymbol{S}_i^* \subset \boldsymbol{S}_j^* \cup \boldsymbol{T}_j^*$ for $j < i$ according to (31). Therefore $\boldsymbol{S}_i^* \subset D$ for $i = 1, 2, ...$ It follows from (32) that

$$(34)\qquad \text{if} \quad \overline{pc} \in \boldsymbol{S}_i, \quad \text{then} \quad p \in D^e$$

[10] If $\boldsymbol{A}$ is a family of sets, the union of all elements of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^*$.

for $i = 1, 2, \ldots$ Thus by (29) and (33) we obtain $\bigcup_{i=1}^{\infty} S_i^* \subset \overline{D^e}$. But it is easy to verify by (30) and (33) that the union $\bigcup_{i=1}^{\infty} S_i^*$ is a dense subset of D, i.e.

$$D = \overline{\bigcup_{i=1}^{\infty} S_i^*}, \tag{35}$$

whence (ii) follows.

To prove (v), first let us observe that no non-degenerate subset A of D^e is connected. Indeed, if $p, q \in A$ and $p \neq q$, the points p, q belong to different elements of $S_i \cup T_i$ for some i. Therefore there exists, by (32), a straight line L such that p and q lie on different sides of L and $L \cap (S_i^* \cup T_i^*) = \{c\}$. Hence $L \cap D = \{c\}$ by (31). But it is evident that c is not an end point of D, and so $c \notin A$, whence $L \cap A = 0$. It follows that A is not connected, its points p and q being separated by L.

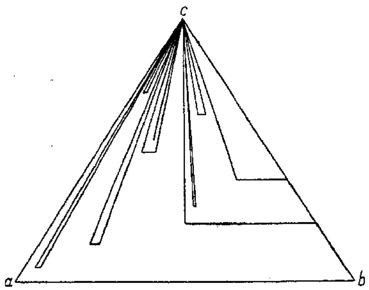

Fig. 4

Thus, to complete the proof of (v), it is enough to prove that $D^e \cup \{c\}$ is a connected set. For this purpose the following lemma will be needed:

LEMMA. *If $T(pqc)$ is an oriented triangle and C is a continuum lying in the same plane as $T(pqc)$ and such that $p, c \notin C$, then there exists a point $q' \in \overline{pq}$ such that $p \neq q'$ and either*

(I) $C \cap \operatorname{Int} T(pq'c) = 0$

or

(II) *some component of the set $C \cap T(pq'c)$ separates $T(pq'c)$ between c and $\overline{pq'}$.*

Proof of the lemma. Since $p \notin C$, a point $r \in \overline{pq}$ exists such that $p \neq r$ and $C \cap \overline{pr} = 0$. If $C \cap \operatorname{Int} T(prc) = 0$, the lemma is true for $q' = r$. We may assume that $C \cap \operatorname{Int} T(prc) \neq 0$. Let r' be a point such that $r' \in \overline{pr}$, $p \neq r'$ and $C \cap \overline{r'c} \neq 0$. Hence every component of the set $C \cap \operatorname{Int} T(pr'c)$ has a limit point on the boundary of $T(pr'c)$. If none of these components have a limit point on the side $\overline{pc}$, then we have $\varepsilon < \varrho[\overline{pc}, C \cap \operatorname{Int} T(pr'c)]$ for some $\varepsilon > 0$. Hence for a point $q' \in \overline{pr'}$ such that $p \neq q'$ and $\varrho(p, q') < \varepsilon$ we obtain $C \cap \operatorname{Int} T(pq'c) = 0$, i.e. (I) is satisfied. Finally, if some component K of the set $C \cap \operatorname{Int} T(pr'c)$ has a limit point on $\overline{pc}$, then the conditions $q' \in \overline{pr'}$, $p \neq q$, $K \cap \overline{q'c} \neq 0$ imply

the existence of component K' of $C \cap T(pq'c)$ such that $K' \cap \overline{q'c} \neq 0$ and $K' \cap \overline{pc} \neq 0$. Then K' must separate $T(pq'c)$ between c and $\overline{pq'}$, i.e. (II) is satisfied. Thus our lemma is proved.

Now let C be an arbitrary plane continuum such that $D - C$ is not connected. To prove the connectedness of $D^e \cup \{c\}$, it is sufficient to show that

$$C \cap (D^e \cup \{c\}) \neq 0. \tag{36}$$

By (34) the end points of every segment of family S_i, $i = 1, 2, \ldots$, belong to $D^e \cup \{c\}$. Therefore if one of them belongs to C, (36) is proved. Thus we may assume that they do not belong to C. It follows from (33) that $T(pqc) \in T_i$ implies $\overline{pc} \in S_{i+1}$ for $i = 1, 2, \ldots$, whence $p, c \notin C$.

Applying the lemma, let us suppose that (I) holds for every $T(pqc) \in T_i$, $i = 1, 2, \ldots$ So $D \cap \operatorname{Int} T(pq'c) \subset D - C$ for $T(pqc) \in T_i$, $i = 1, 2, \ldots$ We infer from (29) that the closure of $D \cap \operatorname{Int} T(pq'c)$ contains the side $\overline{pc}$ of $T(pq'c)$. Thus, putting $V = \bigcup [D \cap \operatorname{Int} T(pq'c)]$, where $T(pqc) \in T_i$, $i = 1, 2, \ldots$, we have $V \subset D - C$ and $\bigcup_{i=1}^{\infty} S_i^* \subset \overline{V}$ by (33), whence $D = \overline{V}$ by (35). However, it is easy to verify that the set V is composed of straight line segments (perhaps without end points) such that the closure of each of them contains the point c (as its end point). Therefore $V \cup \{c\}$ is an arcwise connected set dense in D. Furthermore $V \cup \{c\} \subset D - C \subset \overline{V \cup \{c\}}$. It follows (see [4], p. 83) that $D - C$ is a connected set, contrary to the hypothesis concerning C.

Hence (I) does not hold for some $T(pqc) \in T_j$, and we have (II) by virtue of the lemma. We shall define for every $k = 1, 2, \ldots$ an oriented triangle $T_k = T_k(p_k q_k c)$ such that

$$T_1 \supset T_2 \supset \ldots, \tag{37}$$

$$T_k \in T_{j+k}, \tag{38}$$

(39) some component of $C \cap T_k$ separates T_k between c and $\overline{p_k q_k}$,

and

$$\varrho(C, \overline{p_k q_k}) < 1/k \tag{40}$$

for $k = 1, 2, \ldots$

Namely, it follows from (28), (29) and (II) that there exists a $T_1(p_1 q_1 c) \in T(pqc)$ satisfying (39) and (40) for $k = 1$. Since $T(pqc) \in T_j$, we have $T_1 \in T_{j+1}$ according to (33), and (38) holds for $k = 1$.

Suppose that T_n is given. The existence of T_{n+1} belonging to $T(p_n q_n c)$ and satisfying (39) and (40) for $k = n+1$ is a consequence of (28), (29) and (39) for $k = n$. Therefore $T_{n+1} \subset T_n$ by virtue of the definition of the family $T(p_n q_n c)$, and $T_{n+1} \in T_{j+n+1}$ by virtue of (33) and (38) for $k = n$. So T_{n+1} satisfies all conditions (37)-(40) for $k = n+1$.

The triangles T_k being defined, let us observe that (30), (33) and (38) imply

$$\varrho(p_{k+1}, q_{k+1}) \leqslant \varrho(p_k, q_k)/2$$

for $k = 1, 2, \ldots$ Therefore $T_1, T_2, \ldots$ form, by (37), a descendent sequence of triangles each of which is thinner than the preceding one. It follows that the topological limit $\operatorname{Lim}_{k\to\infty} \overline{p_k q_k}$ of the sides $\overline{p_k q_k}$ exists and is an end point of the dendroid D according to (32). However, we conclude from (40) that $\operatorname{Lim}_{k\to\infty} \overline{p_k q_k} \in C$, C being a compact set. Hence $C \cap D^e \neq 0$ and (36) is proved.

Remark. In the dendroid D the set $D^e \cup \{c\}$ is connected by (v), but, as we have shown, D^e is not a connected set; what is more, it does not contain any non-degenerate connected subset. The question of the existence of a plane dendroid $\varDelta$ such that $\varDelta^e$ is a connected set can be solved in the affirmative by the following additional constructions:

Let $D' = D - \overline{ac} \cup \{c\}$ and let Q be an open plane connected neighbourhood of c such that $\delta(Q) < 1/2^0$ and $D - Q \neq 0$. Let $R_1, R_2, \ldots$ be the sequence of component regions of $Q - D$. Then by (vi) there exists an end point $p_i \in D^e$ $(i = 1, 2, \ldots)$ such that $\overline{p_i c} \cap \operatorname{Fr} R_i \neq 0$. We find a plane homeomorphical image $h_i(D')$ of D' such that $h_i(D') \cap D = \{h_i(c)\}$, $h_i(c) \in \overline{R_i}$ and $\overline{h_i(D')} - h_i(D') = \overline{p_i c} \cup (D \cap \operatorname{Fr} R_i)$ for $i = 1, 2, \ldots$ Then $D_i = \overline{h_i(D')}$ are dendroids and putting $c_i = h_i(c)$ we obtain $\varrho(c, c_i) < 1/2^0$ for $i = 1, 2, \ldots$ Furthermore,

$$(D \cup \bigcup_{i=1}^{\infty} D_i)^e$$

is a connected set *between* every two points belonging to D^e (see [4], p. 89).

Continuing this construction for D_i instead of D and c_i instead of c, etc., we can find sequences of dendroids $D_{i_1 \ldots i_n}$ and their points $c_{i_1 \ldots i_n}$ such that

$$\varrho(c_{i_1 \ldots i_n i}, c_{i_1 \ldots i_n}) < 1/2^n$$

for natural i_j, i and n, and that the closure $\varDelta$ of the union of all $D_{i_1 \ldots i_n}$ $(i_j, n = 1, 2, \ldots)$ is a dendroid. Then, furthermore,

$$(D \cup \bigcup_{i_j} D_{i_1 \ldots i_{n+1}})^e$$

is a connected set between every two points belonging to

$$(D \cup \bigcup_{i_j} D_{i_1 \ldots i_n})^e$$

for $n = 1, 2, \ldots$ It follows that

$$S = (D \cup \bigcup_{n=1}^{\infty} \bigcup_{i_j} D_{i_1 \ldots i_n})^e$$

is a connected set between every two of its points. Thus also $\varDelta^e$ is a connected set, since $S \subset \varDelta^e \subset \overline{S}$.

The above constructions, presented in a few words and without proper precision, are really more complicated and require some other restrictions. I do not know whether they can be simplified and therefore I only announce their possibility.

§ 10. The plane dendroid D for which D^e is not of the 1-st Borel class. Such a dendroid can be obtained from the examples of dendroids given in §§ 8 and 9 by identifying their points $(0, 0)$ and c respectively. Conditions (iv) in §§ 8 and 9 imply that the dendroid D constructed in this manner is such that D^e is neither a G_δ-set nor a F_σ-set. It is of the 2-nd Borel class by virtue of conditions (iii) in §§ 8 and 9.

In § 6 we have proved that the set of the end points of every plane dendroid is of the 3-rd Borel class. The question whether it is always of the 2-nd Borel class remains open.

Similarly the valuation of the Borel class of sets D^e, when D is *not a plane dendroid* or is an arbitrary arcwise connected continuum, is unknown.

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