Collections of convex sets which cover a Banach space

by

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1. Introduction. It is a well known theorem of A. H. Stone ([3], p. 160) that each open cover of a metric space has a locally finite refinement [1]. However, an example has recently been found [4] of a metric space M with a base $\mathcal{B}$ such that there is an open cover of M which does not have a locally finite refinement consisting of elements in $\mathcal{B}$. This example suggested the theorem which is the subject of this paper.

THEOREM. *For any cover $\mathcal{U}$ of a reflexive, infinite dimensional Banach space B, where $\mathcal{U}$ consists of bounded convex sets, there is a point x in B such that each neighborhood of x meets infinitely many elements of $\mathcal{U}$. That is, $\mathcal{U}$ is not locally finite.*

In the proof of the Theorem which is given in section 3, the fact that each closed, convex set which does not contain 0 lies on one side of a hyper-space is used to reduce the problem to a finite dimensional one. A rather technical consequence of Brouwer's fixed point-theorem, which is proved in section 2, completes the proof.

For convenience, only real Banach spaces will be considered.

2. Lemma 1 and notation. The subject of this section is a lemma which is similar to ([2], Proposition IV LD), but as will be noted, more information is required about a situation which is slightly different from that treated in this result.

First, some notation. Let $\{x_i\colon i=1,2,\ldots\}$ be a countable collection of linearly independent points of a Banach space. For each integer $s>0$, let I^s be the set of those elements of the form $\sum_1^s a_i x_i$ with $0 \leqslant a_i \leqslant 1$ for $i=1,2,\ldots,s$. Of course, I^s is homeomorphic to an s-dimensional cube. Let $I^0=0$ and $I^{-1}=\emptyset$. Let $C_i=\{x\in I^s\colon a_i=0\}$ and $C_i'=\{x\in I^s\colon a_i=1\}$. If $\mathcal{U}$ is a cover of I^s, let $\mathcal{U}_i=\{U\in\mathcal{U}\colon U^-\cap I^i\neq\emptyset$, but $U^-\cap I^{i-1}=\emptyset\}$ where the closure of a subset A of I^s is written A^-.

(1) A collection $\mathcal{V}$ of subsets of a topological space X is a locally finite refinement of a cover $\mathcal{U}$ of X, if $\mathcal{V}$ is a cover for X, if each $V\in\mathcal{V}$ is contained in some $U\in\mathcal{U}$, and if for each $x\in X$ there is a neighborhood N of x such that N intersects only finitely many elements of $\mathcal{V}$.

Lemma 1. *Let $\mathcal{U}$ be a finite open cover of I^s such that* (a) *if $U \in \mathcal{U}$ and $U^- \cap C_i \neq \emptyset$, then $U^- \cap C_i' = \emptyset$, and* (b) *if $U \in \mathcal{U}_i$, then $U^- \cap C_i = \emptyset$. Then there is a $U_i \in \mathcal{U}_i$ for $i = 0, 1, \ldots, s$ such that $\bigcap\{\overline{U_i}\colon i = 0, 1, \ldots, s\} \neq \emptyset$.*

Proof. Since each $U \in \mathcal{U}_i$ has the property that $U \cap C_{i+1} \neq \emptyset$, then $\mathcal{U}_{i+1} \neq \emptyset$. Define $f_i(x)$ to be the point in I^s whose coordinates are the same as those of x, except for the ith coordinate. The ith coordinate of $f_i(x)$ is $x_i + d(x, \operatorname{bdry}(\bigcup \mathcal{U}_{i-1}))$, if $x \in \bigcup \mathcal{U}_{i-1}$; if $x \notin \bigcup \mathcal{U}_{i-1}$, the ith coordinate of $f_i(x)$ is $x_i - \min\{d(x, I^i), d(x, \operatorname{bdry}(\bigcup \mathcal{U}_{i-1}))\}$. It is easily seen that f_i is a continuous function from I^s to I^s. Let $\varphi = f_s(f_{s-1}(\ldots(f_1)\ldots))$.

Suppose that $z = (z_1, z_2, \ldots, z_s)$ is a fixed point of φ. (Of course, this notation means $z = \sum_1^s z_i x_i$.) Then z is a fixed point of each f_i. However, z is a fixed point of f_i if and only if $z \in \operatorname{bdry}(\bigcup \mathcal{U}_{i-1})$ or $z_i = 0$. Suppose $z_n = 0$ for some $1 \leqslant n < s$. Since $z \in C_n$ implies that $z \notin \operatorname{bdry}(\bigcup \mathcal{U}_n)$ by virtue of (b), it follows that $z_{n+1} = 0$ because z is a fixed point for f_{n+1}. Hence suppose that z has the form $(z_1, z_2, \ldots, z_p, 0, \ldots, 0)$ where $z_i \neq 0$ for $i \leqslant p$. It follows that $z \in U$ for $U \in \mathcal{U}_i$, at least for some $i \leqslant p$. For this i, $f_i(z) \neq z$. Hence $z_i \neq 0$ for any i, and $z \in \operatorname{bdry}(\bigcup \mathcal{U}_i)$ for $0 \leqslant i < n$. Since there is a $U \in \mathcal{U}_n$ such that $z \in U$, the Lemma follows.

3. Proof of the Theorem. The proof is by contradiction. It will be assumed that $\mathcal{U}$ is a locally finite collection of bounded, convex sets which cover B. One may even assume that each $U \in \mathcal{U}$ is open since (a) it suffices to prove the theorem for separable B, (b) for separable B, $\mathcal{U}$ must be countable, and (c) the ith member of $\mathcal{U}$ may by expanded by $1/i$, and the resulting collection is a locally finite cover with open, bounded, convex sets. Using this, a countable number of linearly independent points $x_1, x_2, \ldots$ will be chosen such that the intersections of members of $\mathcal{U}$ with I^s form a collection which satisfies the conditions of Lemma 1. This will complete the proof, as the following argument shows. Suppose that the x_i have been chosen in the manner indicated. Let $\mathcal{U}_i$, $i = 0, 1, \ldots$ be defined as in section 2, except we are interested only in that part of each U which lies in $\bigcup\{I^s\colon s = 1, 2, \ldots\}$. Each $\mathcal{U}_i$ is a finite collection since $\mathcal{U}$ is locally finite. A collection $U_0, U_1, \ldots, U_s$ will be said to be a chain if $U_i \in \mathcal{U}_i$ and $\bigcap\{\overline{U_i}\colon i = 0, 1, \ldots, s\} \neq 0$. Lemma 1 states that there are arbitrarily long chains. Hence, since $\mathcal{U}_i$ is finite, a standard argument shows that there is an infinite chain $U_0, U_1, U_2, \ldots$, that is, for this collection $\bigcap\{\overline{U_i}\colon i = 0, 1, \ldots, s\} \neq \emptyset$ for $s = 0, 1, \ldots$ Because each U_i is convex, $\overline{U_i}$ is weakly closed ([1], p. 2), and hence weakly compact, since U_i is bounded and B is reflexive ([1], p. 56). Therefore, there is an $x \in \bigcap\{\overline{U_i}\colon i = 0, 1, \ldots\}$, and this contradicts the assumption that $\mathcal{U}$ is locally finite.

All that remains is to choose the x_i. Let $\mathcal{U}_0 = \{U \in \mathcal{U}\colon 0 \in U^-\}$. Pick an $x_1 \in B$ such that $\|x_1\| > \max[\text{diameter } (U)\colon U \in \mathcal{U}_0]$. Obviously $\mathcal{U}_0 \cup \mathcal{U}_1$ has the required property with respect to I^1. Suppose that $x_1, x_2, \ldots, x_s$ is chosen such that $\{U \cap I^s\colon U \in \mathcal{U}$ and $U \cap I^s \neq \emptyset\}$ is a collection of Lemma 1. Also assume that for each $1 < i \leqslant s$ there is an infinite dimensional subspace B_i such that $B_2 \supset B_3 \supset \ldots \supset B_s$ and $(\bigcup \mathcal{U}_i)^- \cap (B_{i+1} + I^{i-1}) = \emptyset$. Here, $B_{i+1} + I^{i-1}$ means the set of all $x \in B$ such that $x = u + v$, $u \in B_{i+1}$ and $v \in I^{i-1}$. Suppose that B_i is chosen such that $x_i \notin B_i$ for $j < i$, but suppose that $x_i \in B_i$, $i = 2, 3, \ldots, s$.

Let us show how B_{s+1} is chosen. By the definition of $\mathcal{U}_s$, $(\bigcup \mathcal{U}_s)^- \cap I^{s-1} = \emptyset$. Let π be the natural projection of $B^s + I^{s-1}$ onto B_s. Then $0 \notin \pi\left([\bigcup \mathcal{U}_s \cap (B_s + I^{s-1})]^-\right)$; hence for $U \in \mathcal{U}_s$ there is a hyper-space h_U in B_s such that $h_U \cap \pi([U \cap B_s + I^{s-1}]^-) = \emptyset$ [1]. Define B_{s+1} to be $\bigcap\{h_U\colon U \in \mathcal{U}_s\}$. Since $x_s \in$ some U in $\mathcal{U}_s$, $x_s \notin B_{s+1}$. Choose x_{s+1} to be an element of B_{s+1} such that $\|x_{s+1}\| > \max[\text{diameter}(U)\colon U \in \bigcup\{\mathcal{U}_i\colon i = 1, 2, \ldots, s\}]$.

It can now be shown easily that the conditions of Lemma 1 are satisfied, and hence the theorem follows as we have seen.

Remarks. A slightly stronger result has been proved than was claimed. It has been shown that there is a s-dimensional cube $I^s \subset B$ such that infinitiely many members of $\mathcal{U}$ meet I^s. I do not know if s may be always chosen to be 0. That is, is there a point in infinitely many members of $\mathcal{U}$?

Moreover, it is easy to see that the same approach establishes the analogous result for a covering of an arbitrary infinite-dimensional normed linear space by open convex sets, if the family $\mathcal{F}$ of their closures has the following property. Whenever a subfamily of $\mathcal{F}$ has the finite intersection property, then it has a nonempty intersection.

References

[1] M. M. Day, *Normed linear spaces*, Berlin 1958.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1948.

[3] J. L. Kelly, *General topology*, New York 1955.

[4] L. B. Trybig, *Concerning certain locally peripherally separable spaces*, Pacific Journ. Math. (To appear in the June 1960 issue.)

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