icm

A nowhere differentiable and except on a denumerable set everywhere continuous function

by

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In what follows, a very simple example of a function f defined on the open unit interval is given, and it is shown that f is continuous everywhere except on a denumerable set and that f is *not differentiable even one-sidedly, either finitely or infinitely*, at any point where it is continuous.

We shall assume the decimal expansion of the reals of the open unit interval I, where infinite succession of 9's is not allowed. Thus every real number belonging to I is either a finite or an infinite decimal.

Consider the one-to-one mapping f from I onto I given by

$$f(0.a_1a_2\ldots a_{2n+1}a_{2n+2}\ldots) = 0.a_2a_1\ldots a_{2n+2}a_{2n+1}\ldots$$

Thus for example:

$$f(0.3540151515\ldots) = 0.5304515151\ldots$$

We assert that

(1) *f is continuous everywhere in I except on the set of all finite decimals where it is continuous only on the right.*

(2) *f is not differentiable even one-sidedly anywhere in I.*

Proof. Let $a = 0.a_1a_2\ldots$ be an infinite decimal. Let $k \geqslant 2n+1$ be such that $a_k \neq 9$ and such that there exists an m with $2n+1 \leqslant m \leqslant k$ and $a_m \neq 0$. Such an a_k always exists since infinite succession of 9's is not allowed and since a is an infinite decimal. Then

$$|f(a+h)-f(a)| < 10^{-2n} \quad \text{for} \quad |h| < 10^{-k},$$

since in our case

$$\sum_{i=2n+1}^{\infty} a_i 10^{-i} < 10^{-2n}.$$

Thus, f is continuous at every infinite decimal. Also the same proof shows that f is continuous on the right at every finite decimal.

Now, let $a = 0.a_1a_2\ldots a_n$ be a finite decimal where naturally $a_n \geqslant 1$.

(i) If n is odd, then $f(a) = 0.a_2a_1 \ldots 0a_n$; but

$$\lim_{i\to\infty} f(a-10^{-n-i}) = 0.a_2a_1 \ldots 9a_n > f(a) .$$

(ii) If n is even, then $f(a) = 0.a_2a_1 \ldots a_na_{n-1}$; but

$$\lim_{i\to\infty} f(a-10^{-n-i}) = 0.a_2a_1 \ldots (a_n-1)(a_{n-1}+1) < f(a)$$

if $a_{n-1} \neq 9$, and $= 0.a_2a_1 \ldots a_n0 < f(a)$ if $a_{n-1} = 9$.

Hence f is not continuous on the left at any finite decimal. Thus (1) is proved.

To prove (2), let us observe that if a is any decimal, then in its expansion $a = 0.a_1a_2 \ldots$, a fixed quadruplet $a'a''a'''a''''$ with $a'a''a'''a'''' \leqslant 9899$ will occur infinitely many times and this for infinitely many i's with $a' = a_{2i+1}$. Now let $\bar{a}$ denote the decimal which is obtained by replacing the quadruplet $a'a''a'''a''''$ in $0.a_1a_2 \ldots$ by a quadruplet $b'b''b'''b'''' > 9899$ at only one place. Then

$$(*) \qquad \frac{f(\bar{a})-f(a)}{\bar{a}-a} = \frac{b''b'b''''b'''-a''a'a''''a'''}{b'b''b'''b''''-a'a''a'''a''''} .$$

Since $\bar{a}-a$ can be made as small as we please, if $f'(a)$ existed on the right, then for various choices of the quadruplet $b'b''b'''b'''' > 9899$, the right hand side of $(*)$ would be a constant. But the three particular choices 9999, 9900 and 9901 for the quadruplet $b'b''b'''b''''$ show the contrary. Hence, as $(*)$ shows, f is not differentiable either finitely or infinitely on the right at any decimal.

Now, if a is an infinite decimal, then in its expansion a fixed quadruplet $a'a''a'''a''''$ with $a'a''a'''a'''' \geqslant 100$ will occur infinitely many times. By reasoning analogous to the above, we see that the three particular choices 99, 88 and 90 for the quadruplet $b'b''b'''b''''$ show that f is not differentiable either finitelly or infinitely on the left at any infinite decimal; and since, as shown earlier, f is not continuous on the left at any finite decimal, the proof of (2) is complete.

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On convex metric spaces I

by

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§ 1. Introduction. In February 1959 Professor Borsuk presented to us the following three problems (for an explanation of the notions see §§ 3 and 4 of this paper):

I. *Is it true that every n-dimensional continuum* (connected compact metric space) *which is strongly convex and without ramifications must be topologically n-cell?*

II. *Has every n-cell ($n = 2, 3, \ldots$) with an arbitrary strongly convex metric (topology preserving) at least $n+1$ terminal points?*

III. *Does there exist a compact metric space whose every point is a ramification point or a frontier point?*

The purpose of this paper is to give a partial (for $n = 2$) positive solution of problem I (see § 9), a general (for $n = 2, 3, \ldots$) negative solution of problem II (see § 14) and a positive solution of problem III (see § 12).

§ 2. Betweenness and linearity. We consider a space X with a metric ϱ and write shortly that (X, ϱ) is a *metric space*. Let $p, q, r \in X$. We say (compare [1], p. 317) that the point q is *between* the points p and r (writing pqr) provided that

$$\varrho(p, r) = \varrho(p, q) + \varrho(q, r) .$$

Evidently pqr is equivalent to rqp, and we have psr provided that pqr and psq or qsr.

We say that the set $A \subset X$ is *linear* if there exists an isometrical transformation i: $A \to \mathcal{E}^1$ of A into the set $\mathcal{E}^1$ of all real numbers, i. e. $\varrho(p, q) = |i(p)-i(q)|$ for every $p, q \in A$.

Hence

2.1. *The set $\{p, q, r\}$ composed of three points is linear if and only if one of the points p, q, r is between two others.*

Let us note that there exists a metric space $(\{p, q, r, s\}, \varrho)$ composed of four points which is not linear, but every proper subset of which is linear. Namely put: $\varrho(p, r) = \varrho(r, q) = \varrho(q, s) = \varrho(s, p) = 1$ and $\varrho(p, q) = \varrho(r, s) = 2$.