Neutral ideals and congruences

by

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Introduction. This is an attempt to find whether there is a (1-1)-correspondence between the congruence relations and neutral ideals of a lattice (1) in general, and of a modular lattice in particular (Problem 73 of [1], p. 161). This problem, though not solved fully in the course of this paper, has led to certain results which are interesting in themselves.

By defining a neutral ideal in a particular way (as in [1], p. 80), it is shown that the neutral ideals of a lattice L correspond (1-1) to a sublattice of the lattice of congruences on L. Hence to every neutral ideal of L corresponds a lattice congruence θ on L. It is therefore natural to ask whether every congruence θ on L has a neutral ideal as the ideal of elements congruent to zero. As is shown in Section 4 below, the answer to this question is in the negative.

In Section 2, I discuss the properties of neutral ideals. In Section 3, I prove the correspondence between neutral ideals and congruences; finally, in Section 4, I give counter examples to show the inconsistency of certain questions which arise naturally.

1. Preliminaries. The symbols $\leqslant$, $\nleqslant$, $+$, $\cdot$ will denote inclusion, non-inclusion, sum (least upper bound) and product (greatest lower bound) in any lattice L; while the symbols $\subseteq$, $\cup$, $\cap$, ϵ, $\notin$ will refer to set-inclusion union (set sum), intersection (set product), membership and non-membership, respectively. Small letters $a, b, \ldots$ will denote elements of the lattice and capital letters $A, B, \ldots$ will stand for ideals of the lattice.

All lattices which are considered during the course of this paper have 0, the null element of the lattice.

A non-null subset $\mathcal{I}$ of elements of a lattice L is called an *ideal* if and only if

(i) $x \in \mathcal{I},\ y \in \mathcal{I} \Rightarrow x+y \in \mathcal{I}$

and

(ii) $x \in \mathcal{I}$ and $t \leqslant x \Rightarrow t \in \mathcal{I}$. (Stone [6]).

(1) For general information regarding lattices, see Birkhoff [1].

An ideal $\mathcal{J}$ of a lattice L is called *neutral* if and only if

$$t \leqslant (x+a)(y+b)(x, y \in \mathcal{J}) \Rightarrow t \leqslant z + a\cdot b \quad \text{for some} \quad z \in \mathcal{J}.$$

An ideal $\mathcal{J}$ of a lattice L is called *m-neutral* if and only if $\mathcal{J}$ as an element of the lattice of ideals of L distributes finite products.

A binary relation θ on L is said to be an *equivalence relation* if it satisfies

(i) $$x \equiv x(\theta) \quad \text{(reflexive)},$$

(ii) $$x \equiv y(\theta) \Rightarrow y \equiv x(\theta) \quad \text{(symmetric)},$$

(iii) $$x \equiv y(\theta),\ y \equiv z(\theta) \Rightarrow x \equiv z(\theta) \quad \text{(transitive)}.$$

If it further satisfies the substitution property

(iv) $$x \equiv x'(\theta),\ y \equiv y'(\theta) \Rightarrow x+y \equiv x'+y'(\theta),$$

then it is called an *additive congruence*.

An equivalence relation θ which has the substitution property

(v) $$x \equiv x'(\theta),\ y \equiv y'(\theta) \Rightarrow x\cdot y \equiv x'\cdot y'(\theta)$$

is called a *multiplicative congruence*.

If the binary relation θ satisfies the conditions (i)-(v) then it is said to be a *lattice congruence* or merely a *congruence on L*. The congruences on a lattice L form a complete lattice (see [1], p. 24).

The sum and product of an arbitrary family of congruences are defined as follows:

$$a \equiv b(\textstyle\bigcup \theta_i)$$

if there exists a finite sequence $a = x_0, x_1, \ldots, x_n = b$ such that $x_{j-1} \equiv x_j(\theta_{ij})$ for some θ_{ij}, $j = (1, 2, \ldots, n)$, and

$$a \equiv b(\textstyle\bigcap \theta_i)$$

if $a \equiv b(\theta_i)$ for every i.

2. The purpose of this section is to show that the neutral ideals of a modular lattice form a Σ-distributive lattice. Certain preliminary results are needed.

LEMMA 1. *The principal ideal generated by a single element c of a lattice L is neutral if and only if c distributes finite products.*

Proof. Let the principal ideal I generated by c be neutral; then

$$t \leqslant (x+a)(y+b)(x, y \in I) \Rightarrow t \leqslant z + a\cdot b$$

for some $z \in I$. But since $x, y \in I$, we can take $x = c$ and $y = c$. Therefore $t \leqslant (c+a)(c+b) \Rightarrow t \leqslant z + a\cdot b$ for some $z \in I$.

Now, any $z \in I$ is less than c and hence $t \leqslant z + a\cdot b \Rightarrow t \leqslant c + a\cdot b$. Therefore $t \leqslant (c+a)(c+b) \Rightarrow t \leqslant c + a\cdot b$. Hence $(c+a)(c+b) \leqslant c + a\cdot b$. But $(c+a)(c+b) \geqslant c + a\cdot b$ for all a, b, c in L. Therefore $(c+a)(c+b) = c + a\cdot b$. Hence c distributes finite products.

Conversely, let c distribute finite products. Then

$$(c+a)(c+b) \leqslant c + a\cdot b$$

and when

$$t \leqslant (x+a)(y+b)(x\cdot y \in I), \quad t \leqslant (c+a)(c+b)$$

since I is the principal ideal generated by c. Therefore $t \leqslant (x+a)(y+b) \Rightarrow$ $\Rightarrow t \leqslant (c+a)(c+b) \Rightarrow t \leqslant c + a\cdot b$; that is, $t \leqslant z + a\cdot b$ for $z = c \in I$. Hence I is a neutral ideal of L.

COROLLARY. *When the lattice L is modular, if c distributes finite products, it distributes finite sums also* (see [1], p. 78) *and hence is a neutral element* (see [3]).

Hence in the case of a modular lattice L the lemma reads as follows:

The principal ideal generated by an element $c \in L$ is neutral if and only if c is neutral.

LEMMA 2. *The sum of any family of neutral ideals of a lattice L is a neutral ideal.*

Proof. Let $N = \sum N_i$, each N_i being a neutral ideal. Elements $x, y \in N$ are given by

$$x \leqslant x_1 + x_2 + \ldots + x_s, \quad y \leqslant y_1 + y_2 + \ldots + y_{s'},$$

s, s' being finite and $x_s \in N_s$ and $y_{s'} \in N_{s'}$ (vide [1], p. 140).

Since 0 belongs to every ideal, we can add as many zeros as are necessary and write

$$x \leqslant x_1 + x_2 + \ldots + x_\nu, \quad y \leqslant y_1 + y_2 + \ldots + y_\nu,$$

such that $x_i, y_i \in N_i$ for each i. Therefore

$$\begin{aligned}
t &\leqslant (x+a)(y+b) \quad \text{for} \quad x, y \in N \\
&\Rightarrow t \leqslant (x_1 + x_2 + \ldots + x_\nu + a)(y_1 + y_2 + \ldots + y_\nu + b) \quad (x_i, y_i \in N_i \text{ for each } i) \\
&\Rightarrow t \leqslant z_1 + (x_2 + \ldots + x_\nu + a)(y_2 + \ldots + y_\nu + b) \\
&\qquad\qquad (\text{for some } z_1 \in N_1, \text{ since } x_1, y_1 \in N_1) \\
&\Rightarrow t \leqslant z_1 + z_2 + (x_3 + \ldots + x_\nu + a)(y_3 + \ldots + y_\nu + b) \quad (z_1 \in N_1;\ z_2 \in N_2) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\Rightarrow t \leqslant z_1 + z_2 + \ldots + z_\nu + a\cdot b \quad (z_i \in N_i \text{ for each } i) \\
&\Rightarrow t \leqslant z + a\cdot b \quad (z \in N)
\end{aligned}$$

where $z = z_1 + z_2 + \ldots + z_\nu$.

Therefore N is a neutral ideal.

It is easily seen that 0 is a neutral ideal. Therefore the neutral ideals on any lattice L form a partially ordered set with 0, and every non-void subset of L has a least upper bound (by Lemma 2). Hence the neutral ideals of any lattice L form a complete lattice (vide [1], p. 49).

LEMMA 3. *The product of two and hence also of a finite number of neutral ideals of a modular lattice L is a neutral ideal.*

Proof. Let X and Y be neutral ideals of L. Let $x, y \in X \cap Y \Rightarrow x, y \in X$ and $x, y \in Y$. Now

$$t \leqslant (x+a)(y+b) \Rightarrow t \leqslant z_1 + a\cdot b \quad (z_1 \in X);$$

also

$$t \leqslant (x+a)(y+b) \Rightarrow t \leqslant z_2 + a\cdot b \quad (z_2 \in Y).$$

Therefore

$$\begin{aligned} t \leqslant (x+a)(y+b) &\Rightarrow t \leqslant (z_1+a\cdot b)(z_2+a\cdot b) \quad (z_1 \in x;\ z_2 \in Y)\\ &\Rightarrow t \leqslant a\cdot b + z_1(z_2+a\cdot b) \quad (L \text{ is modular})\\ &\Rightarrow t \leqslant a\cdot b + z_1(0+z_1)(z_2+a\cdot b)\\ &\Rightarrow t \leqslant a\cdot b + z_1(z_3+z_1\cdot a\cdot b) \quad (0, z_2 \in Y \text{ for some } z_3 \in Y)\\ &\Rightarrow t \leqslant a\cdot b + z_1\cdot z_3 + z_1\cdot a\cdot b \quad (L \text{ is modular})\\ &\Rightarrow t \leqslant z_1\cdot z_3 + a\cdot b \quad (z_1\cdot z_3 \in X \cap Y)\\ &\Rightarrow t \leqslant z + a\cdot b \quad \text{for some} \quad z \in X \cap Y. \end{aligned}$$

Therefore $X \cap Y$ is a neutral ideal. This proof can be extended to a finite number of neutral ideals.

THEOREM 1. *The neutral ideals of a modular lattice satisfy the infinite distributive law: $N \cap (\sum N_i) = \sum (N \cap N_i)$.*

Proof. Now $N \cap (\sum N_i) \supseteq \sum (N \cap N_i)$. Therefore it is enough if we prove that

$$N \cap (\sum N_i) \subseteq \sum (N \cap N_i).$$

Let $a \in N \cap (\sum N_i)$; that is, $a \in N$ and $a \in \sum N_i$, $a \leqslant x_1 + x_2 + \ldots + x_\nu$ for finite ν and $x_\nu \in N_\nu$. Therefore

$$\begin{aligned} a &= a\cdot(x_1+x_2+\ldots+x_\nu)\\ &= (0+a)(x_1+x_2+\ldots+x_\nu) \quad (0, x_1 \in N_1)\\ &\Rightarrow a \leqslant y_1 + a\cdot(x_2+\ldots+x_\nu) \quad (\text{for some } y_1 \in N_1)\\ &\Rightarrow a = a\cdot\big(y_1 + a(x_2+\ldots+x_\nu)\big)\\ &\Rightarrow a = a\cdot y_1 + a\cdot(x_2+\ldots+x_\nu) \quad (L \text{ is modular})\\ &\Rightarrow a = ay_1 + (0+a)(x_2+\ldots+x_\nu) \quad (0, x_2 \in N_2)\\ &\Rightarrow a \leqslant ay_1 + y_2 + a(x_3+\ldots+x_\nu) \quad (y_2 \in N_2)\\ &\Rightarrow a = a\cdot\big(a\cdot y_1 + y_2 + a\cdot(x_3+\ldots+x_\nu)\big)\\ &\Rightarrow a = ay_1 + a\cdot y_2 + a\cdot(x_3+\ldots+x_\nu) \quad (L \text{ is modular})\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\Rightarrow a = ay_1 + ay_2 + \ldots + ay_\nu \quad (y_i \in N_i \text{ for each } i)\\ &\Rightarrow a \in \sum (N \cap N_i) \quad (a\cdot y_i \in N \cap N_i \text{ for each } i). \end{aligned}$$

Therefore $N \cap (\sum N_i) \subseteq \sum (N \cap N_i)$ and hence

$$N \cap (\sum N_i) = \sum (N \cap N_i).$$

COROLLARY 1. *The neutral ideals of a modular lattice form a complete $\sum$-distributive lattice. Hence it is pseudocomplemented* (for definition, see [1], p. 147).

COROLLARY 2. *When the lattice is distributive, all ideals are neutral. Therefore the ideals of a distributive lattice form a pseudo-complemented distributive lattice.*

LEMMA 4. *An ideal I of a lattice L is neutral if and only if I is m-neutral.*

Proof. Let I be neutral; that is,

$$t \leqslant (x+a)(y+b)(x, y \in I) \Rightarrow t \leqslant z + a\cdot b \quad (z \in I).$$

Consider two ideals A, B of L such that $a \in A$ and $b \in B$: then $a\cdot b \in A \cap B$. Hence

$$\begin{aligned} &t \in (I \cup A) \cap (I \cup B)\\ &\Rightarrow t \leqslant (x+a)(y+b)\\ &\Rightarrow t \leqslant z + a\cdot b \quad (z \in I)\\ &\Rightarrow t \in I \cup (A \cap B). \end{aligned}$$

Therefore $(I \cup A) \cap (I \cup B) = I \cup (A \cap B)$.

Thus I distributes finite products and hence is m-neutral.

Conversely, let I be m-neutral; that is,

$$(I \cup A) \cap (I \cup B) = I \cup (A \cap B)$$

for any two ideals A, B of L. Therefore

$$(I \cup A) \cap (I \cup B) \subseteq I \cup (A \cap B).$$

Choose A, B as the principal ideals generated by a, b respectively. Let $x, y \in I$. Then

$$\begin{aligned} &t \leqslant (x+a)(y+b)\\ &\Rightarrow t \in (I \cup A) \cap (I \cup B)\\ &\Rightarrow t \in I \cup (A \cap B)\\ &\Rightarrow t \leqslant z + a_1\cdot b_1 \quad (a_1 \in A;\ b_1 \in B \text{ and } z \in I)\\ &\Rightarrow t \leqslant z + a\cdot b \quad (\text{since } a_1 \leqslant a \text{ and } b_1 \leqslant b). \end{aligned}$$

Therefore, I is a neutral ideal.

COROLLARY. *When the lattice L is modular, an m-neutral ideal is neutral as an element of the lattice of ideals of L. Hence an ideal of a modular lattice L is neutral if and only if it is neutral as an element of the lattice of ideals of L.*

When the lattice L is non-modular, a neutral ideal of L need not be neutral as an element of the lattice of ideals of L. For, consider the ideal I of the non-modular lattice L as shown in figure 1.

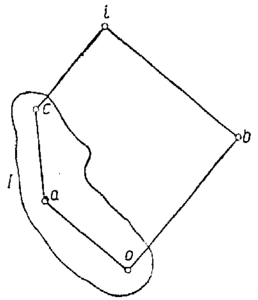


Fig. 1

I is a neutral ideal of L, but it is not neutral as an element of the lattice of ideals of L, since $I \cap (A \cup B) \neq (I \cap A) \cup (I \cap B)$ where $A = (0, a)$; $B = (0, b)$ for, $I \cap (A \cup B) = I$ and $(I \cap A) \cup (I \cap B) = A$.

Further, when the lattice is non-modular, the principal ideal generated by a single element c can be neutral even though c is not neutral in L. For, I is a neutral ideal of L, but c is not neutral, since $c = c\cdot(a+b) \neq c\cdot a + c\cdot b = a$.

3. The purpose of this section is to point out the correspondence between neutral ideals and congruences on a lattice L.

LEMMA 5. *The congruence modulo an ideal is a lattice congruence if and only if the ideal is neutral* (see Theorem 1 of [5]).

Proof. Let I be a neutral ideal. Define a binary relation: $x \equiv y$ if and only if $x+a = y+a$ for some $a \in I$. Clearly this relation is

(i) reflexive: $x \equiv x$ for $x+a = x+a$ for all $a \in I$;

(ii) symmetric: $x \equiv y \Rightarrow y \equiv x$, for $x+a = y+a \Rightarrow y+a = x+a$ $(a \in I)$;

(iii) transitive: $x \equiv y$; $y \equiv z \Rightarrow x \equiv z$ for $x+a = y+a$; $y+b = z+b \Rightarrow x+a+b = z+a+b$, where $a, b, (a+b) \in I$.

Therefore, "$\equiv$" is an equivalence relation.

(iv) an additive congruence: $x \equiv y \Rightarrow x+z \equiv y+z$ for

$$\begin{aligned} x \equiv y &\Rightarrow x+a = y+a \quad \text{for} \quad a \in I \\ &\Rightarrow x+z+a = y+z+a \quad \text{for all } z \\ &\Rightarrow x+z \equiv y+z . \end{aligned}$$

(v) a multiplicative congruence:

$$x \equiv y \Rightarrow x\cdot z \equiv y \cdot z .$$

By (iv), $\quad x \equiv y \Rightarrow x+a = y+a \quad$ for $\quad a \in I$.

So $x\cdot z + a\cdot b \leqslant (x+a)(z+b)$ for all $z \in L$ and $a, b \in I$, or $x\cdot z+a\cdot b \leqslant (y+a)(z+b)$ or $x\cdot z + a\cdot b \leqslant y\cdot z + c$ for some $c \in I$.

Similarly,

$$\begin{aligned} y\cdot z + a\cdot b &\leqslant (y+a)(z+b) \\ &\leqslant (x+a)(z+b) \\ &\leqslant x\cdot z + d \quad \text{for some } d \in I . \end{aligned}$$

Therefore,

$$x\cdot z + a\cdot b + c + d \leqslant y\cdot z + a\cdot b + c + d$$

and

$$x\cdot z + a\cdot b + c + d \geqslant y\cdot z + a\cdot b + c + d ;$$

that is, $x\cdot z + a\cdot b + c + d = y\cdot z + a\cdot b + c + d$ and $a\cdot b + c + d \in I$. Hence $x\cdot z \equiv y\cdot z$.

Thus, $x \equiv y$ is a lattice congruence on L; that is, any neutral ideal is a congruence class (congruent to zero) under some lattice congruence.

Conversely, let θ be a lattice congruence modulo an ideal I; that is, $x \equiv y$ if and only if $x+a = y+a$ for some $a \in I$. Let $t \leqslant (a+x)(b+y)$ $(a, b \in I)$. Then

$$\begin{aligned} t &= t\cdot(a+x)(b+y) , \\ t &\equiv t\cdot x\cdot y \cdot (\theta) \end{aligned}$$

$$\Rightarrow t + x\cdot y \equiv x\cdot y (\theta) \text{ since } \theta \text{ is a lattice congruence.}$$

Therefore $t + x\cdot y + c = x\cdot y + c$ for some $c \in I$; that is, $t \leqslant x\cdot y + c$. Hence

$$\begin{aligned} t &\leqslant (a+x)(b+y) \quad (a, b \in I) \\ \Rightarrow t &\leqslant c + x\cdot y \quad \text{for some} \quad c \in I . \end{aligned}$$

Therefore I is a neutral ideal.

COROLLARY. *Any ideal of a distributive lattice L is a zero class under some suitable congruence relation on L.*

Definition. The congruence modulo a neutral ideal is called a *neutral congruence.*

It is easily seen that there is a (1-1)-correspondence between neutral ideals and neutral congruences. If a non-neutral congruence Φ has a neutral ideal N as its zero class and if θ is the neutral congruence corresponding to the neutral ideal N, then $\theta \subseteq \Phi$.

For if $a \equiv b(\theta)$ then $a+t = b+t$ for some $t \in N$; also $t \equiv 0(\Phi)$. Therefore $a+t = b+t \Rightarrow a \equiv b(\Phi)$. Hence $\theta \subseteq \Phi$.

THEOREM 2. *The neutral congruences on a lattice L form a complete Σ-distributive lattice.*

Proof. Let $\theta_1, \theta_2, \ldots$ be an arbitrary family of neutral congruences on L. Let $\theta = \sum \theta_i$. Let $N_1, N_2, \ldots$ be the corresponding neutral ideals and let $N = \sum N_i$. Let Φ be the neutral congruence corresponding to the neutral ideal N. Then $\theta = \Phi$; for, $a \equiv b(\theta)$ implies that there exists a finite sequence $a = x_0, x_1, \ldots, x_\nu = b$ such that $x_{i-1} \equiv x_i(\theta_i) \Rightarrow x_{i-1}+t_i = x_i+t_i$ for some $t_i \in N_i$. Therefore,

$$\begin{aligned} x_0+t_1 &= x_1+t_1 \quad (t_1 \in N_1), \\ x_1+t_2 &= x_2+t_2 \quad (t_2 \in N_2), \\ &\ldots\ldots\ldots\ldots \\ x_{\nu-1}+t_\nu &= x_\nu+t_\nu \quad (t_\nu \in N_\nu). \end{aligned}$$

That is,

$$\begin{aligned} x_0+t_1+\ldots+t_\nu &= x_\nu+t_1+\ldots+t_\nu, \\ a+t_1+\ldots+t_\nu &= b+t_1+\ldots+t_\nu, \end{aligned}$$

and

$$t_1+\ldots+t_\nu \in \sum N_i.$$

Therefore $a \equiv b(\Phi)$ and hence $\theta \subseteq \Phi$.

Next let

$$a \equiv b(\Phi).$$

$$\Rightarrow a+t_1+\ldots+t_k = b+t_1+\ldots+t_k, \quad t_i \in N_i \text{ for some } i \text{ and finite } k.$$

$$\begin{aligned} a+t_1+\ldots\quad +t_k &\equiv a+t_1+\ldots\quad +t_{k-1}(\theta_k) \\ a+t_1+\ldots+t_{k-1} &\equiv a+t_1+\ldots+t_{k-2}(\theta_{k-1}) \\ &\ldots\ldots\ldots\ldots \\ a+t_1 &\equiv a(\theta_1). \end{aligned}$$

Therefore, $a+t_1+\ldots+t_k \equiv a(\theta)$.

Similarly $b+t_1+\ldots+t_k \equiv b(\theta)$. But $a+t_1+\ldots+t_k = b+t_1+\ldots+t_k$. Hence $a \equiv b(\theta)$. Therefore $\Phi \subseteq \theta$. Hence $\theta = \Phi$.

Therefore, an arbitrary sum of a number of neutral congruences is neutral and the identity congruence is also a neutral congruence. Therefore the neutral congruences on any lattice L form a complete lattice (see [1], p. 49). Further, the congruences on any lattice L, and in particular the neutral congruences, satisfy the infinite distributive law $\theta \cap (\sum \theta_i) = \sum (\theta \wedge \theta_i)$ ([1], p. 24). Therefore the neutral congruences on any lattice form a complete Σ-distributive lattice L_1.

COROLLARY 1. *The distributive lattice L_1 is a sublattice of the lattice of all congruence relations on L* (see [4]).

COROLLARY 2. *The number of neutral elements of L is at most equal to the number of neutral congruences on L.*

LEMMA 6. *The product of a finite number of neutral congruences on a modular lattice L is a neutral congruence.*

Proof. Let θ_1, θ_2 be two neutral congruences on L. Let N_1, N_2 be the corresponding neutral ideals. Then $N_1 \cap N_2$ is a neutral ideal (Lemma 3). Let θ be the neutral congruence corresponding to $N_1 \cap N_2$. Then $\theta = \theta_1 \cap \theta_2$. For let

$$a \equiv b(\theta) \Rightarrow a+x = b+x \quad \text{for} \quad x \in N_1 \wedge N_2.$$

Therefore $x \in N_1$ and $x \in N_2$. Hence $a \equiv b(\theta_1)$ and $a \equiv b(\theta_2)$. Hence $\theta \subseteq \theta_1 \wedge \theta_2$.

Next let $a \equiv b(\theta_1)$ and $a \equiv b(\theta_2)$. Thus $a+x = b+x$ for some $x \in N_1$ and $a+y = b+y$ for some $y \in N_2$. Therefore

$$\begin{aligned} (a+x)(a+y) &= (b+x)(b+y), \\ a+x\cdot(a+y) &= b+x\cdot(b+y) \quad (L \text{ is modular}). \end{aligned}$$

Now

$$\begin{aligned} a+x\cdot y &\leqslant a+x\cdot(a+y) \\ &\leqslant b+x\cdot(b+y) \\ &\leqslant b+x\cdot(0+x)(y+b) \\ &\leqslant b+x\cdot(y_1+b\cdot x) \quad (\text{for some } y_1 \in N_2) \\ &\leqslant b+x\cdot y_1+b\cdot x \quad (L \text{ is modular}) \\ &\leqslant b+x\cdot y_1. \end{aligned}$$

That is, $a+z \leqslant b+z_1$ for $z, z_1 \in N$.

Similarly $b+z \leqslant a+z_2$ for $z, z_2 \in N$. Hence $a+z+z_1+z_2 = b+z+z_1+z_2$, and $z+z_1+z_2 \in N$. Therefore $a \equiv b(\theta)$; that is, $\theta_1 \cap \theta_2 \subseteq \theta$. Hence $\theta = \theta_1 \cap \theta_2$.

Therefore the product of two and hence of a finite number of neutral congruences of a modular lattice L is a neutral congruence.

Remark. *When L is a simple modular lattice with* $(0, 1)$, *no element of L is neutral* (*except* 0 *and* 1).

Proof. Let L be a simple lattice. Then there are only two congruences and both of them are neutral. Therefore L_1 (the lattice of congruences on L) is simple.

Further, the number of neutral elements of $L \leqslant$ the number of number of neutral congruences on L, i. e. 2 in this case (Corollary 2 of Theorem 2). But any lattice has at least two neutral elements, viz., the zero and one of the lattice. Therefore L has no neutral elements excepting 0 and 1.

The converse is not necessarily true; that is, when L has only 0 and 1 as neutral elements, L need not be simple. This is shown in figure 2.

4. In this section we give certain examples to prove certain negative statements.

Example 1. The kernel of every congruence on a lattice L (even when L is modular) need not necessarily be neutral.

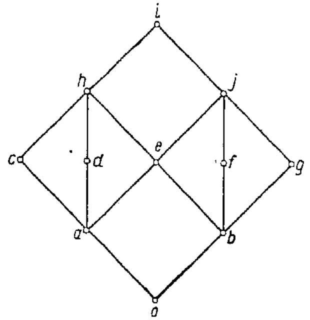
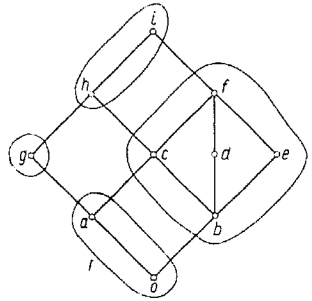


Fig. 2



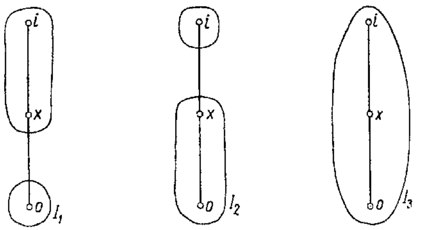
θ on L

Fig. 3

Let θ be a congruence on the lattice of fig. 3. I is the zero class of the congruence θ on L. I is not a neutral ideal, for though $f \leqslant (a+d)(a+e)$

$$f \nleqslant z + d \cdot e \quad \text{for all} \quad z \in I.$$

Example 2. If θ_1 and θ_2 are two congruences on a lattice L such that the kernel of θ_1 and the kernel of θ_2 are neutral ideals; then the kernel of $\theta_1 \cup \theta_2$ need not be equal to (the kernel of θ_1) $\cup$ (the kernel of θ_2).

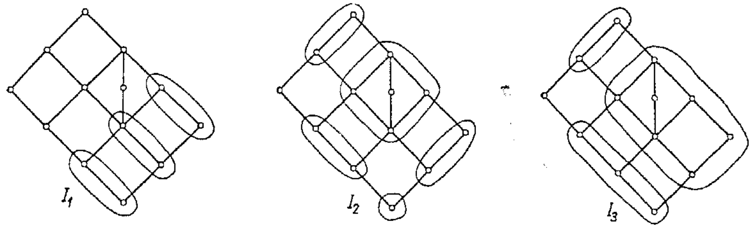


θ_1 on L $\quad$ θ_2 on L $\quad$ $\theta_1 \cup \theta_2$ on L

Fig. 4

For, θ_1 is a congruence on L with I_1 as its kernel and θ_2 a second congruence with I_2 as kernel. But the kernel of $\theta_1 \cup \theta_2$ is equal to the whole of L, whereas the (kernel of θ_1) $\cup$ (kernel of θ_2) $= I_2$.

Example 3. If θ_1 and θ_2 are two congruences on a lattice L such that the kernel of θ_1 and the kernel of θ_2 are neutral ideals, the kernel of $\theta_1 \cup \theta_2$ need not be neutral.



θ_1 on L $\quad$ θ_2 on L $\quad$ $\theta_1 \cup \theta_2$ on L

Fig. 5

I_1 and I_2 are neutral ideals on L and are the kernels of two congruences θ_1 and θ_2 on L. But the kernel I_3 of the congruence $\theta_1 \cup \theta_2$ is not a neutral ideal.

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