Independence in algebras of sets and Boolean algebras

by

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The following notion of *set-theoretical independence* has been introduced by G. Fichtenholz and L. Kantorovitch [4], p. 78: the sets $E_1, \ldots, E_n$ are called *independent* if none of their atoms (for the definition of atom see below § 2) is empty. This notion, as well as some of its variants, has interesting applications in measure theory, probability theory, and functional analysis and has been treated by several authors. (See e. g. Fichtenholz and Kantorovitch [4], Tarski [15], p. 53, Marczewski [6] and [7], Banach [1], Sikorski [11] and [12].)

A related condition was considered by S. Mazurkiewicz ([9], p. 86): none of the sets $E_1, \ldots, E_n$ is contained in the union of the remaining ones. This condition is weaker than the preceding one.

A dual condition to that of Mazurkiewicz is the following: none of sets $E_1, \ldots, E_n$ contains the intersection of the remaining ones. This is of course, the set-theoretical formulation of *logical independence* (independence of axioms, independence of conditions, independence of equations, etc.).

The principal aim of this paper is to prove that the notions listed above fall under the same algebraic scheme, presented in § 1. The corresponding theorems are: 4 (i), 5 (i) and 5 (iii). The main results presented here have been given without proof in my paper [8], which includes other examples, from various branches of mathematics, of the notions called independence, which also fall under the scheme of § 1 [1].

1. Independence in abstract algebras. Let $\mathfrak{A} = (A; \boldsymbol{F})$ be an *abstract algebra* (see, e. g., Birkhoff [2], p. vii), i. e. a set A of *elements* and a class $\boldsymbol{F}$ of *fundamental operations*. Every $f \in \boldsymbol{F}$ is a function of several variables which associates with each sequence $(x_1, \ldots, x_n)$ of elements of A an element $f(x_1, \ldots, x_n) \in A$. We denote by $A^{(n)}(\mathfrak{A})$, or briefly by $A^{(n)}$, the class of all *algebraic operations* (cf. McKinsey and

[1] For other investigations concerning that scheme see Świerczkowski [13] and [14].

Tarski [10], p. 160) of n variables, i. e. the smallest class of functions containing n "identity functions":

$$e_k^{(n)}(x_1, \ldots, x_n) = x_k, \quad k = 1, 2, \ldots, n,$$

(defined for $x_j \in A$) and closed under the composition with the fundamental operations. The set of all values of all constant algebraic operations (called *algebraic constants*) will be denoted by $\boldsymbol{A}^{(0)}(\mathfrak{A})$ or $\boldsymbol{A}^{(0)}$.

A set $N \subset A$ will be called a *set of independent elements* (²), whenever, for each sequence $a_1, \ldots, a_n$ of different elements of N and each pair $f, g \in \boldsymbol{A}^{(n)}$, if $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ then f and g are identical in A.

Every fundamental operation f of k variables may be treated as a function which associates with k algebraic operation $f_1, \ldots, f_k \in \boldsymbol{A}^{(n)}$ an algebraic operation $g = f(f_1, \ldots, f_k)$ defined by the formula

$$g(x_1, \ldots, x_n) = f\big(f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)\big).$$

Consequently the class of all algebraic operations of n variables forms a new algebra: $(\boldsymbol{A}^{(n)}; \boldsymbol{F})$.

2. Atoms and their unions. Let $\boldsymbol{B}(X)$, or briefly $\boldsymbol{B}$, denote the class of all subsets of a non-empty set X (or, more generally, the class of all elements of a non-trivial Boolean algebra, i. e. consisting of at least two elements).

If X is fixed, it will be called a *space* and denoted by 1. Further, let us denote by 0 the empty set, and by $\cup$, $\cap$, $'$, $\setminus$, and $\dot{-}$ the elementary set operations: union, intersection, complementation, subtraction and symmetric subtraction. In accordance to the preceding paragraph, the same symbols will denote operations on set-valued functions.

The following notation will be very useful in the sequel:

$$(1) \qquad E^0 = E' \quad \text{and} \quad E^1 = E \quad \text{for} \quad E \in \boldsymbol{B}.$$

In the whole paper the letters i and j will be used as indices running over 0 and 1 only. Thus we have

$$(2) \qquad i^j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

The set

$$(3) \qquad A_{(i_1, \ldots, i_n)}(E_1, \ldots, E_n) = \bigcap_{k=1}^{n} E_k^{i_k}$$

(²) The notion of independence can be considered as a case of the notion of free algebra, defined by G. Birkhoff. Cf. Birkhoff [2], p. viii and McKinsey-Tarski [10], p. 170.

will be called the *atom* of the sequence $E_1, \ldots, E_n$, corresponding to the sequence of indices $i_1, \ldots, i_n$.

Obviously

$$(4) \qquad A_{(i_1, \ldots, i_n)}(E_1, \ldots, E_n) \cap A_{(j_1, \ldots, j_n)}(E_1, \ldots, E_n) = 0 \quad \text{if} \quad (i_1, \ldots, i_n) \neq (j_1, \ldots, j_n).$$

For a fixed sequence $i_1, \ldots, i_n$, formula (3) determines a function $A_{(i_1, \ldots, i_n)}$ of n set-variables $E_1, \ldots, E_n$. It follows from (2) that

$$(5) \qquad A_{(i_1, \ldots, i_n)}(j_1, \ldots, j_n) = \begin{cases} 0 & \text{if} \quad (i_1, \ldots, i_n) \neq (j_1, \ldots, j_n), \\ 1 & \text{if} \quad (i_1, \ldots, i_n) = (j_1, \ldots, j_n). \end{cases}$$

Let us denote by T^n the set of all sequences $(i_1, \ldots, i_n)$ and let us set for every non-empty $J \subset T^n$:

$$(6) \qquad A_J(E_1, \ldots, E_n) = \bigcup_{(i_1, \ldots, i_n) \in J} A_{(i_1, \ldots, i_n)}(E_1, \ldots, E_n).$$

We may complete this definition in the case of the empty set, putting

$$(7) \qquad A_0(E_1, \ldots, E_n) = 0.$$

Definition (6) implies

$$(8) \qquad A_{T^n}(E_1, \ldots, E_n) = 1.$$

It follows easily from (5), (6) and (7) that

$$(9) \qquad A_J(j_1, \ldots, j_n) = \begin{cases} 0 & \text{if} \quad (j_1, \ldots, j_n) \notin J, \\ 1 & \text{if} \quad (j_1, \ldots, j_n) \in J. \end{cases}$$

Among the functions A_J there are all identity functions: denoting by $I_k^{(n)}$ the set of all sequences $(i_1, \ldots, i_n)$ with $i_k = 1$, we have

$$(10) \qquad A_{I_k^{(n)}} = e_k^n.$$

Definitions (6) and (7) and formulas (4) and (8) imply for every $J_1 \subset T^n$ and $J_2 \subset T^n$:

$$(11) \qquad \begin{aligned} A_{J_1} \cup A_{J_2} &= A_{J_1 \cup J_2}, \\ A_{J_1} \cap A_{J_2} &= A_{J_1 \cap J_2}, \\ A_{J_1} \setminus A_{J_2} &= A_{J_1 \setminus J_2}, \\ A_{J_1} \dot{-} A_{J_2} &= A_{J_1 \dot{-} J_2}, \\ (A_J)' &= A_{J'}, \end{aligned}$$

where obviously J' denotes $T^n \setminus J$.

Let us notice finally that

(12) $A_{J_1} = A_{J_2}$ if and only if $J_1 = J_2$.

In fact, in view of (9), the relation $A_J = 0$ is equivalent to the relation $J = 0$. Consequently, the relation $A_{J_1} = A_{J_2}$ is equivalent successively to the following ones: $A_{J_1} \dot{-} A_{J_2} = 0$, $A_{J_1 \dot{-} J_2} = 0$, $J_1 \dot{-} J_2 = 0$, $J_1 = J_2$.

3. Operations in algebras of sets. We consider the following five algebras:

$$\mathfrak{B} = (\boldsymbol{B}; \cup, '),$$
$$\mathfrak{B}_1 = (\boldsymbol{B}; \cup, \setminus), \qquad \mathfrak{B}_2 = (\boldsymbol{B}; \cup, \cap),$$
$$\mathfrak{B}_3 = (\boldsymbol{B}; \cup), \qquad \mathfrak{B}_4 = (\boldsymbol{B}; \cap).$$

(i) 0 *and* 1 *are algebraic constants in* $\mathfrak{B}$ [3]. *Intersection and subtraction are algebraic operations in* $\mathfrak{B}$.

In fact,

$$0 = (E \cup E')', \qquad 1 = E \cup E',$$
$$E \cap F = (E' \cup F')', \qquad E \setminus F = (E' \cup F)'.$$

(ii) 0 *is an algebraic constant in* $\mathfrak{B}_1$ [4]. *Intersection is an algebraic operation in* $\mathfrak{B}_1$.

In fact,

$$0 = E \setminus E, \qquad E \cap F = E \setminus (E \setminus F).$$

Propositions (i) and (ii) imply that

(iii) *Every algebraic operation in* $\mathfrak{B}_3$ *or in* $\mathfrak{B}_4$ *is algebraic in* $\mathfrak{B}_2$. *Every operation algebraic in* $\mathfrak{B}_2$ *is algebraic in* $\mathfrak{B}_1$. *Every operation algebraic in* $\mathfrak{B}_1$ *is algebraic in* $\mathfrak{B}$.

The following well-known theorem (cf. e. g. Birkhoff-Mc Lane [3], p. 322) concerns the algebraic operations in $\mathfrak{B}$, called by several authors *Boolean polynomials*:

(iv) *A function is an algebraic operation in* $\mathfrak{B}$ *if and only if it is of the form* A_J.

It follows from (i) that every function A_J is algebraic in $\mathfrak{B}$. To prove the converse relation it suffices to remark that the class of all functions of the form A_J with $J \subset T^n$ contains all identity functions of n variables (see (10)) and that it is closed under union and complementation (in view of (11)).

[3] It follows from (9) and 3 (iv) that there exist no other algebraic constants in $\mathfrak{B}$.
[4] It follows from (9) and 3 (v) that there exist no other algebraic constants in $\mathfrak{B}_1$.

(v) *A function of* n *set-variables is an algebraic operation in* $\mathfrak{B}_1$ *if and only if it is of the form* A_J *where* J *does not contain the* n*-tuple* $(0, 0, \ldots, 0)$.

Let us consider $A_{(i_1, \ldots, i_n)}$ where not all the indices vanish. Then we have

(*) $$i_{k_1} = i_{k_2} = \ldots = i_{k_m} = 1, \qquad i_{l_1} = i_{l_2} = \ldots = i_{l_{n-m}} = 0$$

where k_s and l_s are two increasing sequences, the first of which is non-empty (whereas the second may be empty). Thus

(**) $$A_{(i_1, \ldots, i_n)}(E_1, \ldots, E_n) = \bigcap_{s=1}^{n} E_s^{i_s} = \bigcap_{s=1}^{m} E_{k_s} \cap \bigcap_{s=1}^{n-m} E'_{l_s}$$
$$= \bigcap_{s=1}^{n} E_{k_s} \cap \Big(\bigcup_{s=1}^{n-m} E_{l_s}\Big)' = \bigcap_{s=1}^{m} E_{k_s} \setminus \bigcup_{s=1}^{n-m} E_{l_s},$$

whence, in view of (ii), $A_{(i_1, \ldots, i_n)}$ is an algebraic operation in $\mathfrak{B}_2$. Consequently, if

$$(0, 0, \ldots, 0) \notin J \neq 0,$$

then the operation A_J is algebraic. In the case of $J = 0$ the operation A_J is also algebraic, in view of (7) and (ii).

In order to prove the converse implication it suffices to prove that the class of all operations A_J with $(0, 0, \ldots, 0) \notin J$ contains all identity functions (see (10)) and that it is closed under union and subtraction (in view of (11)).

(vi) *Every algebraic operation in* $\mathfrak{B}_2$ *is of the form* A_J, *where* $(0, 0, \ldots, 0) \notin J$ *and* $(1, 1, \ldots, 1) \in J$.

It follows from (iii), (iv) and (v), that every algebraic operation in $\mathfrak{B}_2$ is of the form A_J with $(0, 0, \ldots, 0) \notin J$.

Since the class of all functions f of n variables fulfilling the condition

$$f(E_1, \ldots, E_n) \supset \bigcap_{k=1}^{n} E_k$$

contains all identity functions of n variables and is closed under union and intersection, every algebraic operation in $\mathfrak{B}_2$ satisfies this condition.

Consequently, if A_J is an algebraic operation in $\mathfrak{B}_2$, then

$$A_J(1, 1, \ldots, 1) = 1,$$

whence, in view of (9), $(1, 1, \ldots, 1) \in J$, q. e. d.

Theorem (vi) does not give a complete characterization of algebraic operations in $\mathfrak{B}_2$. We formulate such a characterization (without proof), but it will be not used in the sequel:

(vi′) *A function is an algebraic operation in* $\mathfrak{B}_2$ *if and only if it is of the form* A_J, *where* J *is a non-empty set which does not contain* $(0,\dots,0)$ *and satisfies the following condition: if* $(i_1,\dots,i_{k-1},0,i_k,\dots,i_n)\in J$, *then* $(i_1,\dots,i_{k-1},1,i_{k+1},\dots,i_n)\in J$.

The following lemma concerning the algebraic operations in $\mathfrak{B}_2$ will be used in the sequel.

(vii) *If* $(i_1,\dots,i_n)$ *is a sequence of non-identical terms, then the function* $A_{(i_1,\dots,i_n)}$ *is the symmetric difference of two different algebraic operations in* $\mathfrak{B}_2$.

The assumption implies the existence of two non-empty increasing sequences k_s and l_s satisfying $(*)$. Modifying $(**)$, we obtain

$$A_{(i_1,\dots,i_n)}(E_1,\dots,E_n)=\bigcap_{s=1}^{m}E_{k_s}\dot{-}\Big(\bigcap_{s=1}^{m}E_{k_s}\cap\bigcup_{s=1}^{n-m}E_{l_s}\Big).$$

Putting for every sequence $E_1,\dots,E_n$

$$f(E_1,\dots,E_n)=\bigcap_{s=1}^{m}E_{k_s}\quad\text{and}\quad g(E_1,\dots,E_n)=\bigcap_{s=1}^{m}E_{k_s}\cap\bigcup_{s=1}^{n-m}E_{l_s},$$

we obtain

$$A_{(i_1,\dots,i_n)}(E_1,\dots,E_n)=f(E_1,\dots,E_n)\dot{-}g(E_1,\dots,E_n)$$

where, of course, f and g are two different algebraic operations in $\mathfrak{B}_2$.

4. Independence in algebras of sets: $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$.

(i) *The sets* $F_1,\dots,F_n$ *are independent in* $\mathfrak{B}$ *if and only if every atom of this sequence is non-empty* ([5]).

Let us suppose that an atom of $F_1,\dots,F_n$ is void:

$$(+)\qquad A_{(i_1,\dots,i_n)}(F_1,\dots,F_n)=0.$$

By 3 (iv), $A_{(i_1,\dots,i_n)}$ and 0 are algebraic operations in $\mathfrak{B}$. They are different, because, by (5),

$$A_{(i_1,\dots,i_n)}(i_1,\dots,i_n)=1.$$

Thus, formula (+) expresses the equality for $F_1,\dots,F_n$ of two different algebraic operations in $\mathfrak{B}$, or, in other words, the dependence of $F_1,\dots,F_n$ in $\mathfrak{B}$.

([5]) Let us notice that the so-called denumerable independence of sets (cf. e. g. Marczewski [7], p. 123) could also be treated as the independence in an algebra (with operations of infinitely many variables).

Let us suppose, conversely, that no atom of $F_1,\dots,F_n$ is void. If we have for two algebraic operations A_{J_1} and A_{J_2}:

$$A_{J_1}(F_1,\dots,F_n)=A_{J_2}(F_1,\dots,F_n),$$

then

$$A_{J_1}(F_1,\dots,F_n)\dot{-}A_{J_2}(F_1,\dots,F_n)=0$$

and, by (11),

$$A_{J_1\dot{-}J_2}(F_1,\dots,F_n)=0.$$

Since, for every non-void J,

$$A_J(F_1,\dots,F_n)\neq 0,$$

we obtain $J_1\dot{-}J_2=0$, or else $J_1=J_2$. Thus $F_1,\dots,F_n$ are independent in $\mathfrak{B}$, q. e. d.

(ii) *The sets* $F_1,\dots,F_n$ *are independent in* $\mathfrak{B}_1$ *if and only if each atom* $A_{(i_1,\dots,i_n)}(F_1,\dots,F_n)$ *with indices* i_k *which are not all vanishing is non-empty.*

Let us suppose that there exists a sequence $i_1,\dots,i_n$, with terms which are not all vanishing, such that (+).

As in the preceding proof, we may treat $A_{(i_1,\dots,i_n)}$ and 0 as two different algebraic functions in $\mathfrak{B}_1$ (in view of 3 (v)). Thus $F_1,\dots,F_n$ are dependent in $\mathfrak{B}_1$.

The second part of proof is also analogous to the case of the algebra $\mathfrak{B}$. In order to apply the same argument to $\mathfrak{B}_1$ it suffices to remark that if $(0,0,\dots,0)\notin J_1$ and, simultaneously, $(0,0,\dots,0)\notin J_2$, then $(0,0,\dots,0)\notin J_1\dot{-}J_2$.

(iii) *The sets* $F_1,\dots,F_n$ *are independent in* $\mathfrak{B}_2$ *if and only if each atom* $A_{(i_1,\dots,i_n)}(F_1,\dots,F_n)$ *with indices that are not all identical is non-empty.*

Let us suppose that there exists a sequence $i_1,\dots,i_n$, with indices that are not all identical, such that (+). By lemma 3 (vii) there exist two different operations f and g which are algebraic in $\mathfrak{B}_2$ and such that $A_{(i_1,\dots,i_n)}=f\dot{-}g$. Therefore

$$f(F_1,\dots,F_n)=g(F_1,\dots,F_n),$$

whence $F_1,\dots,F_n$ are dependent in $\mathfrak{B}_2$.

The second part of the proof is analogous to the cases of $\mathfrak{B}$ and $\mathfrak{B}_1$. In order to apply the same argument to $\mathfrak{B}_2$ it suffices to remark that if $(0,\dots,0)\notin J_k$ and $(1,\dots,1)\in J_k$ for $k=1,2$, then $(0,\dots,0)\notin J_1\dot{-}J_2$ and $(1,\dots,1)\notin J_1\dot{-}J_2$.

(iv) *In the class* $\boldsymbol{B}$ *of all subsets of a space there exist* n *sets which are independent in* $\mathfrak{B}$, $\mathfrak{B}_1$ *or* $\mathfrak{B}_2$ *if and only if the space contains at least* 2^n, 2^n-1 *or* 2^n-2 *elements respectively.*

Let $F_1, \ldots, F_n$ be independent sets in $\mathfrak{B}$. It follows from (i) that all 2^n atoms of this sequence are non-empty. Since the atoms are disjoint, the space contains at least 2^n elements. Analogously we infer from (ii) or (iii) that if $F_1, \ldots, F_n$ are independent in $\mathfrak{B}_1$ or $\mathfrak{B}_2$, then the space contains at most 2^n-1 or 2^n-2 elements.

In order to prove the converse implications, let us use the notation of § 2 and put

$$X = T_n, \quad Y = T_n \setminus \{(0, 0, \ldots, 0)\}, \quad Z = T_n \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}.$$

Therefore X, Y and Z have 2^n, 2^n-1 and 2^n-2 elements respectively and it follows directly from the equality

$$A_{(i_1, \ldots, i_n)}(I_1^{(n)}, \ldots, I_n^{(n)}) = \{(i_1, \ldots, i_n)\}$$

that the sets

$$I_1^{(n)}, I_2^{(n)}, \ldots, I_n^{(n)},$$

$$I_1^{(n)} \cap Y, I_2^{(n)} \cap Y, \ldots, I_n^{(n)} \cap Y,$$

$$I_1^{(n)} \cap Z, I_2^{(n)} \cap Z, \ldots, I_n^{(n)} \cap Z$$

are independent in the algebras

$$\mathfrak{B} = (\boldsymbol{B}(X), \cup, '), \quad \mathfrak{B}_1 = (\boldsymbol{B}(Y), \cup, \setminus), \quad \text{and} \quad \mathfrak{B}_2 = (\boldsymbol{B}(Z), \cup, \cap)$$

respectively.

Theorems 3 (iii) and 4 (iv) imply that

(v) *The independence in $\mathfrak{B}$ implies the independence in $\mathfrak{B}_1$ and the independence in $\mathfrak{B}_1$ implies the independence in $\mathfrak{B}_2$, but not conversely.*

Nevertheless

(vi) *For an infinite class of sets the notions of independence in $\mathfrak{B}$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent.*

In view of (v) it is to prove that if an infinite class $\boldsymbol{N}$ is a class of sets independent in $\mathfrak{B}_2$, then these sets are independent in $\mathfrak{B}$. Let us suppose that $F_1, F_2, \ldots, F_n, F_{n+1}$ are different elements of $\boldsymbol{N}$ and let $i_1, i_2, \ldots, i_n$ be an arbitrary sequence of 0's and 1's. Putting either $i_{n+1} = 0$ or $i_{n+1} = 1$ we obviously obtain a sequence $i_1, i_2, \ldots, i_n, i_{n+1}$ of terms that are not all identical. Since $\boldsymbol{N}$ is a class of sets independent in $\mathfrak{B}_2$, we have by (iii)

$$A_{(i_1, \ldots, i_n, i_{n+1})}(F_1, \ldots, F_n, F_{n+1}) \neq 0$$

and since of course

$$A_{(i_1, \ldots, i_n)}(F_1, \ldots, F_n) \supset A_{(i_1, \ldots, i_n, i_{n+1})}(F_1, \ldots, F_n, F_{n+1}),$$

the atom $A_{(i_1, \ldots, i_n)}(F_1, \ldots, F_n)$ is non void.

The sequence $i_1, \ldots, i_n$ has been arbitrary; therefore, in view of (i), we obtain the independence in $\mathfrak{B}$. The theorem is thus proved.

Formulas (11) and (12) prove that the correspondence $J \to A_J$ for J running over $\boldsymbol{B}(T^n)$ is an isomorphism with respect to all elementary set operations. Consequently

(vii) *The algebraic operations $A_{J_1}, \ldots, A_{J_m}$ in $\mathfrak{B}$ are independent in the algebra $(\boldsymbol{A}^{(n)}(\mathfrak{B}); \cup, ')$ if and only if the sets $J_1, \ldots, J_m$ are independent in $(\boldsymbol{B}(T^n); \cup, ')$.*

Analogous theorems are valid for the algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

5. Independence in algebras of sets: $\mathfrak{B}_3$ and $\mathfrak{B}_4$.

(i) *The sets $F_1, \ldots, F_n$ are independent in $\mathfrak{B}_3$ if and only if none of them is contained in the union of the remaining ones.*

If the set F_k is contained in the union of the remaining ones, then

$$F_1 \cup \ldots \cup F_n = F_1 \cup \ldots \cup F_{k-1} \cup F_{k+1} \cup \ldots \cup F_n.$$

So we see that two different operations algebraic in $\mathfrak{B}_3$ are equal for $F_1, \ldots, F_n$ and consequently $F_1, \ldots, F_n$ are dependent in $\mathfrak{B}_3$.

In order to prove the converse implication, let us remark that every algebraic operation of n variables in $\mathfrak{B}_3$ is of the form

$$f(E_1, \ldots, E_n) = E_{l_1} \cup \ldots \cup E_{l_m} \quad \text{where} \quad 1 \leqslant l_1 < \ldots l_m \leqslant n.$$

Thus, if $F_1, \ldots, F_n$ are dependent in $\mathfrak{B}_3$, then there are two different sequences $1 \leqslant l_1 < \ldots < l_m \leqslant n$ and $1 \leqslant l_1' < \ldots < l_{m'}' \leqslant n$ such that

$$F_{l_1} \cup \ldots \cup F_{l_m} = F_{l_1'} \cup \ldots \cup F_{l_{m'}'}.$$

There exists of course an index k which appears on one and only one side in this equality. If it appears on the left, then we have

$$F_k \subset F_{l_1} \cup \ldots \cup F_{l_m} = F_{l_1'} \cup \ldots \cup F_{l_{m'}'} \subset F_1 \cup \ldots \cup F_{k-1} \cup F_{k+1} \cup \ldots \cup F_n.$$

Of course,

(ii) *The sets $F_1, \ldots, F_n$ are independent in $\mathfrak{B}_4$ if and only if their complements are independent in $\mathfrak{B}_3$,*

whence

(iii) *The sets $F_1, \ldots, F_n$ are independent in $\mathfrak{B}_4$ if and only if none of them contains the intersection of the remaining ones.*

Now we shall prove that

(iv) *In the class $\boldsymbol{B}$ of all subsets of a space there exist n independent sets in $\mathfrak{B}_3$ and $\mathfrak{B}_4$ if and only if the space contains at least n elements.*

If there exist n independent sets in $\mathfrak{B}_3$ then, in view of (i), the space contains at least n points. In view of (ii) the existence of n independent sets in $\mathfrak{B}_4$ implies the same conclusion.

In order to prove the converse implications it suffices to state that n different one-point sets form a class of sets independent in $\mathfrak{B}_3$ and their complements — a class of sets independent in $\mathfrak{B}_4$.

Let us also note the following relations, easy to prove:

(v) *The independence in $\mathfrak{B}_2$ implies the independence in $\mathfrak{B}_3$ and in $\mathfrak{B}_4$ but not conversely. Neither of the independences in $\mathfrak{B}_3$ and $\mathfrak{B}_4$ implies the other one.*

6. Some properties of linear independence. Linear independence (e. g. in a vector space) possesses some properties which have served several authors as the definition of "abstract linear independence". It is interesting that the algebraic independence of numbers falls under this notion.

The condition considered by H. Whitney [16] is the following:

(W) If A is a set containing n independent elements and B — a set containing $n+1$ independent elements, then there exists an element $b \in B \setminus A$ such that $A \cup \{b\}$ is a set of independent elements.

The condition considered by O. Haupt, G. Nöbeling and C. Pauc [5] is the following:

(H) If a belongs to a set A of independent elements, and neither of the sets $A \cup \{b\}$ and $A \cup \{c\}$ is a set of independent elements, then $(A \setminus \{a\}) \cup \{b, c\}$ is not a set of independent elements.

We shall prove that the above conditions are not fulfilled by the independence in $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ and $\mathfrak{B}_4$.

Let us consider the "space" consisting of four elements: $X = \{t, u, v, w\}$, and three sets: $U = \{u, v, w\}$, $V = \{u, v\}$, $W = \{u, w\}$. Then, in the algebra $\mathfrak{B} = (B(X), \cup, ')$ the class $\{U\}$ consists of one independent set, $\{V, W\}$ is a class of two independent sets, but $\{U, V\}$ and $\{U, W\}$ are classes of dependent sets since $U \supset V$ and $U \supset W$.

Therefore, the independence in $\mathfrak{B}$ does not fulfil condition (W).

The same example shows that the independence in $\mathfrak{B}$ does not fulfil condition (H).

By the same argument the notions of independence in $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ and $\mathfrak{B}_4$ do not fulfil conditions (W) and (H).

References

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