Distributivity and representability

by

R. Sikorski (Warszawa)

The main theorem of this paper is theorem 3.2 on the representation of Boolean algebras as factor algebras $\mathfrak{F}/\mathfrak{J}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets and $\mathfrak{J}$ is an $\mathfrak{m}$-ideal (1). Some necessary and sufficient conditions for the existence of such a representation were given by Chang [1], [2] (2). A sufficient condition formulated as a distributivity property was given by Smith [9], who has proved that his condition is also necessary if $\mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}$ for all cardinals $\mathfrak{n} < \mathfrak{m}$. The necessity of Smith's condition for regular cardinals is equivalent to the generalized continuum hypothesis.

In this paper I shall give a condition both necessary and sufficient (see 3.2 (r_1), (r_2)) which is a simple modification of Smith's condition but omits the additional hypothesis on $\mathfrak{m}$. This condition has a very simple topological interpretation (3) (see 3.2 (r_3), (r_4)). For completeness I also quote Chang's conditions (3.2 (r_5), (r_5')). Combining (r_2) with Chang's condition (r_5') I obtain a new condition (r_2'). Conditions (r_3') and (r_4') are topological interpretations of (r_2').

The fundamental conditions (r_1), (r_2) have the character of a distributivity property. To underline the analogy between the representation problem and the distributivity, § 1 and § 2 on distributivity have been added (some of the theorems in §§ 1, 2 are known).

Recently Kelley [4] has given a simple topological condition for a Boolean σ-algebra satisfying the σ-chain condition to be weakly σ-distributive (see 4.2) (4). To explain the topological character of the $\mathfrak{m}$-chain condition, theorem 4.1 has been added.

---

(1) We use the abbrevations: $\mathfrak{m}$-algebra, $\mathfrak{m}$-field, $\mathfrak{m}$-ideal, $\mathfrak{m}$-filter for: $\mathfrak{m}$-complete algebra, $\mathfrak{m}$-complete field, $\mathfrak{m}$-complete ideal, $\mathfrak{m}$-complete filter respectively. $\mathfrak{m}$ always denotes an infinite cardinal.

(2) A simple proof of the necessity and sufficiency of Chang's [1] condition (see 3.2 (r_5)) has been communicated to me by A. Białynicki-Birula before Chang's [2] proof was published. The proof of the implication $(\mathrm{r}_2) \to (\mathrm{r}_2')$ on p. 100-101 is a slight modification of a part of Białynicki's proof.

(3) During the print of this paper I observed that the topological interpretation was also given by Pierce [6].

(4) σ is the cardinal of the set of all integers.

In the main theorems 1.1, 2.1, 3.2, 4.1 no hypotheses of a higher completeness of Boolean algebras are necessary. In theorem 3.2 we assume the following definition of $\mathfrak{m}$-filters ($\mathfrak{m}$-ideals) in arbitrary (not necessarily $\mathfrak{m}$-complete) Boolean algebras:

A filter $\mathfrak{J}$ (an ideal $\mathfrak{I}$) is said to be an $\mathfrak{m}$-*filter* (an $\mathfrak{m}$-*ideal*) provided, for every indexed set $A_t \in \mathfrak{J}$ ($A_t \in \mathfrak{I}$), $t \in T$, $\overline{\overline{T}} \leqslant \mathfrak{m}$, there exists an element $A \in \mathfrak{J}$ ($A \in \mathfrak{I}$) such that $A \subset A_t$ ($A_t \subset A$) for every $t \in T$.

To express topological interpretations of some distributivity properties, it is convenient to introduce the notions of $\mathfrak{m}$-closed and $\mathfrak{m}$-open sets, of $\mathfrak{m}$-nowhere dense sets, of sets of $\mathfrak{m}$-category and of closed sets of character $\mathfrak{m}$. The definitions of these notions are given below (see p. 93).

Terminology and notation. S and T always denote some non-empty sets. S^T denotes the set of all mappings f from T into S.

Boolean algebras are denoted by letters $\mathfrak{A}$, $\mathfrak{B}$ (with indexes, if necessary), fields of sets by $\mathfrak{F}$, ideals by $\mathfrak{I}$ and filters by $\mathfrak{J}$. Elements of Boolean algebras or fields of sets are denoted by $A, B, \ldots$

We use the symbols $\cup$, $\bigcup$, $\cap$, $\bigcap$ both for set-theoretical operations and for the corresponding Boolean operations. Sometimes we write $\bigcup^{\mathfrak{A}}$, $\bigcap^{\mathfrak{A}}$ instead of $\bigcup$ and $\bigcap$ respectively in order to underline then the (infinite) Boolean joins and meets under consideration are taken relative to the Boolean algebra $\mathfrak{A}$. The sign $-$ denotes complementation and subtraction. The sign $\subset$ denotes both the set-theoretical inclusion and the Boolean ordering relation. The sign $\bigvee$ denotes the unit element of the Boolean algebra in question. The sign $\bigwedge$ denotes both the zero element of a Boolean algebra and the empty set. $[A]$ denotes the element (of a Boolean factor algebra $\mathfrak{A}/\mathfrak{J}$ or $\mathfrak{A}/\mathfrak{I}$) determined by $A \in \mathfrak{A}$.

A filter $\mathfrak{J}$ (an ideal $\mathfrak{I}$) in a Boolean algebra $\mathfrak{A}$ is said to *preserve* a given join $A = \bigcup^{\mathfrak{A}}_{t\in T} A_t$ or meet $B = \bigcap^{\mathfrak{A}}_{t\in T} B_t$ provided $[A] = \bigcup_{t\in T}[A_t]$ or $[B] = \bigcap_{t\in T}[B_t]$ in $\mathfrak{A}/\mathfrak{J}$ (in $\mathfrak{A}/\mathfrak{I}$).

An indexed set $\{A_t\}_{t\in T}$ of elements of a Boolean algebra $\mathfrak{A}$ is said to be an $\mathfrak{m}$-*indexed set* provided $\overline{\overline{T}} \leqslant \mathfrak{m}$. The same terminology is used for doubly indexed sets: $\{A_{t,s}\}_{t\in T, s\in S}$ is said to be an $\mathfrak{m}$-*indexed set* if $\overline{\overline{T}} \leqslant \mathfrak{m}$ and $\overline{\overline{S}} \leqslant \mathfrak{m}$.

A homomorphism (isomorphism) h of a Boolean algebra $\mathfrak{A}$ into another Boolean algebra $\mathfrak{A}'$ is said to be an $\mathfrak{m}$-*homomorphism* ($\mathfrak{m}$-*isomorphism*) *of* $\mathfrak{A}$ *into* $\mathfrak{A}'$ provided, for every $\mathfrak{m}$-indexed set $\{A_t\}_{t\in T}$ of elements in $\mathfrak{A}$, if $\bigcup^{\mathfrak{A}}_{t\in T} A_t$ exists, then $\bigcup^{\mathfrak{A}'}_{t\in T} h(A_t)$ also exists and

$$h\big(\textstyle\bigcup^{\mathfrak{A}}_{t\in S} A_t\big) = \bigcup^{\mathfrak{A}'}_{t\in T} h(A_t)\,.$$

By the de Morgan formulas, we obtain an equivalent definition by replacing everywhere $\bigcup$ by $\bigcap$. A necessary and sufficient condition for a homomorphism (an isomorphism) h to be an $\mathfrak{m}$-homomorphism (an $\mathfrak{m}$-isomorphism) is that

$$\textstyle\bigcap^{\mathfrak{A}}_{t\in T} A_t = \bigwedge \quad \text{imply} \quad \bigcap^{\mathfrak{A}'}_{t\in T} h(A_t) = \bigwedge$$

for every $\mathfrak{m}$-indexed set $\{A_t\}_{t\in T}$ of elements in $\mathfrak{A}$.

A subalgebra $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is said to be an $\mathfrak{m}$-*regular subalgebra* provided the identity mapping of $\mathfrak{B}$ into $\mathfrak{A}$ is an $\mathfrak{m}$-isomorphism (i. e. if, for every $\mathfrak{m}$-indexed set $\{A_t\}_{t\in T}$ of elements in $\mathfrak{B}$, $\bigcup^{\mathfrak{B}}_{t\in T} A_t$ exists, then $\bigcup^{\mathfrak{A}}_{t\in T} A_t$ exists also and

$$\textstyle\bigcup^{\mathfrak{A}}_{t\in T} A = \bigcup^{\mathfrak{B}}_{t\in T} A_t\,;$$

and the same holds for meets). For instance, if h is an $\mathfrak{m}$-isomorphism of $\mathfrak{A}$ into another Boolean algebra $\mathfrak{A}'$, then the set $h(\mathfrak{A})$ is an $\mathfrak{m}$-regular subalgebra of $\mathfrak{A}'$.

The *Stone space* X of a Boolean algebra $\mathfrak{A}$ is the set of all maximal filters in $\mathfrak{A}$. The mapping h_0:

$$h_0(A) = \text{the set of all } \mathfrak{J} \in X \text{ such that } A \in \mathfrak{J} \qquad (A \in \mathfrak{A})$$

is the *Stone isomorphism* of $\mathfrak{A}$ onto the field $\mathfrak{F}_0$ of both open and closed subsets of X.

A subset B of X is said to be $\mathfrak{m}$-*open* ($\mathfrak{m}$-*closed*) provided it is the union (the intersection) of at most $\mathfrak{m}$ sets in $\mathfrak{F}_0$.

A subset B of X is said to be $\mathfrak{m}$-*nowhere dense* provided it is a subset of a nowhere dense $\mathfrak{m}$-closed set. For instance, for any $\mathfrak{m}$-indexed set $\{A_t\}_{t\in T}$ of elements in $\mathfrak{A}$, if $A = \bigcup^{\mathfrak{A}}_{t\in T} A_t$, then the set

$$h_0(A) - \textstyle\bigcup_{t\in T} h_0(A_t)$$

(where $\bigcup$ denotes the set-theoretical union) is $\mathfrak{m}$-closed and nowhere dense, and therefore it is $\mathfrak{m}$-nowhere dense. Similarly, if $A = \bigcap^{\mathfrak{A}}_{t\in T} A_t$ ($\overline{\overline{T}} \leqslant \mathfrak{m}$), then the set

$$\textstyle\bigcap_{t\in T} h_0(A_t) - h_0(A)$$

is $\mathfrak{m}$-closed and nowhere dense, and therefore it is $\mathfrak{m}$-nowhere dense. Conversely, for every $\mathfrak{m}$-nowhere dense set B there exists an $\mathfrak{m}$-indexed set $\{A_t\}_{t\in T}$ of elements in $\mathfrak{A}$ such that

$$\textstyle\bigcap^{\mathfrak{A}}_{t\in T} A_t = \bigwedge \qquad \text{and} \qquad B \subset \bigcap_{t\in T} h_0(A_t)\,.$$

A subset B of X is said to be of the $\mathfrak{m}$-*category* if it is the union of at most $\mathfrak{m}$ sets $\mathfrak{m}$-nowhere dense in X.

A closed subset B of X is said to be *of the character* $\mathfrak{m}$ if for every $\mathfrak{m}$-indexed set $\{B_t\}_{t\in T}$ of sets in $\mathfrak{F}_0$, such that $B \subset B_t$ for every $t \in T$, the interior of the intersection of all B_t contains B. A closed set $B \subset X$ is of the character $\mathfrak{m}$ if and only if the class of all elements $A \in \mathfrak{A}$ such that $B \subset h_0(A)$ is an $\mathfrak{m}$-filter (see the definition on p. 92).

§ 1. The $\mathfrak{m}$-distributivity. A Boolean algebra $\mathfrak{A}$ is said to be $\mathfrak{m}$-*distributive* if

$$(1)\qquad \bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}=\bigcup_{f\in S^T}\bigcap_{t\in T}A_{t,f(t)}$$

for every $\mathfrak{m}$-indexed set $\{A_{t,s}\}_{t\in T,s\in S}$ of elements in $\mathfrak{A}$ such that

(2) all the joins $\bigcup_{s\in S}A_{t,s}$ $(t\in T)$ and the meet $\bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}$ exist,

and

(2′) all the meets $\bigcap_{t\in T}A_{t,f(t)}$ $(f\in S^T)$ exist.

1.1. *The following three conditions are equivalent* [5] *for any Boolean algebra* $\mathfrak{A}$:

(d) $\mathfrak{A}$ *is* $\mathfrak{m}$-*distributive*;

(d_1) *for every* $\mathfrak{m}$-*indexed set* $\{A_{t,s}\}_{t\in T,s\in S}$ *satisfying* (2), *if*

$$(3)\qquad \bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}\neq\wedge\,,$$

then there exists a mapping $f\in S^T$ *such that* [6]

$$(4)\qquad \bigcap_{t\in T}A_{t,f(t)}\neq\wedge\,;$$

(d_2) *for every* $\mathfrak{m}$-*indexed set* $\{A_{t,s}\}_{t\in T,s\in S}$ *satisfying* (2), *if*

$$(5)\qquad \bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}=\vee\,,$$

then, for every $A\neq\wedge$, *there exists a mapping* $f\in S^T$ *such that*

$$(6)\qquad A\cap\bigcap_{t\in T}A_{t,f(t)}\neq\wedge\,.$$

(d) implies (d_1) since (1) and (3) implies (4).

To deduce (d_2) from (d_1) it suffices to augment the set T by a new element t_0, to assume $A_{t_0,s}=A$ for all $s\in S$, and to apply (d_1) to $\{A_{t,s}\}_{t\in T\cup(t_0),s\in S}$ under the hypothesis that (5) holds.

(d_2) implies (d). In fact, let $\{A_{t,s}\}_{t\in T,s\in S}$ satisfy (2) and (2′), and $B=\bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}$. Suppose that (1) does not hold, i. e. there exists an element $A\neq\wedge$ such that

$$(7)\qquad A\subset B\quad\text{and}\quad\bigcap_{t\in T}A_{t,f(t)}\subset B-A\quad\text{for every}\quad f\in S^T.$$

Augment the set S by a new element s_0 and write

$$B_{t,s_0}=-B\quad\text{for every}\quad t\in T\,,$$

$$B_{t,s}=B\cap A_{t,s}\quad\text{for every}\quad t\in T\quad\text{and every}\quad s\in S\,.$$

[5] The equivalence of (d) and (d_1) was proved by Smith and Tarski [10].

[6] Inequality (4) should be read: either the infinite meet (4) does not exist, or it exists and is not equal to $\wedge$. The same remark should be applied to (6), (12), (14), (19), (21), (22)

The $\mathfrak{m}$-indexed set $\{B_{t,s}\}_{t\in T,s\in S\cup(s_0)}$ satisfies (5). Applying (d_2) to this indexed set, we infer that there exists an $f\in S^T$ such that $A\cap\bigcap_{t\in T}A_{t,f(t)}\neq\wedge$. This is a contradiction of (7).

1.2. *For every Boolean* $\mathfrak{m}$-*algebra* $\mathfrak{A}$ *with at most* $\mathfrak{m}$ *generators* [7], *the following conditions are equivalent:*

(i) $\mathfrak{A}$ *is* $\mathfrak{m}$-*distributive*;

(ii) $\mathfrak{A}$ *is atomic*;

(iii) $\mathfrak{A}$ *is isomorphic to an* $\mathfrak{m}$-*field of sets.*

Only the implication (i)→(ii) ought to be proved. Assume the notation

$$\varepsilon\cdot A=\begin{cases}A & \text{if}\quad \varepsilon=1\,,\\ -A & \text{if}\quad \varepsilon=-1\,,\end{cases}$$

for every $A\in\mathfrak{A}$.

If an $\mathfrak{m}$-indexed set $\{A_t\}_{t\in T}$ generates $\mathfrak{A}$, then each element of the form

$$(8)\qquad \boldsymbol{a}=\bigcup_{t\in T}\varepsilon(t)\cdot A_t$$

where $\varepsilon(t)=\pm1$ is either the zero element or an atom since, for every $A\in\mathfrak{A}$, either $\boldsymbol{a}\cap A=\wedge$ or $\boldsymbol{a}\subset A$. Since

$$\bigcap_{t\in T}(A_t\cup -A)=\vee\,,$$

it follows from 1.1 (d_2) that every element $A\neq\wedge$ contains an atom $\boldsymbol{a}$ of the form (8). Thus $\mathfrak{A}$ is atomic.

1.3. *For every Boolean* $\mathfrak{m}$-*algebra* $\mathfrak{A}$, *the following conditions are equivalent*:

(i) $\mathfrak{A}$ *is* $\mathfrak{m}$-*distributive*;

(ii) *every* $\mathfrak{m}$-*subalgebra generated by at most* $\mathfrak{m}$ *elements is atomic*;

(iii) *every* $\mathfrak{m}$-*subalgebra generated by at most* $\mathfrak{m}$ *elements is isomorphic to an* $\mathfrak{m}$-*field of sets.*

This immediately follows from 1.2 since $\mathfrak{A}$ is $\mathfrak{m}$-distributive if and only if each of its $\mathfrak{m}$-subalgebras generated by at most $\mathfrak{m}$ elements is distributive.

§ 2. The weak $\mathfrak{m}$-distributivity. The letters T and S will denote, as previously, non-empty sets of power $\leqslant\mathfrak{m}$. The letter $\boldsymbol{S}$ will denote in this section the class of all finite subsets of S. According to the convention assumed on p. 92, $\boldsymbol{S}^T$ will denote the set of all functions F defined on T with values in $\boldsymbol{S}$, i. e. such that, for every $t\in T$, $F(t)$ is

[7] This means: The least $\mathfrak{m}$-subalgebra containing a given set of generators of power $<\mathfrak{m}$ coincides with $\mathfrak{A}$. Conditions (ii) and (iii) in 1.3 should be understood in a similar way.

a finite subset of S. If $\{A_{t,s}\}_{t\in T, s\in S}$ is any $\mathfrak{m}$-indexed set of elements in a Boolean algebra $\mathfrak{A}$, and $F\in S^T$, then $A_{t,F(t)}$ will denote the element

$$A_{t,F(t)} = \bigcup_{s\in F(t)} A_{t,s}\,.$$

A Boolean algebra $\mathfrak{A}$ is said to be *weakly* $\mathfrak{m}$-*distributive* [8] if

$$\bigcap_{t\in T}\bigcup_{s\in S} A_{t,s} = \bigcup_{F\in S^T}\bigcap_{t\in T} A_{t,F(t)} \tag{9}$$

for every $\mathfrak{m}$-indexed set $\{A_{t,s}\}_{t\in T, s\in S}$ of elements in $\mathfrak{A}$ such that

(10) all the joins $\bigcup_{s\in S} A_{t,s}$ $(t\in T)$ and the meet $\bigcap_{t\in T}\bigcup_{s\in S} A_{t,s}$ exist,

and

(10′) all the meets $\bigcap_{t\in T} A_{t,F(t)}$ $(F\in S^T)$ exist.

2.1. *The following conditions are equivalent for any Boolean algebra* $\mathfrak{A}$:

(w) $\mathfrak{A}$ *is weakly* $\mathfrak{m}$-*distributive*;

(w_1) *for every* $\mathfrak{m}$-*indexed set* $\{A_{t,s}\}_{t\in T, s\in S}$ *satisfying* (10) *if*

$$\bigcap_{t\in T}\bigcup_{s\in S} A_{t,s} \neq \wedge\,, \tag{11}$$

then there exists an $F\in S^T$ *such that*

$$\bigcap_{t\in T} A_{t,F(t)} \neq \wedge\,; \tag{12}$$

(w_2) *for every* $\mathfrak{m}$-*indexed set* $\{A_{t,s}\}_{t\in T, s\in S}$ *satisfying* (10), *if*

$$\bigcap_{t\in T}\bigcup_{s\in S} A_{t,s} = \vee\,, \tag{13}$$

then for every $A\neq\wedge$ *there exists an* $F\in S^T$ *such that*

$$A\cap\bigcap_{t\in T} A_{t,F(t)} \neq \wedge\,; \tag{14}$$

(w_3) *in the Stone space of* $\mathfrak{A}$ *the interior of any intersection of at most* $\mathfrak{m}$ *dense* $\mathfrak{m}$-*open subsets is dense*;

(w_4) *in the Stone space of* $\mathfrak{A}$ *every set of the* $\mathfrak{m}$-*category is nowhere dense*;

(w_5) *for every set of infinite joins and meets in* $\mathfrak{A}$:

$$\begin{aligned} &\bigcup_{s\in S'_t} A_{t,s} = A_t \quad \textit{where} \quad \overline{\overline{S'_t}}\leqslant\mathfrak{m}, \quad t\in T', \quad \overline{\overline{T'}}\leqslant\mathfrak{m}\,,\\ &\bigcap_{s\in S''_t} B_{t,s} = B_t \quad \textit{where} \quad \overline{\overline{S''_t}}\leqslant\mathfrak{m}, \quad t\in T'', \quad \overline{\overline{T''}}\leqslant\mathfrak{m}\,, \end{aligned} \tag{15}$$

each element $A\neq\wedge$ *contains a subelement* $B\neq\wedge$ *such that every maximal filter containing* B *preserves all the joins and meets* (15).

[8] For some examples of weakly σ-distributive Boolean algebras, see e. g. Horn and Tarski [3].

(w) implies (w_1), (w_1) implies (w_2), (w_2) implies (w). The proof of these implications is similar to the proof of the implications (d)$\to$(d_1), (d_1)$\to$(d_2) (d_2)$\to$(d) in 1.1.

In the proof of the next implications, h_0 denotes the isomorphism (defined on p. 93) of $\mathfrak{A}$ onto the field $\mathfrak{F}_0$ of all both open and closed subsets of the Stone space X of $\mathfrak{A}$.

(w_2) implies (w_3). For suppose that, for every $t\in T$, G is a dense $\mathfrak{m}$-open subset of X, i. e.

$$G_t = \bigcup_{s\in S} h_0(A_{t,s}) \quad \text{where} \quad \bigcup^{\mathfrak{A}}_{s\in S} A_{t,s} = \vee$$

($\overline{\overline{T}}\leqslant\mathfrak{m}$, $\overline{\overline{S}}\leqslant\mathfrak{m}$). Let G_0 be the interior of the intersection of all sets G_t $(t\in T)$ and let $G\subset X$ be any open non-empty set. There exists a non-zero element A in $\mathfrak{A}$ such that $h_0(A)\subset G$. By (w_2), there exists an $F\in S^T$ such that the interior H of the intersection

$$h_0(A)\cap\bigcap_{t\in T} h_0(A_{t,F(t)})$$

is not empty. Since H is open and $H\subset h_0(A)\cap\bigcap_{t\in T} G_t$, we infer that $H\subset G\cap G_0$. The intersection of G_0 with any non-empty open set G being non-empty, the set G_0 is dense.

(w_3) implies (w_4) by passing to complements.

(w_4) implies (w_5). In fact, the set N of all maximal filters which do not preserve any of the joins or meets (15) is of the $\mathfrak{m}$-category (see p. 93). By (w_4), N is nowhere dense. Thus the set $h_0(A)-N$ has a non-empty interior, i. e. there exists an element $B\neq\wedge$ $(B\subset A)$ in $\mathfrak{A}$ such that $h_0(B)\subset h_0(A)-N$. The element B has all the required properties.

(w_5) implies (w_2). Suppose that (13) holds and apply (w_5) to the joins

$$\bigcup_{s\in S} A_{t,s} = \vee \qquad (t\in T)\,. \tag{16}$$

Since all maximal filters containing B (i. e. belonging to $h_0(B)$) preserve all the joins (16), we have

$$h_0(B)\subset\bigcup_{s\in S} h_0(A_{t,s}) \quad \text{for every} \quad t\in T\,.$$

Since $h_0(B)$ is closed and $h_0(A_{t,s})$ are open in the compact space X, there exists a finite set $F(t)\subset S$ such that

$$h_0(B)\subset\bigcup_{s\in F(t)} h_0(A_{t,s}) = h_0(A_{t,F(t)})\,.$$

Since

$$\wedge\neq B\subset A \quad \text{and} \quad B\subset A_{t,F(t)} \quad \text{for all} \quad t\in T\,,$$

(14) holds.

2.2. *Every* $\mathfrak{m}$-*distributive Boolean algebra is weakly* $\mathfrak{m}$-*distributive.*

2.3. *A Boolean* $\mathfrak{m}$*-algebra is weakly* $\mathfrak{m}$*-distributive if and only if each of its* $\mathfrak{m}$*-subalgebras generated by at most* $\mathfrak{m}$ *elements is weakly* $\mathfrak{m}$*-distributive.*

The easy proof is left to the reader.

§ 3. The $\mathfrak{m}$-representable algebras. A Boolean algebra is said to be $\mathfrak{m}$*-representable* if it is isomorphic to an $\mathfrak{m}$-regular subalgebra of a factor algebra $\mathfrak{F}/\mathfrak{I}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets, and $\mathfrak{I}$ is an $\mathfrak{m}$-ideal (otherwise speaking, if there exists an $\mathfrak{m}$-isomorphism of $\mathfrak{A}$ into $\mathfrak{F}/\mathfrak{I}$ where $\mathfrak{F}$ and $\mathfrak{I}$ have the properties mentioned above).

Thus a Boolean $\mathfrak{m}$-algebra is $\mathfrak{m}$-representable if and only if it is isomorphic to a factor algebra $\mathfrak{F}/\mathfrak{I}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets, and $\mathfrak{I}$ is an $\mathfrak{m}$-ideal of $\mathfrak{F}$.

In the sequel, $\mathfrak{A}$ denotes a fixed Boolean algebra, h_0 is the Stone isomorphism (defined on p. 93) of $\mathfrak{A}$ onto the field of all both open and closed subsets of the Stone space X of $\mathfrak{A}$, $\mathfrak{F}_\mathfrak{m}$ denotes the least $\mathfrak{m}$-field containing $\mathfrak{F}_0$, $\mathfrak{I}_\mathfrak{m}$ is the $\mathfrak{m}$-ideal of all sets $B\in\mathfrak{F}_\mathfrak{m}$ of the $\mathfrak{m}$-category in X, and $\mathfrak{F}'_\mathfrak{m}$ is the field of all sets of the form

$$(B_0\cup B_1)-B_2 \quad \text{where} \quad B_0\in\mathfrak{F}_0 \quad \text{and} \quad B_1,B_2\in\mathfrak{I}_\mathfrak{m}.$$

By definition, $\mathfrak{F}'_\mathfrak{m}$ is a subfield of $\mathfrak{F}_\mathfrak{m}$, and $\mathfrak{F}'_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$ is a subalgebra of $\mathfrak{F}_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$.

The factor algebra $\mathfrak{F}'_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$ is called the *canonical* $\mathfrak{m}$*-representation* for $\mathfrak{A}$. The following homomorphism h of $\mathfrak{A}$ onto $\mathfrak{F}'_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$:

$$h(A)=[h_0(A)] \quad \text{for} \quad A\in\mathfrak{A}$$

is called the *canonical homomorphism*. If h is one-to-one, it is called the *canonical isomorphism.*

Using the above terminology we shall prove the following two theorems.

3.1. *The canonical homomorphism* h *is an* $\mathfrak{m}$*-homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{F}_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$. *Consequently, if* h *is an isomorphism, then* $\mathfrak{F}'_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$ *is an* $\mathfrak{m}$*-regular subalgebra of* $\mathfrak{F}_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$.

If $\mathfrak{A}$ *is an* $\mathfrak{m}$*-algebra, then* $\mathfrak{F}'_\mathfrak{m}=\mathfrak{F}_\mathfrak{m}$ *and consequently* $\mathfrak{F}'_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}=\mathfrak{F}_\mathfrak{m}/\mathfrak{I}_\mathfrak{m}$.

If $\bigcap^{\mathfrak{A}}_{t\in T}A_t=\wedge$ $(\overline{\overline{T}}\leqslant\mathfrak{m})$, then the intersection of all sets $h_0(A_t)$ belongs to $\mathfrak{I}_\mathfrak{m}$ and consequently

$$\bigcap_{t\in T}h(A_t)=[\bigcap_{t\in T}h_0(A_t)]=\wedge.$$

This proves the first part of 3.1.

Suppose now that $\mathfrak{A}$ is $\mathfrak{m}$-complete. To prove that $\mathfrak{F}'_\mathfrak{m}=\mathfrak{F}_\mathfrak{m}$ it suffices to show that $\mathfrak{F}'_\mathfrak{m}$ is an $\mathfrak{m}$-field.

Observe that, by definition, $B\in\mathfrak{F}'_\mathfrak{m}$ if and only if there exists an element $A\in\mathfrak{A}$ such that

$$h_0(A)-B\in\mathfrak{I}_\mathfrak{m} \quad \text{and} \quad B-h_0(A)\in\mathfrak{I}_\mathfrak{m}. \tag{17}$$

Suppose that $B_t\in\mathfrak{F}'_\mathfrak{m}$ for every $t\in T$ $(\overline{\overline{T}}\leqslant\mathfrak{m})$, i. e.

$$h_0(A_t)-B_t\in\mathfrak{I}_\mathfrak{m} \quad \text{and} \quad B_t-h_0(A_t)\in\mathfrak{I}_\mathfrak{m}$$

for an $A_t\in\mathfrak{A}$. Let $A=\bigcup^{\mathfrak{A}}_{t\in T}A_t$, and let B the the set-theoretical union of all B_t $(t\in T)$. We have

$$h_0(A)-B\subset\big(h_0(A)-\bigcup_{t\in T}h_0(A_t)\big)\cup\big(\bigcup_{t\in T}(h_0(A)-B_t)\big)$$

and

$$B-h_0(A)\subset\bigcup_{t\in T}(B_t-h_0(A_t))$$

where $\bigcup_{t\in T}$ denotes the set-theoretical union. This proves that A and B satisfy (17). Hence it follows that $B\in\mathfrak{F}'_\mathfrak{m}$. Thus $\mathfrak{F}'_\mathfrak{m}$ is an $\mathfrak{m}$-field.

3.2. *The following conditions are equivalent for every Boolean algebra* $\mathfrak{A}$:

(r) $\mathfrak{A}$ *is* $\mathfrak{m}$*-representable;*

(r_0) *the canonical homomorphism is an isomorphism;*

(r_1) *for every* $\mathfrak{m}$*-indexed set* $\{A_{t,s}\}_{t\in T,s\in S}$ *satisfying* (2), *if*

$$\bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}\neq\wedge, \tag{18}$$

then there exists an $f\in S^T$ *such that*

$$\bigcup_{t\in T'}A_{t,f(t)}\neq\wedge \quad \textit{for every finite set} \quad T'\subset T; \tag{19}$$

(r_2) *for every* $\mathfrak{m}$*-indexed set* $\{A_{t,s}\}_{t\in T,s\in S}$ *satisfying* (2), *if*

$$\bigcap_{t\in T}\bigcup_{s\in S}A_{t,s}=\vee, \tag{20}$$

then, for every element $A\neq\wedge$, *there exists an* $f\in S^T$ *such that*

$$A\cap\bigcap_{t\in T'}A_{t,f(t)}\neq\wedge \quad \textit{for every finite set} \quad T'\subset T; \tag{21}$$

(r_2') *for every* $\mathfrak{m}$*-indexed set* $\{A_{t,s}\}_{t\in T,s\in S}$ *satisfying* (2), *if* (20) *holds, then for every proper* $\mathfrak{m}$*-filter* $\mathfrak{I}$ *of* $\mathfrak{A}$ *there exists an* $f\in S^T$ *such that*

$$A\cap\bigcap_{t\in T'}A_{t,f(t)}\neq\wedge \ \textit{for every}\ A\in\mathfrak{I}\ \textit{and every finite set}\ T'\subset T; \tag{22}$$

(r_3) *in the Stone space of* $\mathfrak{A}$, *any intersection of at most* $\mathfrak{m}$ *dense* $\mathfrak{m}$*-open sets is dense, i. e. any intersection of at most* $\mathfrak{m}$ *dense* $\mathfrak{m}$*-open sets intersects every non-empty open set;*

(r_3') *in the Stone space of* $\mathfrak{A}$, *any intersection of at most* $\mathfrak{m}$ *dense* $\mathfrak{m}$*-open sets intersects any non-empty closed set of character* $\mathfrak{m}$;

(r_4) *in the Stone space of* $\mathfrak{A}$, *every set of the* $\mathfrak{m}$*-category is a boundary set, i. e. no open non-empty set is of the* $\mathfrak{m}$*-category;*

(r_4') *in the Stone space of* $\mathfrak{A}$, *no non-empty closed set of character* $\mathfrak{m}$ *is of the* $\mathfrak{m}$*-category;*

(r_5) *for every set* (15) *of infinite joins and meets in* $\mathfrak{A}$, *and for every* $A \neq \wedge$ *there exists a maximal filter containing* A *and preserving all the joins and meets* (15);

(r_5') *for every set* (15) *of infinite joins and meets in* $\mathfrak{A}$, *and for every proper* $\mathfrak{m}$*-filter* $\mathfrak{J}$ *there exists a maximal filter containing* $\mathfrak{J}$ *and preserving all the joins and meets* (15).

(r_1) implies (r_2). The proof is similar to the proof of the implication $(d_1) \to (d_2)$ in 1.1.

(r_2) implies (r_3). In fact, suppose that, for every $t \in \overline{\overline{T}}$ ($T \leqslant \mathfrak{m}$), G_t is a dense $\mathfrak{m}$-open subset of X, i. e.

$$G_t = \bigcup_{s \in S} h_0(A_{t,s}) \quad \text{where} \quad \bigcup_{s \in S}^{\mathfrak{A}} A_{t,s} = \vee \quad (\overline{\overline{S}} \leqslant \mathfrak{m}).$$

Let G be any non-empty open subset of X. There exists an element $A \neq \wedge$ such that $h_0(A) \subset G$. By (r_2), there exists an $f \in S^T$ such that (21) holds, i. e.

$$h_0(A) \cap \bigcap_{t \in T'} h_0(A_{t,f(t)}) \neq \wedge$$

for every finite set $T' \subset T$. Since all the sets $h_0(A)$, $h_0(A_{t,s})$ are closed in the compact space X, we obtain

$$\wedge \neq h_0(A) \cap \bigcap_{t \in T} h_0(A_{t,f(t)}) \subset G \cap \bigcap_{t \in T} G_t .$$

(r_3) implies (r_4) by passing to complements.

(r_4) implies (r_5). In fact, the set of all maximal filters which do not preserve a join or meet in (15) is of the $\mathfrak{m}$-category. By (r_4), there exists a point in $h_0(A)$ which does not belong to this set of the $\mathfrak{m}$-category. This point is a maximal filter preserving all the joins and meets (15).

(r_5) implies (r_2). Suppose that (20) holds and apply (r_5) to the joins

$$\bigcup_{s \in S} A_{t,s} = \vee \quad (t \in T). \tag{23}$$

Let $\mathfrak{J}_0$ be a maximal filter preserving all the joins (23) and containing A. By definition, $\mathfrak{J}_0 \in h_0(A)$ and $\mathfrak{J}_0 \in \bigcup_{s \in S} h_0(A_{t,s})$ for every $t \in T$. Thus exists an $s = f(t)$ such that $\mathfrak{J}_0 \in h_0(A_{t,f(t)})$. Consequently

$$h_0(A) \cap \bigcap_{t \in T'} h_0(A_{t,f(t)}) \neq \wedge \quad \text{for every finite set} \quad T' \subset T,$$

i. e. (21) holds.

(r_2') implies (r_3'), (r_3') implies (r_4'), (r_4') implies (r_5'), (r_5') implies (r_2'). The proof of these implications is similar to the proof of the implications $(r_2) \to (r_3)$, $(r_3) \to (r_4)$, $(r_4) \to (r_5)$, $(r_5) \to (r_2)$ respectively.

(r_2) implies (r_2'). For suppose that (20) holds but (22) does not hold, i. e. for every $f \in S^T$ there exists a finite set $T_f \subset T$ and, for the set T_f, there exists an element $A_{T_f} \in \mathfrak{J}$ such that

$$A_{T_f} \cap \bigcap_{t \in T_f} A_{t,f(t)} = \wedge .$$

The set of all elements A_{T_f} has a power $\leqslant \mathfrak{m}$ (since the class of all finite subsets of T has a cardinal $\leqslant \mathfrak{m}$). $\mathfrak{J}$ being an $\mathfrak{m}$-filter, there exists an element $A \in \mathfrak{J}$ such that $A \subset A_{T_f}$ for every $f \in S^T$. We have $A \neq \wedge$ since $\mathfrak{J}$ is proper. Thus

$$A \cap \bigcap_{t \in T_f} A_{t,f(t)} = \wedge$$

for every $f \in S^T$, i. e. (r_2) does not hold.

(r_2') implies (r_2) (take as $\mathfrak{J}$ the principal filter generated by A).

(r_4) implies (r_0). In fact, if $A \neq \wedge$, then $h_0(A)$ is open and non-empty. By (r_4), $h_0(A) \notin \mathfrak{J}_{\mathfrak{m}}$, i. e. $h(A) \neq \wedge$. This proves that the canonical homomorphism h is an isomorphism.

(r_0) implies (r). This immediately follows from 3.1.

(r) implies (r_1). It suffices to prove this implication in the case where $\mathfrak{A}$ is an $\mathfrak{m}$-regular subalgebra of $\mathfrak{F}/\mathfrak{J}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets and $\mathfrak{J}$ is an $\mathfrak{m}$-ideal of $\mathfrak{F}$.

Suppose that (18) holds in $\mathfrak{A}$. Since $\mathfrak{A}$ is an $\mathfrak{m}$-regular subalgebra of $\mathfrak{F}/\mathfrak{J}$, all joins and meets in (18) can be considered as joins and meets in $\mathfrak{F}/\mathfrak{J}$. We have

$$A_{t,s} = [B_{t,s}] \quad \text{for some sets} \quad B_{t,s} \in \mathfrak{F} .$$

Let B be the union of all finite intersections

$$B_{t_1,s_1} \cap \ldots \cap B_{t_n,s_n}$$

which belong to $\mathfrak{J}$, and let

$$C_{t,s} = B_{t,s} - B .$$

We have

$$A_{t,s} = [C_{t,s}] ,$$

since $B \in \mathfrak{J}$. Moreover, for any finite intersection,

$$\text{if} \quad C_{t_1,s_1} \cap \ldots \cap C_{t_n,s_n} \neq \wedge , \quad \text{then} \quad C_{t_1,s_1} \cap \ldots \cap C_{t_n,s_n} \notin \mathfrak{J} , \tag{24}$$

i. e.

$$A_{t_1,s_1} \cap \ldots \cap A_{t_n,s_n} \neq \wedge .$$

By (18), $[\bigcap_{t \in T} \bigcup_{s \in S} C_{t,s}] \neq \wedge$. Thus the set $\bigcap_{t \in T} \bigcup_{s \in S} C_{t,s}$ contains a point x. Consequently, for every $t \in T$ there exists an $s = f(t)$ such that $x \in C_{t,f(t)}$. Therefore

$$C_{t_1,f(t_1)} \cap \ldots \cap C_{t_n,f(t_n)} \neq \wedge .$$

This implies (19) on account of (24).

3.3. *Every weakly* $\mathfrak{m}$*-distributive Boolean algebra is* $\mathfrak{m}$*-representable.*

This follows from 3.2 and 2.1 since (w_3) implies (r_3).

3.4. *A Boolean $\mathfrak{m}$-algebra is $\mathfrak{m}$-representable if and only if each of its $\mathfrak{m}$-subalgebras generated by at most $\mathfrak{m}$ elements is $\mathfrak{m}$-representable.*

This follows from 3.2 since $\mathfrak{A}$ satisfies (r_1) if and only if each of its $\mathfrak{m}$-subalgebras generated by at most $\mathfrak{m}$ elements satisfies (r_1).

3.5. *Every Boolean algebra is σ-representable* (9).

This follows immediately from 3.2 (r_4) since every set of σ-category is a set of the first category (i. e. the union of a sequence of nowhere dense sets), and no open non-empty subset of a compact Hausdorff space is of the first category.

§ 4. The $\mathfrak{m}$-chain condition. A Boolean algebra $\mathfrak{A}$ is said to satisfy the $\mathfrak{m}$-*chain condition* if every class of disjoint elements in A has a power $\leqslant \mathfrak{m}$.

4.1. *A Boolean algebra $\mathfrak{A}$ satisfies the $\mathfrak{m}$-chain condition if and only if, in its Stone space X, every nowhere dense set is $\mathfrak{m}$-nowhere dense.*

Let h_0 be the Stone isomorphism defined on p. 93.

Suppose that N is a nowhere dense subset of X. Let $\{A_t\}_{t\in T}$ be a maximal class of non-zero disjoint elements in $\mathfrak{A}$ such that the sets $h_0(A_t)$ do not intersect N. Since the class is maximal, the union G of all sets $h_0(A_t)$ is dense in X, i. e. its complement $N_0 = X-G$ is a nowhere dense set. We have $N \subset N_0$. If $\mathfrak{A}$ satisfies the $\mathfrak{m}$-chain condition, then $\overline{\overline{T}} \leqslant \mathfrak{m}$, and consequently the set $N_0 = \bigcap_{t\in T} h_0(-A_t)$ is $\mathfrak{m}$-closed. This proves that N is then $\mathfrak{m}$-nowhere dense.

Suppose now that every nowhere dense subset of X is $\mathfrak{m}$-nowhere dense. We shall prove that, for every indexed set $\{A_t\}_{t\in T}$ of disjoint non-zero elements in $\mathfrak{A}$, we have $\overline{\overline{T}} \leqslant \mathfrak{m}$. It sufficies to prove it in the case where $\{A_t\}_{t\in T}$ is a maximal class of disjoint elements, i. e. the union G of all sets $h_0(A_t)$ is dense in X. The nowhere dense set $X-G$ is contained in a nowhere dense $\mathfrak{m}$-closed set N. Thus there exists an $\mathfrak{m}$-indexed set $\{B_s\}_{s\in S}$ of elements in $\mathfrak{A}$, such that the union G_0 of all sets $h_0(B_s)$ $(s \in S)$ satisfies

$$G_0 = X - N \subset G\,.$$

Since $h_0(B_s)$ is compact and $h_0(A_t)$ are disjoint and open in the compact space X, for every fixed s there exists only a finite number of indexes t such that $h_0(A_t)$ intersects $h_0(B_s)$. Since G_0 is dense in X, every set $h_0(A_t)$ intersects at least one set $h_0(B_s)$. This proves that $\overline{\overline{T}} \leqslant \mathfrak{m}$.

(9) This theorem is known. For the case of Boolean σ-algebras, see Loomis [5] and Sikorski [7]. For the case of arbitrary Boolean algebras, see Sikorski [8].

4.2. *A Boolean algebra $\mathfrak{A}$ satisfying the σ-chain condition is weakly σ-distributive if and only if, in the Stone space of $\mathfrak{A}$, every set of the first category is nowhere dense* (10).

In fact, if follows from 4.1 that if $\mathfrak{A}$ satisfies the σ-chain condition, then the notion of σ-nowhere dense set and the notion of nowhere dense set coincide in the Stone space of $\mathfrak{A}$. Consequently sets of the σ-category coincide with sets of the first category. Therefore 4.2 follows from 2.1 $(w) \equiv (w_4)$.

References

[1] C. C. Chang, Bull. Amer. Math. Soc. (Abstract) 61 (1955), p. 325.

[2] — *On the representation of α-complete Boolean algebras*, Trans. Amer. Math. Soc. 85 (1957), p. 208-218.

[3] A. Horn and A. Tarski, *Measures in Boolean algebras*, Trans. Amer. Math. Soc. 64 (1948), p. 467-497.

[4] J. L. Kelley, *Measures in Boolean algebras*, in print.

[5] L. H. Loomis, *On the representation of δ-complete Boolean algebras*, Bull. Amer. Math. Soc. 53 (1947), p. 757-760.

[6] R. S. Pierce, *Representation theorems for certain Boolean algebras*, Bull. Amer. Math. Soc. 10 (1959), p. 42-50.

[7] R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fund. Math. 35 (1948), p. 247-254.

[8] — *Cartesian products of Boodean algebras*, Fund. Math. 37 (1950), p. 25-54.

[9] E. C. Smith (Jr.), *A distributivity condition for Boolean algebras*, Ann. of Math. 64 (1956), p. 551-561.

[10] E. C. Smith (Jr.) and A. Tarski, *Higher degrees of distributivity and completeness in Boolean algebras*, Trans. Amer. Math. Soc. 84 (1957), p. 230-257.

(10) This theorem is due to Kelley [4] and J. Oxtoby.

Reçu par la Rédaction le 17. 3. 1959