The space of prime ideals of a ring *

by

C. W. Kohls (Lafayette, Ind.)

1. Introduction. Jacobson showed [4] that the set of primitive ideals of an arbitrary ring may be made into a topological space by means of a closure operator defined in terms of intersection and inclusion relations among ideals of the ring. It was observed by McCoy in [11] that the set of generalized prime ideals defined therein may be treated in exactly the same way.

In the present paper, we shall primarily consider subspaces of this latter space, among which the space of primitive ideals is the most important. In section 2, we review the basic results of the subject and establish notation and terminology. In section 3, we present some simple extensions of the discussion of Jacobson [4] on the connection between compactness of a general space of ideals and restrictions on the ring. The following section treats the relation of other topological properties on appropriate spaces of prime ideals to algebraic conditions on a commutative ring. Section 5 is devoted to an examination of the connection between the prime and primitive ideals of an *ideal* of an arbitrary ring and those of the whole ring. These results are applied in the last section to the situation in which the ideal is viewed as given, and the containing ring is a ring with identity into which it has been imbedded by a standard process.

Several of the results of sections 3 and 4 (in particular, 3.1, 4.1 and 4.7) were obtained independently, in a slightly different form, by McKnight [12]. Since none of his results have been published, we have included a complete discussion.

2. Preliminary concepts. Throughout this paper, the word "ideal" always means "proper two-sided ideal". For the definition of a primitive ideal, see [4], and for the notion of prime ideal in an arbitrary ring,

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see [11]. It is shown in [11], Theorem 1, that an ideal P in an arbitrary ring A is prime if and only if whenever $axb \in P$ for all $x \in A$, then $a \in P$ or $b \in P$.

We shall use the same notation and terminology as [2]. For an arbitrary ring A, $\mathfrak{S}(A)$ denotes a general structure space of A; $\mathfrak{P}(A)$, the space of primitive ideals; $\mathfrak{Q}(A)$, the space of prime ideals; and $\mathfrak{R}(A)$, either $\mathfrak{P}(A)$ or $\mathfrak{Q}(A)$.

For each $a \in A$, we call $\mathfrak{S}(a) = \{S \in \mathfrak{S}(A)\colon a \in S\}$ the $\mathfrak{S}$-*set of* a, and $\mathfrak{C}\mathfrak{S}(a) = \mathfrak{S}(A) - \mathfrak{S}(a) = \{S \in \mathfrak{S}(A)\colon a \notin S\}$ (¹), the $\mathfrak{C}\mathfrak{S}$-*set of* a. It is easily seen that $\mathfrak{S}(a)$ is closed, and thus that $\mathfrak{C}\mathfrak{S}(a)$ is open.

A ring A will be called $\mathfrak{S}$-*semi-simple* if $\Delta\mathfrak{S}(A) = (0)$. The notion of $\mathfrak{P}$-*semi-simplicity* coincides with the usual semi-simplicity in the sense of Jacobson. The ideal $\Delta\mathfrak{Q}(A)$ is the radical defined by McCoy in [11]. It is contained in the Perlis-Jacobson radical $\Delta\mathfrak{P}(A)$; so $\mathfrak{P}$-semi-simplicity implies $\mathfrak{Q}$-semi-simplicity. More generally, for any $\mathfrak{S}(A)$ containing $\mathfrak{P}(A)$, $\mathfrak{P}$-semi-simplicity implies $\mathfrak{S}$-semi-simplicity.

Many investigations about topological properties of $\mathfrak{S}(A)$ are simplified by use of the observation that the $\mathfrak{C}\mathfrak{S}$-sets form a base for the open sets of $\mathfrak{S}(A)$. This can be established by noting that for any subset $\mathfrak{B}$ of $\mathfrak{S}(A)$, the relations $\mathfrak{B} \subseteq \bigcap_{a \in \Delta\mathfrak{B}} \mathfrak{S}(a) \subseteq \overline{\mathfrak{B}}$, together with the fact that every $\mathfrak{S}$-set is closed, imply that $\overline{\mathfrak{B}} = \bigcap_{a \in \Delta\mathfrak{B}} \mathfrak{S}(a)$. (The advantage of this approach was observed also by McKnight, [12].)

It is now easily shown that $\mathfrak{S}(A)$ is always a T_0-space. For, given distinct $S, T \in \mathfrak{S}(A)$, there is an element a in either $S - T$ or $T - S$, say the former; and then $\mathfrak{C}\mathfrak{S}(a)$ is a neighborhood of T not containing S. If A has an identity, then every maximal ideal of A is primitive. Now, in general, there may be a primitive ideal contained properly in another; but when A is commutative, every primitive ideal is maximal, so that $\mathfrak{P}(A)$ is a T_1-space. However, this need not be the case for $\mathfrak{Q}(A)$. For example, if A is the ring of integers, every non-empty $\mathfrak{C}\mathfrak{Q}$-set contains (0).

The set of primitive ideals can be neatly characterized when A is commutative: it consists of those maximal ideals M which are prime, *i. e.*, such that A/M is a field. (It is well known that this is equivalent to the statement that A/M is not a zero-ring.) For, if A/M is a field, it is a simple ring with identity, and hence a primitive ring, so M is a primitive ideal. Conversely, if A/M is not a field, *i. e.*, if A/M is a zero-ring, it is a radical ring, which can contain no primitive ideals; so M is not a primitive ideal.

(¹) The symbol $-$ will be used for set complement.

3. General structure spaces. In this section we study the relation between algebraic conditions on the ring A and compactness of a general structure space $\mathfrak{S}(A)$.

LEMMA 3.1. *The space* $\mathfrak{S}(A)$ *is compact if and only if whenever the set* $\{a_\varphi\}_{\varphi \in \Phi}$ *is contained in no* $\mathfrak{S}$-*ideal, then there is a finite subset* $\{a_1, \ldots, a_n\}$ *that is contained in no* $\mathfrak{S}$-*ideal.*

Proof. The condition stated is equivalent to: For every collection $\{a_\varphi\}_{\varphi \in \Phi}$ of elements of A such that for each $S \in \mathfrak{S}(A)$ there is an $a_\varphi \notin S$, there is a finite subcollection $\{a_1, \ldots, a_n\}$ such that for each $S \in \mathfrak{S}(A)$, there is an $a_i \notin S$. This is equivalent to: Every cover by $\mathfrak{C}\mathfrak{S}$-sets has a finite subcover; which is the same as requiring that $\mathfrak{S}(A)$ be compact.

It follows that for every Noetherian ring A, the space $\mathfrak{S}(A)$ is compact. For, the ideal generated by any set $\{a_\varphi\}_{\varphi \in \Phi}$ has a finite basis, and it is easy to show that the basis elements may be selected from the a_φ's. Also, if every non-zero element belongs to at most a finite number of $\mathfrak{S}$-ideals (as in the ring of integers, for instance) then $\mathfrak{S}(A)$ is compact.

We shall say that A *is finitely generated* if A is a finitely generated ideal of A.

THEOREM 3.2. *Let* A *be a ring in which every proper ideal is contained in an* $\mathfrak{S}$-*ideal. Then* $\mathfrak{S}(A)$ *is compact if and only if* A *is finitely generated.*

Proof. With the stated condition on A, Lemma 3.1 now becomes: $\mathfrak{S}(A)$ is compact if and only if whenever $\{a_\varphi\}_{\varphi \in \Phi}$ is a set of generators for A, then there is a finite subset $\{a_1, \ldots, a_n\}$ such that $(a_1, \ldots, a_n) = A$.

We note that if A is finitely generated, then every proper ideal is contained in a maximal ideal. For, let I be any proper ideal, and let U be the union of any maximal chain of proper ideals containing I. Then U is a proper ideal; for otherwise, each one of the finite number of generators of A would be in some ideal of the chain, and thus in the largest ideal of this finite subchain, which is impossible. So we have:

COROLLARY 3.3. *If* A *is finitely generated, and every maximal ideal of* A *is an* $\mathfrak{S}$-*ideal, then* $\mathfrak{S}(A)$ *is compact.*

Now when A has an identity, every maximal ideal of A is primitive. Hence:

COROLLARY 3.4 (Jacobson). *If* A *has an identity, and* $\mathfrak{S}(A) \supseteq \mathfrak{P}(A)$, *then* $\mathfrak{S}(A)$ *is compact.*

It is *not* true that if every proper ideal is contained in a maximal ideal, then A is finitely generated, as shown for instance by the discrete direct sum of an infinite number of fields.

4. Subspaces of $\mathfrak{Q}(A)$. Throughout this section, we require $\mathfrak{S}(A)$ to be a subspace of $\mathfrak{Q}(A)$; and except in Theorem 4.1 (a), we assume that A is a commutative ring.

The first two theorems are concerned with conditions which make $\mathfrak{S}(A)$ a Hausdorff space.

THEOREM 4.1. (a) *Let A be an arbitrary ring, and let $\mathfrak{S}(A)\subseteq\mathfrak{Q}(A)$. The space $\mathfrak{S}(A)$ is Hausdorff if and only if for each distinct pair $S,T\in\mathfrak{S}(A)$, there exist $a,b\in A$ such that $a\notin S$, $b\notin T$ and $axb\in\Delta\mathfrak{S}(A)$ for all $x\in A$.*

(b) *In particular, if A is commutative, the last requirement simplifies to $ab\in\Delta\mathfrak{S}(A)$.*

Proof. (a) In view of [11], Theorem 1, the condition stated is equivalent to the statement that for each distinct pair $S,T\in\mathfrak{S}(A)$, there exist disjoint $\mathfrak{C}\mathfrak{S}$-sets containing S and T respectively. This in turn is equivalent to the requirement that $\mathfrak{S}(A)$ be Hausdorff.

(b) Since $\Delta\mathfrak{S}(A)$ is an intersection of prime ideals, this is evident.

It follows easily that if A is an integral domain with more than one $\mathfrak{S}$-ideal, and if A is $\mathfrak{S}$-semi-simple, then $\mathfrak{S}(A)$ is not Hausdorff. This is true, in particular, for the ring of integers whenever $\mathfrak{S}$ is infinite. The same result was obtained by different means in [2], Theorem 3.4 ff.

THEOREM 4.2. *Let A be a commutative ring, let $\mathfrak{S}(A)\subseteq\mathfrak{Q}(A)$ and suppose that for each distinct pair $S,T\in\mathfrak{S}(A)$, there are $a,b\in A$ satisfying the following conditions: $a\notin S$, $b\in S-T$, and for some $x,y\in A$ and integers m,n,j,k, with $m>j$, $n>k$, we have $a^mx=a^j$, $b^ny=b^k$. Then $\mathfrak{S}(A)$ is Hausdorff.*

Proof. We note that $b^{n-k}y\in S$; and $b^{n-k}y\notin T$, since the contrary implies $b^ny=b^k\in T$, whence $b\in T$, since T is prime. Select any positive integer h satisfying $(n-k)h>k$. Then for all $p\geqslant h$, we have $(n-k)p>k$, so $(b^{n-k}y)^p=b^{(n-k)p-k}y^pb^k=b^{(n-k)p-k}y^pb^ny=(b^{n-k}y)^{p+1}$. Hence, by iteration, $d=(b^{n-k}y)^h=(b^{n-k}y)^{2h}=d^2$. Thus, we have an idempotent $d\in S-T$. Similarly, there is an idempotent $c\notin S$. Now $c-d\notin S$, for otherwise, $c\in S$. Thus $c(c-d)\notin S$, since S is prime; and $[c(c-d)]d=0\in\Delta\mathfrak{S}(A)$. By Theorem 4.1, $\mathfrak{S}(A)$ is Hausdorff.

DEFINITION 4.3. Let A be a commutative ring, and let $a\in A$. We shall say that an element $e\in A$ is a *relative identity* for a if $ae=a$.

Evidently if A has an identity e, then e is a relative identity for each element of A. We shall be primarily interested in rings without an identity which have relative identities for sufficiently many (or perhaps all) of their elements. An example of such a ring is the ring of all continuous real-valued functions with compact supports on a locally compact, non-compact Hausdorff space. This ring has no identity, but each element has a relative identity.

Another example is provided by π-regular rings. A commutative ring A is said to be *π-regular* (see [7], p. 62) if for each $a\in A$, there is an $x\in A$ and an integer n (depending on a) such that $a^{2n}x=a^n$. Thus, a^nx is a relative identity for a^n. In case A is *regular*, *i. e.*, $n=1$ for all a, then every $a\in A$ has a relative identity.

If A is a commutative ring such that for each maximal ideal M, there is an $a\notin M$ having a relative identity e, then each maximal ideal is prime. For, $ae=a\notin M$, so A/M is not a zero-ring. In [8], 3.8 we give an example of a ring which satisfies this condition, but possesses elements without relative identities.

The following lemma amounts to the observation that a well known process for "enlarging" idempotents can be applied to relative identities as well ([2]).

LEMMA 4.4. *Let A be a commutative ring. If $a_0,\dots,a_n$ have relative identities $e_0,\dots,e_n$, respectively, then there is a common relative identity for the set $\{a_0,\dots,a_n\}$.*

Proof. Define a sequence of elements of A as follows: $f_1=e_0+e_1-e_0e_1$, $f_2=f_1+e_2-f_1e_2$, and in general, $f_k=f_{k-1}+e_k-f_{k-1}e_k$, $(k=2,\dots,n)$. It can be verified immediately that f_k is a relative identity for $\{a_0,\dots,a_k\}$. Thus, f_n is the desired element.

LEMMA 4.5. *Let A be a commutative ring which is finitely generated: $A=(a_0,\dots,a_n)$. If $a_i\in(a_i^2)$, $(i=0,\dots,n)$, then A has an identity element.*

Proof. Let $y_i\in A$ and integers m_i be such that $a_i=y_ia_i^2+m_ia_i^2$, and set $e_i=y_ia_i+m_ia_i$; then e_i is a relative identity for a_i $(i=0,\dots,n)$. By Lemma 4.4, there is a common relative identity e for the set $\{a_0,\dots,a_n\}$. Since any $b\in A$ may be written in the form $b=\sum_{i=0}^{n}(z_ia_i+n_ia_i)$ for suitable $z_i\in A$ and integers n_i, it follows that e is the identity of A.

THEOREM 4.6. *Let A be a commutative ring without identity, but having a single generator. Then A contains a non-prime maximal ideal.*

Proof. Let $(a)=A$. By Lemma 4.5, $a\notin(a^2)$, so (a^2) is a proper ideal. In view of the remark preceding 3.3, (a^2) can be imbedded in a maximal

([2]) The same device was used in a lemma on Banach algebras by Loomis in [9], p. 83. (Added in proof: It has been called to our attention that 4.7 and 4.9 are very similar to Théorème 4 of K. Fujiwara, *Sur les anneaux des fonctions continues à support compact*, Math. J. Okayama Univ. 3 (1954), p. 175-184. If his more general notion of relative identity is used, the proofs of 4.7 and 4.9 can be carried over to non-commutative rings with only very simple changes.)

ideal M. Now $A/(a^2)$ is a zero-ring; for if $r,s \in A$, *i. e.*, $r,s \in (a)$, then $rs \in (a^2)$. Hence A/M is also a zero-ring, so M is not prime.

For a ring of this type, Corollary 3.3 is inapplicable. An example is the ring E of even integers. In this case $a=2$, and (4) is itself the non-prime maximal ideal. Of course, here we may conclude that $\mathfrak{S}(E)$ is compact from the fact that E is Noetherian.

THEOREM 4.7. *Let A be a commutative ring such that for each $S \in \mathfrak{S}(A)$, there is an $a \notin S$ having a relative identity e; and suppose that $\mathfrak{Q}(A) \supseteq \mathfrak{S}(A) \supseteq \mathfrak{P}(A)$. Then $\mathfrak{S}(A)$ is locally compact.*

Proof. Let $T \in \mathfrak{C}\mathfrak{S}(a)$ be arbitrary. For all $x \in A$, $aex - ax = 0 \in T$. Since $a \notin T$, and T is prime, we have $ex - x \in T$. Thus $ex - x \in \Delta\mathfrak{C}\mathfrak{S}(a)$ for all $x \in A$, *i. e.*, the image of e in $A/\Delta\mathfrak{C}\mathfrak{S}(a)$ is the identity. Since $\overline{\mathfrak{C}\mathfrak{S}(a)}$ is homeomorphic to $\mathfrak{S}(A/\Delta\mathfrak{C}\mathfrak{S}(a))$ ([2], Theorem 1.1) it is compact; and it is the closure of a neighborhood of S.

COROLLARY 4.8. *Let A be a commutative ring such that for each $a \in A$, there exist $x \in A$ and integers n,k with $n > k$, satisfying $a^n x = a^k$; and suppose that $\mathfrak{Q}(A) \supseteq \mathfrak{S}(A) \supseteq \mathfrak{P}(A)$. Then $\mathfrak{S}(A)$ is locally compact.*

Proof. For each $S \in \mathfrak{S}(A)$, pick $a \notin S$. Then a^k has a relative identity $a^{n-k}x$, and $a^k \notin S$.

THEOREM 4.9. *Let A be a commutative $\mathfrak{S}$-semi-simple ring such that for each $S \in \mathfrak{S}(A)$, there is an $a \notin S$ having a relative identity; and suppose that $\mathfrak{Q}(A) \supseteq \mathfrak{S}(A) \supseteq \mathfrak{P}(A)$. Then $\mathfrak{S}(A)$ is compact if and only if A has an identity.*

Proof. Sufficiency. See Corollary 3.4.

Necessity. For each $S \in \mathfrak{S}(A)$, choose an $a \notin S$ having a relative identity. The collection of $\mathfrak{C}\mathfrak{S}$-sets for these elements is an open cover of $\mathfrak{S}(A)$, which by compactness, may be reduced to a finite subcover, say $\{\mathfrak{C}\mathfrak{S}(a_0), \ldots, \mathfrak{C}\mathfrak{S}(a_n)\}$. By Lemma 4.4, there is an $e \in A$ which is a common relative identity for the set $\{a_0, \ldots, a_n\}$.

Now let $S \in \mathfrak{S}(A)$ be given. For some k, $a_k \notin S$. For all $x \in A$, $a_k ex - a_k x = 0 \in S$. Thus, $ex - x \in S$, since S is prime. By $\mathfrak{S}$-semi-simplicity, $ex = x$, *i. e.*, e is the identity of A.

Collecting results from 4.2, 4.8 and 4.9, we have:

COROLLARY 4.10 (cf. [1], Theorem 2.2 and [7], Theorem 4.1). *Let A be a commutative π-regular ring; and suppose that $\mathfrak{Q}(A) \supseteq \mathfrak{S}(A) \supseteq \mathfrak{P}(A)$. Then $\mathfrak{S}(A)$ is a locally compact Hausdorff space. If in addition A is $\mathfrak{S}$-semi-simple, then $\mathfrak{S}(A)$ is compact if and only if A has an identity. In particular, both of these conclusions are valid when A is a commutative regular ring (which is $\mathfrak{P}$-semi-simple, hence $\mathfrak{S}$-semi-simple).*

5. The prime ideals of an ideal. Let I be an ideal of a ring A, and let J be an ideal of I. It is not always true that J is also an ideal of the whole ring A. For example, let A be the ring of all polynomials in an indeterminate x over the integers; I, the ideal of A consisting of all elements of A with the constant term divisible by 6 and the coefficient of x by 2; J, the ideal of I consisting of all elements of I with the coefficient of x divisible by 4. Then J is *not* an ideal of A, since $(6+4x)x = 6x + 4x^2 \notin J$.

Now a condition on J which is weaker than the requirement that J be prime in I was shown by Johnson to be sufficient to insure that J be an ideal of A ([5], Lemma 2.1). However, we shall be interested only in prime ideals. Suppose J is prime in I, and let $j \in J$, $a \in A$ be given. Then for all $x \in I$, $(ja)x(ja) = j(axja) \in J$, so $ja \in J$; and $(aj)x(aj) = (ajxa)j \in J$, so $aj \in J$. Thus, J is an ideal of A.

In general, J will not be a *prime* ideal of A. For example, let A be the ring of integers; I, the ideal (2); and J, the ideal (6) of I.

We recall that the symbol $\mathfrak{R}(A)$ designates either of the spaces $\mathfrak{P}(A)$ or $\mathfrak{Q}(A)$. The first theorem below is concerned with enlarging ideals of $\mathfrak{R}(I)$ to ideals of $\mathfrak{R}(A)$. The result for the spaces $\mathfrak{P}(I)$ and $\mathfrak{P}(A)$ was recently published by Goldie ([3], Theorem 1). And the statement for the spaces $\mathfrak{Q}(I)$ and $\mathfrak{Q}(A)$ is almost contained in [6], Theorem 2.5. In both cases, however, the proofs are different from the one we present.

THEOREM 5.1. *Let I be any proper ideal of a ring A, and let $P \in \mathfrak{R}(I)$. Set $Q = \{a \in A\colon Ia \subseteq P\}$. Then $Q \in \mathfrak{R}(A)$, and $P = Q \cap I$.*

Proof (for the spaces $\mathfrak{Q}(I)$ and $\mathfrak{Q}(A)$). Let $a,b \in Q$. Then $I(a-b) \subseteq P$, whence $a - b \in Q$.

For every $x \in A$, we have $Ixa \subseteq Ia \subseteq P$; and since P is an ideal of A (as remarked above), we have $Iax \subseteq Px \subseteq P$. Hence $xa \in Q$ and $ax \in Q$. Therefore Q is an ideal of A.

Now let $c,d \notin Q$ be arbitrary. Then there exist $i,j \in I$ such that $ic, jd \notin P$. Since $P \in \mathfrak{Q}(I)$, and $ic, jd \in I$, there is a $z \in I$ for which $(ic)z(jd) \notin P$. Thus $I(czjd) \not\subseteq P$, so $c(zj)d \notin Q$. Hence $Q \in \mathfrak{Q}(A)$ [11].

Finally, if $p \in P$, then $Ip \subseteq P$, so $p \in Q \cap I$. And if $k \in I - P$, then there is an $x \in I$ such that $kxk \notin P$. Since $kx \in I$, we have $Ik \not\subseteq P$; so $k \notin Q$. Thus, $P = Q \cap I$.

An alternative approach to the result proved above is possible. We show first that $\{a \in A\colon Ia \subseteq P\} = \{a \in A\colon IaI \subseteq P\}$. Suppose that $Ia \not\subseteq P$, and let $i \in I$ satisfy $ia \notin P$. Then for some $x \in I$, $iaxia \notin P$. Since $xia \in I$, we have $IaI \not\subseteq P$. Thus, $\{a \in A\colon IaI \subseteq P\} \subseteq \{a \in A\colon Ia \subseteq P\}$; and the reverse inclusion is obvious. (Clearly we could show in the same manner that $\{a \in A\colon aI \subseteq P\} = \{a \in A\colon IaI \subseteq P\}$.)

The proof can then be carried through in substantially the same way, except that the proof that Q is a right ideal is now direct. Having obtained 5.1 for the spaces $\mathfrak{Q}(I)$ and $\mathfrak{Q}(A)$, we conclude that P is an ideal of A as a *corollary* (cf. [3]).

When A is commutative, we can also describe Q in another manner, which will be useful in the next section. Let c be any fixed element of $I-P$, and set $Q_c=\{a \in A\colon\ ca \in P\}$. Let $a \in Q_c$. Then $ca \in P$, so $Iac=Ica\subseteq IP\subseteq P$. Since $c \notin P$, it follows that $Ia\subseteq P$, *i. e.*, $a \in Q$. Hence $Q_c\subseteq Q$. But it is obvious that $Q\subseteq Q_c$. Consequently, $Q=Q_c$.

An alternative proof of Goldie's result that $P \in \mathfrak{P}(I)$ implies $Q \in \mathfrak{P}(A)$ for commutative A can be given by use of this fact. Let $P \in \mathfrak{P}(I)$, and let $a \notin Q=Q_c$, $b \in A$ be arbitrary. Since $ca, cb \in I$, and $ca \notin P$, the congruence $(ca)x\equiv cb \pmod P$ has a solution x (in I). Thus $c(ax-b) \in P$, so $ax\equiv b \pmod Q$. Hence $Q \in \mathfrak{P}(A)$.

THEOREM 5.2. *Let I be any proper ideal of a ring A, and let $\mathfrak{T}=\{Q \in \mathfrak{R}(A)\colon\ Q \not\supseteq I\}$. Then $\mathfrak{T}$ is an open subset of $\mathfrak{R}(A)$ that is homeomorphic to $\mathfrak{R}(I)$, under the mapping a defined by $a(Q)=Q\cap I$.*

Proof. For the spaces $\mathfrak{P}(I)$ and $\mathfrak{P}(A)$, this is part of a theorem of Kaplansky ([7], Theorem 3.1 (a); cf. also [3]). We consider then the spaces $\mathfrak{Q}(I)$ and $\mathfrak{Q}(A)$. By [11], Lemma 2, the mapping a is into $\mathfrak{Q}(I)$; and by Theorem 5.1, it is onto. The proof that a is one-to-one, continuous and open is identical with that given by Kaplansky.

Combining Theorem 5.2 with [2], Theorem 1.1, we have: $\mathfrak{R}(A)$ is the union of an open set homeomorphic to $\mathfrak{R}(I)$ and a closed set homeomorphic to $\mathfrak{R}(A/I)$.

6. Adjunction of an identity. Let A be a commutative ring without identity. If D is a commutative ring with identity such that A admits D as a ring of operators, then A may be imbedded in the ring $(A;D)$ with identity defined as follows (cf. [10], p. 87-88): Let $(A;D)=\{(a,d)\colon\ a \in A,\ d \in D\}$, and define operations in $(A;D)$ by $(a_1,d_1)+(a_2,d_2)=(a_1+a_2,d_1+d_2)$, $(a_1,d_1)\cdot(a_2,d_2)=(a_1a_2+d_1a_2+d_2a_1,\ d_1d_2)$. The identity of $(A;D)$ is the element $(0,1)$. The subset $A_0=\{(a,d)\colon\ d=0\}$ is easily seen to be an ideal of $(A;D)$ which is isomorphic to the given ring A. The quotient-ring $(A;D)/A_0$ is isomorphic to D. Thus, $A_0 \in \mathfrak{P}(A;D)$ (resp $\mathfrak{Q}(A;D)$) if and only if D is a field (resp. integral domain). More generally, it follows from the well known correspondence between ideals containing a given ideal and the ideals of the quotient-ring modulo this ideal, that there is a one-to-one correspondence between the ideals of D and the ideals of $(A;D)$ containing A_0. The latter have the form $\{(a,d) \in (A;D)\colon\ d\ \ K\}$, where K is an ideal of D. It is also easily verified directly that for every ideal J of $(A;D)$, the set $K_J=\{d \in D\colon (a,d) \in J$ for some $a \in A\}$ is an ideal of D.

Since $\mathfrak{R}(D)$ is homeomorphic to $\mathfrak{R}\big((A;D)/A_0\big)$, the remark at the end of section 5 now specialized to: $\mathfrak{R}(A;D)$ is the union of an open set homeomorphic to $\mathfrak{R}(A)$ and a closed set homeomorphic to $\mathfrak{R}(D)$.

If A has characteristic n, we may always choose D to be I_n, the ring of integers modulo (n). Now if $n\neq 0$, $\mathfrak{R}(I_n)$ is evidently a Hausdorff space. But when $n=0$, we are imbedding $\mathfrak{R}(A)$ in a space containing a subset homeomorphic to a non-Hausdorff space, which in certain instances might be undesirable. However, this choice can be avoided in many cases; for example, if A is an algebra over a field F, we may set $D=F$.

THEOREM 6.1. *If D is $\mathfrak{R}$-semi-simple, then each $P_0 \in \mathfrak{R}(A_0)$ is an intersection of ideals in $\mathfrak{R}(A;D)$. If A is also $\mathfrak{R}$-semi-simple, then $(A;D)$ is $\mathfrak{R}$-semi-simple.*

Proof. Let $Q=\{b \in (A;D)\colon (A_0)b\subseteq P_0\}$. By Theorem 5.1, $Q \in \mathfrak{R}(A;D)$, and $P_0=Q\cap A_0$. From the above discussion it follows easily that the $\mathfrak{R}$-semi-simplicity of D implies that A_0 is the intersection of the ideals of $\mathfrak{R}(A;D)$ which contain it, *i. e.*, that $\Delta\big(\mathfrak{R}(A;D)-\mathfrak{T}\big)=A_0$, where $\mathfrak{T}=\{S \in \mathfrak{R}(A;D)\colon\ S\not\supseteq A_0\}$.

The second statement follows from the equations:

$$\Delta\mathfrak{R}(A;D)=\Delta\mathfrak{T}\cap\Delta\big(\mathfrak{R}(A;D)-\mathfrak{T}\big)$$
$$=\Big(\bigcap_{S\in\mathfrak{T}}S\Big)\cap A_0=\bigcap_{S\in\mathfrak{T}}(S\cap A_0)=\Delta\mathfrak{R}(A_0)=(0)\,.$$

It is a familiar fact that the one-point compactification of a locally compact Hausdorff space is Hausdorff. In the next theorem, we see that for certain spaces $\mathfrak{R}(A)$, a "k-point compactification" (with k finite) will do just as well.

THEOREM 6.2. *Let A be a commutative ring having characteristic $n=p_1^{h_1}\dots p_k^{h_k}$, where the p_i's are distinct primes, with the properties*: (1) *$\mathfrak{R}(A)$ is Hausdorff*; (2) *for each $P \in \mathfrak{R}(A)$, there is an $a \notin P$ having a relative identity. Then $\mathfrak{R}(A;I_n)$ is Hausdorff.*

Proof. We shall show that the condition of Theorem 4.1 (for commutative rings) is satisfied by $\mathfrak{R}(A;I_n)$. The method of verification depends on whether both, one or neither of the ideals under consideration contains A_0. The elements of $\mathfrak{R}(A;I_n)$ containing A_0 have the form

$$Q_i=\{(a,d) \in (A;I_n)\colon\ d \in (p_i)/(n)\}\qquad (i=1,\dots,k)\,.$$

Given Q_i, Q_j, $i\neq j$, let d_i, d_j be the images of the integers $p_i^{h_i}$, $n/p_i^{h_i}$, respectively, in I_n. Then $(0,d_j)\notin Q_i$, $(0,d_i)\notin Q_j$, and $(0,d_j)\cdot(0,d_i)=(0,d_jd_i)=(0,0) \in \Delta\mathfrak{R}(A;I_n)$.

Next, let $Q \in \mathfrak{R}(A;I_n)$ be such that $Q \not\supset A_0$. Then $Q \cap A_0 \in \mathfrak{R}(A_0)$. By hypothesis, there is an $(a,0) \notin Q \cap A_0$ having a relative identity e. Clearly $(a,0) \notin Q$, and for any i, $(e,-1) \notin Q_i$. Furthermore, $(a,0)\cdot(e,-1) = (ae-a,0) = (0,0) \in \Delta\mathfrak{R}(A;I_n)$.

Finally, let $S_1, S_2 \in \mathfrak{R}(A;I_n)$ be distinct ideals such that $S_1 \not\supset A_0$, $S_2 \not\supset A_0$. Then, from Theorem 5.2, $S_1 \cap A_0$ and $S_2 \cap A_0$ are distinct elements of $\mathfrak{R}(A_0)$. Since $\mathfrak{R}(A_0)$ is Hausdorff, it follows from Theorem 4.1 that there exist $(a,0),(b,0) \in A_0$ such that $(a,0) \notin S_1 \cap A_0$, $(b,0) \notin S_2 \cap A_0$, and $(a,0)\cdot(b,0) \in \Delta\mathfrak{R}(A_0)$. Clearly $(a,0) \notin S_1$ and $(b,0) \notin S_2$. Now let $\mathfrak{T} = \{S \in \mathfrak{R}(A;I_n):\ S \not\supset A_0\}$. Then $\Delta\mathfrak{R}(A;I_n) = \Delta\mathfrak{T} \cap \Delta\big(\mathfrak{R}(A;I_n) - \mathfrak{T}\big) \supseteq \big(\bigcap_{S\in\mathfrak{T}} S\big) \cap A_0 = \bigcap_{S\in\mathfrak{T}} (S \cap A_0) = \Delta\mathfrak{R}(A_0)$. Hence $(a,0)\cdot(b,0) \in \Delta\mathfrak{R}(A;I_n)$.

This includes all the possibilities, so the proof is complete.

We conclude with a more specialized result which has applications to problems about rings of continuous functions.

THEOREM 6.3. *Let A be a commutative algebra over a field F such that for each $P \in \mathfrak{P}(A)$, A/P is operator-isomorphic to F. Then for each $Q \in \mathfrak{P}(A;F)$, $(A;F)/Q$ is operator-isomorphic to F, under a mapping which sends the coset of Q containing (a,f) into $a[Q \cap A_0] + f$* (³). *Furthermore, if F is a topological field and A is a ring of continuous functions from $\mathfrak{P}(A)$ to F, then $(A;F)$ is a ring of continuous functions from $\mathfrak{P}(A;F)$ to F.*

Proof. It has already been noted that $(A;F)/A_0 \cong F$; and it is evident that in this case the isomorphism has the stated properties. Now consider any $Q \in \mathfrak{P}(A;F) - \{A_0\}$, let $P_0 = Q \cap A_0$, and define a mapping $\alpha\colon (A;F) \to F$ by $\alpha(a,f) = a[P_0] + f$. It is easily verified that α is a homomorphism onto. Let $(c,0) \notin P_0$ be fixed. The kernel of α is

$$\begin{aligned}\{(a,f) \in (A;F):\ a[P_0]+f=0\} &= \{(a,f):\ a[P_0]c[P_0]+fc[P_0]=0\}\\ &= \{(a,f):\ (ac+fc)[P_0]=0\} = \{(a,f):\ (a,f)\cdot(c,0) \in P_0\} = Q\end{aligned}$$

(see the remarks following Theorem 5.1). Hence $(A;F)/Q$ is operator-isomorphic to F; and the mapping clearly has the form indicated.

Now suppose A is a ring of continuous functions from $\mathfrak{P}(A)$ to F. Then A is semi-simple. Therefore, by Theorem 6.1, $(A;F)$ is semi-simple. It follows that $(A;F)$ is a ring of functions from $\mathfrak{P}(A;F)$ to F. Since $\mathfrak{P}(F)$ consists of only one element, $\mathfrak{P}(A;F)$ is homeomorphic to the one-point compactification of $\mathfrak{P}(A)$. Continuity then follows from the functional relations $(a,f)(A_0) = f$, and $(a,f)(Q) = a[Q \cap A_0] + f$ when $Q \in \mathfrak{P}(A;F) - \{A_0\}$, which are valid for all $(a,f) \in (A;F)$.

(³) For simplicity, we let $a[Q \cap A_0]$ denote the element of F which corresponds to the image of $(a,0)$ in $A_0/(Q \cap A_0)$.

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