On Sierpiński sets in groups

by

E. G. **Straus** (Los Angeles, California)

In a previous paper [1] we defined Sierpiński sets as sets congruent to each maximal proper subset and proved that a group which contains a free subgroup of rank 2 contains Sierpiński sets. (Here "congruence" is given by the group operation).

In this note we wish to prove the following converse, which we had stated as a conjecture in [1].

THEOREM. *If a group G contains a Sierpiński set then it contains a free subgroup of rank 2.*

We first prove the following:

LEMMA. *If S is a Sierpiński set in G then the multiplications which map it into maximal proper subsets are multiplications on the same side.*

Proof. Assume there are elements a, b of S and φ, ψ of G so that

$$\varphi S = S - \{a\}, \qquad S\psi = S - \{b\}.$$

Then

1. $\varphi x \in S$, $x\psi \in S$ for every $x \in S$,
2. $\varphi^{-1}x \in S$ for every $x \neq a$, $x \in S$ and $\varphi^{-1}a \notin S$,
3. $x\psi^{-1} \in S$ for every $x \neq b$, $x \in S$ and $b\psi^{-1} \notin S$.

Hence $a\psi^{-1} \in S$ and since $a\psi^{-1} \neq a$ also $\varphi^{-1}(a\psi^{-1}) = (\varphi^{-1}a)\psi^{-1} \in S$. But this implies $\big((\varphi^{-1}a)\psi^{-1}\big)\psi = \varphi^{-1}a \in S$; a contradiction.

Proof of the Theorem. Let S be a Sierpiński set in G and assume that the mappings into maximal proper subsets are left multiplications. Then any right multiplication maps S into a Sierpiński set. In particular we may therefore assume that the identity, 1, belongs to S.

Let $\varphi S = S - \{1\}$. Then $\varphi 1 \in S$ and there exists a ψ so that $\psi S = S - \{\varphi\}$. We shall prove that φ, ψ generate a free subgroup of rank 2.

First, φ and ψ are of infinite order. For $\varphi^n = 1$, $n > 0$ implies $\varphi^{-1} = \varphi^{n-1} \in S$; a contradiction. Similarly $\psi^n \in 1$, $n > 0$ implies $\psi^{-1}\varphi = \psi^{n-1}\varphi \in S$; a contradiction.

Now, if φ, ψ are not independent then they must satisfy a relation of the form

$$(1) \qquad \varphi^{a_1}\psi^{a_2}\varphi^{a_3}\ldots\psi^{a_k} = 1 \quad (a_i \neq 0).$$

We may normalize (1) by the following conditions:

(i) k is minimal,

(ii) $\sum_{i=1}^{k} |a_i|$ is minimal for the given k,

(iii) $a_1 > 0$ (otherwise take inverses and multiply on the left by $\varphi^{|a_1|}$ and on the right by $\varphi^{-|a_1|}$).

Now every right sub-word of the expression in (1) belongs to S. We prove this by induction:

The first right sub-word is either ψ or ψ^{-1}, both of which belong to S.

If the right sub-word, w, belongs to S then the next sub-word is one of the following

$$\varphi w, \ \psi w, \ \varphi^{-1}w, \ \psi^{-1}w.$$

The first two certainly belong to S. By the minimality conditions (i), (ii) we have $w \neq 1$. Hence $\varphi^{-1}w \in S$. If $\psi^{-1}w \notin S$ then $w = \varphi$. Thus, if $\psi^{-1}w$ is the next right sub-word, we have to compare (1) with

$$(2) \qquad \varphi w^{-1} = \varphi\psi^{-a_k}\psi^{-a_{k-1}}\ldots\varphi^{-a_{l-1}}\psi^{a_l'} = 1.$$

Where $|a_l'| < |a_l|$. If $l > 2$ then (2) violates the minimality condition (i); and if $b = 2$ then

$$1 + |a_k| + \ldots + |a_3| + |a_2'| < 1 + |a_k| + \ldots + |a_2| \leqslant |a_1| + |a_2| + \ldots + |a_k|,$$

in contradiction to condition (ii).

But if every right sub-word of (1) belongs to S we have in particular

$$\varphi^{-1} = \varphi^{a_1 - 1}\psi^{a_2}\ldots\psi^{a_k} \in S,$$

a contradiction.

COROLLARY. *Let G be a group of fixed-point-free transformations on a space Σ. Then Σ contains Sierpiński sets under the transformations of G if and only if G contains a free subgroup of rank 2.*

Proof. If $S \subset \Sigma$ is a Sierpiński set and $p \in S$, then so is $S' = S \cap G_p$, where G is the set of images of p under G.

Since G is fixed-point-free we can now associate to each $p' \in S'$ the unique $\varphi \in G$ for which $p' = \varphi p$. This correspondence constructs a Sierpiński set in G.

Reference

[1] E. G. Straus, *On a problem of W. Sierpiński on the congruence of sets*, Fund. Math. 44 (1957), p. 75-81.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

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