Contribution to the theory of Saks spaces

by

W. Orlicz (Poznań)

1. In the sequel X will denote a linear space in which a norm $\| \ \|$ is defined; this (fixed) norm will be called *fundamental*. Besides this, one or more norms will be defined in X, and they will all be called *starred norms*. All norms are supposed to be of B-type (shortly: to be B-norms), *i. e.*, to be homogeneous, and whenever we speak about normed spaces without further specification, the norms will be supposed to be of this type.

We shall set

$$X_s = \mathop{E}_{x} \{x \in X, \ \|x\| \leqslant 1\}.$$

Now let a starred norm $\| \ \|^*$ be defined in X; we define in X_s the distance

$$(*) \qquad d(x_1, x_2) = \|x_1 - x_2\|^* \quad \text{where} \quad x_1, x_2 \in X_s.$$

If $d(x_n, x_0) = \|x_n - x_0\|^* \to 0$ and $\|x_n\| = O(1)$, where $x_n, x_0 \in X$, the sequence $\{x_n\}$ will be said to be *ω-convergent* to x_0, in symbols: $x_n \xrightarrow{\omega} x_0$ as $n \to \infty$. The space X_s, provided with the distance $(*)$ (given a starred norm), considered as a metric space (in which therefore the ω-convergence is defined) will be denoted by $X_s(\omega)$; if this space is complete, it will be called the *Saks space*. Let us notice that we do not suppose the space X to be complete either with respect to the fundamental norm or to the starred one.

If two norms $\| \ \|_1$ and $\| \ \|_2$ are defined in X and $x_n \in X$, $\|x_n\|_1 \to 0$ as $n \to \infty$ implies $\|x_n\|_2 \to 0$, then the norm $\| \ \|_1$ is called *non-weaker than* $\| \ \|_2$ *in* X. If $\| \ \|_1$ is non-weaker than $\| \ \|_2$ and *vice versa* the norms are called *equivalent in* X. The definition of non-weaker and equivalent norms in X_s is evident. Obviously, equivalent norms in X_s are not necessarily equivalent in X.

The norms $\| \ \|_1$ and $\| \ \|_2$ are equivalent in X if and only if two positive constants k_1 and k_2 exist such that $k_1\|x\|_1 \geqslant \|x\|_2 \geqslant k_2\|x\|_1$ for $x \in X$. The first of these inequalities is necessary and sufficient in order that $\| \ \|_1$ be non-weaker than $\| \ \|_2$.

The space $X_s(\omega)$ [the Saks space] will be called *equivalent* to the normed [Banach] space if the fundamental norm is equivalent in X to the starred norm under consideration. This nomenclature is justified by the fact that if we define in X another starred norm equivalent to the former one and equal to the fundamental norm, then X_s is identical with the sphere $\|x\| \leqslant 1$ in a normed space in which the metric-topology $(*)$ is identical to the topology induced by the norm $\| \ \|$.

In section 1 some supplements to the results of [5], [6], and [9] [1]) are given. In section 2 linear functionals over the space of bounded functions (in an infinite interval) are investigated, the starred norm being defined in an adequate manner. This space supplies also a skilful example of a Saks space for which the space X is not complete either with respect to fundamental or to the starred norm. In section 3 I give some applications of general theorems to the examination of the continuity of distributive operations.

1.1. We begin with the following relation between the fundamental and the starred norm:

If $X_s(\omega)$ is a Saks space, then X is complete with respect to the norm

$$(*) \qquad \|x\|_0 = \sup(\|x\|, \|x\|^*).$$

(See [6], p. 1).

1.2. *Let $X_s(\omega)$ be a Saks space. A necessary and sufficient condition in order that the fundamental norm be non-weaker in X than the starred norm is that X be complete with respect to the norm $\| \ \|$.*

In order that the starred norm be non-weaker in X than the fundamental norm it is necessary and sufficient that the space X be complete with respect to the norm $\| \ \|^*$.

This follows from 1.1 (see [6], p. 2, where the first part of the theorem is proved; the proof of the second proceeds analogously).

In connection with the last theorem let us add the following remark. If X is complete with respect to the starred norm and

$$x_n \in X_s(\omega), \qquad x_n \xrightarrow{\omega} x_0 \quad \text{implies} \quad \varliminf_{n\to\infty} \|x_n\| \geqslant \|x_0\|,$$

then $X_s(\omega)$ is a Saks space.

It follows that, if X is complete with respect to the starred norm and if this norm is non-weaker in X than the fundamental norm, then $X_s(\omega)$ is a Saks space.

Let X be the linear space composed of functions having a continuous derivative in $\langle a, b\rangle$, and vanishing for $t = a$. We define $\|x\| = \sup\limits_{\langle a,b\rangle} |x'(t)|$,

[1]) The number in brackets refer to the bibliography at the end of this paper.

as the fundamental norm, and $\|x\|^* = \sup_{\langle a,b\rangle} |x(t)|$ as the starred one. X is obviously complete with respect to the fundamental norm, and this norm is non-weaker in X than the starred one. Let $y(t)$ be a function having a discontinuous derivative in $\langle a,b\rangle$, let $y(a)=0$ and let $|y'(t)|\leqslant 1$ in $\langle a,b\rangle$. Let us choose $x_n \in X_s(\omega)$ such that $x_n'(t)\to y'(t)$ for $t \in \langle a,b\rangle$.

Then the Cauchy condition $\|x_n - x_m\|^* \to 0$ as $m,n\to\infty$ is satisfied, $\|x_n - y\|^*\to 0$ as $n\to\infty$ and $y \notin X$. This example shows that if X is complete with respect to the fundamental norm and this norm is non-weaker in X than the starred norm, then $X_s(\omega)$ is not necessarily a Saks space.

1.3. Let U be a distributive operation from X to a normed space Y. U is called *linear* in $X_s(\omega)$ or, more precisely, $(X_s(\omega), Y)$-*linear* if $x_n \in X_s(\omega)$, $x_0 \in X_s(\omega)$, $x_n \xrightarrow{\omega} x_0$ implies $U(x_n)\to U(x_0)$. If Y is the linear space of real numbers, then the term *linear operation* in $X_s(\omega)$ will be replaced, as usual, by the term *linear functional* in $X_s(\omega)$.

We shall denote by Ξ or Ξ^* or Ξ_0 respectively the space of linear functionals (*i. e.*, the conjugate space) over the normed space X, $\|\ \|$, or X, $\|\ \|^*$ or X, $\|\ \|_0$, where $\|\ \|_0$ is defined by formula 1.1 $(*)$. In these spaces the norms of the functional ξ will be introduced by the usual definition, and will be denoted by $\|\xi\|$, $\|\xi\|^*$, $\|\xi\|_0$ when ξ is in Ξ, Ξ^*, Ξ_0 respectively. All these spaces (with the usual definitions of addition and multiplication by scalars) supplied with the corresponding norms are Banach spaces.

Evidently $\Xi\subset\Xi_0$, $\|\xi\|_0\leqslant\|\xi\|$, $\Xi^*\subset\Xi_0$, $\|\xi\|_0\leqslant\|\xi\|^*$.

Let $\Xi_s(\omega)$ denote the linear space of linear functionals in $X_s(\omega)$ (this is the conjugate space to $X_s(\omega)$). If $\xi\in\Xi_s(\omega)$ then the norm of the functional ξ (without a further indication of the related conjugate space) will always mean the norm $\|\xi\|_0$.

Moreover the inclusion $\Xi^*\subset\Xi_s(\omega)$ is satisfied.

1.31. (α) *$\Xi_s(\omega)$ is a closed linear subspace of Ξ_0.*

(β) *If X is complete with respect to the fundamental norm and $X_s(\omega)$ is a Saks space, then $\Xi_s(\omega)$ is a closed linear subspace of Ξ, and the norms $\|\xi\|_0$ and $\|\xi\|$ are equivalent in $\Xi_s(\omega)$.*

(γ) *If X is complete with respect to the starred norm and $X_s(\omega)$ is a Saks space, then $\Xi_s(\omega)$ is identical with Ξ^*, and the norms $\|\xi\|_0$ and $\|\xi\|^*$ are equivalent in $\Xi_s(\omega)$.*

Ad (α). The inclusion $\Xi_s(\omega)\subset\Xi_0$ is evident. The closedness of $\Xi_s(\omega)$ in Ξ_0 results from the fact that the relation $\|\xi_n-\xi\|_0\to 0$ as $n\to\infty$, $\xi_n\in\Xi_s(\omega)$, is equivalent to the uniform convergence of the sequence $\xi_n(x)$ to $\xi(x)$ in the set $\|x\|\leqslant 1$, $\|x\|^*\leqslant 1$.

Ad (β). The inclusion $\Xi_s(\omega)\subset\Xi$ results by 1.2. By 1.2 $\|x\|\leqslant 1$ implies $\|x\|^*\leqslant k$, where we may suppose that $k>1$. Since

$$\sup\left|\frac{\xi}{k}(x)\right| = \left\|\frac{\xi}{k}\right\|$$

(the supremum being taken over the set $\|x\|\leqslant 1$) and

$$\sup\left|\frac{\xi}{k}(x)\right| = \sup\left|\xi\left(\frac{x}{k}\right)\right|\leqslant\|\xi\|_0,$$

we get $\|\xi\|\leqslant k\|\xi\|_0$. On the other hand, the inequality $\|\xi_0\|\leqslant\|\xi\|$ is fulfilled.

Ad (γ). The inclusion $\Xi^*\subset\Xi_s(\omega)$ is valid without any hypotheses concerning the norms $\|\ \|$ and $\|\ \|^*$. Let $\xi\in\Xi_s(\omega)$ and let $\|x_n\|^*\to 0$. By 1.2 $\|x_n\|\to 0$, whence $\xi(x_n)\to 0$, and this implies $\xi\in\Xi^*$. Thus $\Xi^*=\Xi_s(\omega)$. The equivalence of the norms $\|\xi\|_0$ and $\|\xi\|^*$ in $\Xi_s(\omega)$ may be proved in the same way as for the case (β).

1.32. (α) *Every linear functional in $X_s(\omega)$ may be represented in the form*

$(*)$ $$\xi=\xi_1+\xi_2 \quad \textit{where} \quad \xi_1\in\Xi,\ \xi_2\in\Xi^*,$$

moreover

$(**)$ $$\|\xi\|_0=\|\xi_1\|+\|\xi_2\|^*.$$

Since, by 1.31 (α), ξ is continuous with respect to the norm $\|\ \|_0$, the representation $(*)$ and formula $(**)$ result from a general theorem on the representation of distributive functionals, continuous with respect to two pseudonorms (compare [2], p. 139). Representation $(*)$ is, of course, not unique. If $X_s(\omega)$ is separable, then among all representations $(*)$ satisfying the condition $(**)$ there are two,

$$\xi=\bar{\xi}_1+\bar{\xi}_2,\qquad \xi=\underline{\xi}_1+\underline{\xi}_2,$$

where $\|\bar{\xi}_1\|$ (or $\|\underline{\xi}_1\|$) is equal to the supremum M (or the infimum m) of all norms $\|\xi_1\|$ of functionals appearing in $(*)$ under the condition $(**)$. Let $\|\xi_1^n\|\to m$, as $n\to\infty$, $\xi=\xi_1^n+\xi_2^n$, $\|\xi_1^n\|+\|\xi_2^n\|^*=\|\xi\|_0$. Since $\|\xi_2^n\|^*\leqslant\|\xi\|_0$, and since the separability of $X_s(\omega)$ implies the separability of X, $\|\ \|^*$, there must exist a subsequence $\xi_2^{n_i}$ converging for every x to $\xi_2\in\Xi^*$.

Let

$$\xi_1(x)=\lim_{i\to\infty}\left(\xi(x)-\xi_2^{n_i}(x)\right)=\lim_{i\to\infty}\xi_1^{n_i}(x).$$

Obviously $\xi=\xi_1+\xi_2$; $\xi_1^{n_i}\in\Xi$, $\|\xi_1^{n_i}\|\leqslant\|\xi\|_0$, implies $\xi_1\in\Xi$. Moreover, $\|\xi\|_0 =\|\xi_1\|+\|\xi_2\|^*$, $m=\underline{\lim}_i\|\xi_1^{n_i}\|\geqslant\|\xi_1\|\geqslant m$. For M the argument is similar. Besides the representation $(*)$ with condition $(**)$, also a representation with the norms surpassing $\|\xi\|_0$ may be useful.

(β) *If there exist such* $\xi_n' \in \Xi^*$ *for* $\xi \in \Xi_s(\omega)$ *that* $\|\xi_n'-\xi\| \to 0$ *as* $n \to \infty$, *then for this* ξ *and every* $\varepsilon > 0$ *a representation* $(*)$ *exists with* $\|\xi_1\| < \varepsilon$.

(β′) *If* ξ *is a distributive functional on* X *and for every* $\varepsilon > 0$ *there exists a representation* $(*)$ *in which* $\|\xi_1\| < \varepsilon$, *then* $\xi \in \Xi_s(\omega)$.

Ad (β). Since $\xi_n' \in \Xi_s(\omega)$, it suffices to set $\xi_1 = \xi - \xi_n'$, n being sufficiently large.

Ad (β′). There exist, for $n = 1, 2, \ldots$, representations $\xi = \xi_1^n + \xi_2^n$ such that $\|\xi - \xi_2^n\|_0 \leqslant \|\xi - \xi_2^n\| < 1/n$, and since $\xi_2^n \in \Xi_s(\omega)$, it is sufficient to make use of 1.31 (α).

1.33. (α) *Let* $X_s(\omega)$ *be a Saks space. Let* $\xi_n \in \Xi_s(\omega)$ *for* $n = 1, 2, \ldots$ *The sequence* $\xi_n(x)$ *is bounded for every* $x \in X$ *if and only if* $\|\xi_n\|_0 = O(1)$.

(β) *Let* U_n *be* $(X_s(\omega), Y)$*-linear operations,* Y *being a linear normed space. Then the sequence* $\|U_n(x)\|$ *is bounded for every* $x \in X$ *if and only if the following inequality is satisfied*:

$$(*) \qquad \|U_n(x)\| \leqslant k\|x\|_0 \quad \text{for} \quad x \in X.$$

Every functional $\xi \in \Xi_s(\omega)$ is in Ξ_0; by 1.1 and by the Banach-Steinhaus theorem the boundedness of $\xi_n(x)$ for every $x \in X$ implies the existence of a constant $k > 0$ such that

$$(**) \qquad |\xi_n(x)| \leqslant k\|x\|_0 \quad \text{for} \quad x \in X.$$

Therefore $\|\xi_n\|_0 \leqslant k$. Conversely, $\|\xi_n\|_0 = O(1)$ implies $(**)$. The proof for (β) is similar.

Remark. For the proof of necessity the hypothesis of the completeness of $X_s(\omega)$ may be replaced by that of the completeness of X, $\| \ \|_0$, which is weaker.

As a consequence of the above theorem it follows that: Every sequence of linear functionals (operations) in the Saks space $X_s(\omega)$, bounded everywhere, is locally uniformly bounded in $X_s(\omega)$.

By $(**)$ it follows for $\|x - x_0\|^* \leqslant 1$, $x, x_0 \in X_s$, that

$$|\xi_n(x)| \leqslant |\xi_n(x - x_0)| + |\xi_n(x_0)| \leqslant 2k + \sup_n |\xi_n(x_0)|.$$

Analogously for the sequence U_n.

Now the question arises under which hypotheses concerning $X_s(\omega)$ the boundedness of $\xi_n(x)$ for every x, where $\xi_n \in \Xi_s(\omega)$, implies the uniform boundedness of $\xi_n(x)$ in X_s.

It is easy to remark that a necessary condition is that the fundamental norm be non-weaker in X than the starred one. If $X_s(\omega)$ is a Saks space, then this condition is also sufficient.

If the fundamental norm is not non-weaker in X than the starred norm, then there exists a sequence $x_n \in X$ such that $\|x_n\| \to 0$, $\|x_n\|^* \to \infty$ for $n \to \infty$.

Let us choose $\xi_n \in \Xi^*$ so as $\|\xi_n\|^* = 1$, $\xi_n(x_n) = \|x_n\|^*$. Then $\xi_n \in \Xi_s(\omega)$, $|\xi_n(x)| \leqslant \|x\|^*$ for $x \in X$, but the sequence $\xi_n(x)$ is not uniformly bounded in X_s.

The sufficiency follows directly from 1.2 and 1.31 (β).

1.4. In this section

$\boldsymbol{K}_1$ will denote the class of all normed linear spaces,

$\boldsymbol{K}_2$ will denote the class of all normed separable linear spaces,

$\boldsymbol{K}_3$ will denote the class constituted of the space of reals.

The following properties of the space $X_s(\omega)$ will be needed:

(A, $\boldsymbol{K}_i$) (*for* i *equal* 1 *or* 2). *Let* U *be a distributive operation from* X *to* Y *where* $Y \in \boldsymbol{K}_i$. *Suppose that for every functional* η *linear on* Y *the functional* $\eta(U(x))$ *is in* $\Xi_s(\omega)$. *Then the operation* U *is continuous in* $X_s(\omega)$.

(B, $\boldsymbol{K}_i$) (*for* $i = 1, 2, 3$). *Let* U_n *be a sequence of linear operations from* $X_s(\omega)$ *to the normed space* $Y \in \boldsymbol{K}_i$. *Suppose that*

$$\lim_{n\to\infty} U_n(x) = U(x)$$

for every $x \in X$. *Then the operations* U_n *are equicontinuous in* $X_s(\omega)$.

If, in the formulation of the property (A, $\boldsymbol{K}_i$) (for $i = 1, 2$), the words "for every functional linear on Y" are replaced by "for every $\eta \in H_n$ where H_n is a fundamental set of linear functionals on Y" [2]), the obtained property will be denoted as (A′, $\boldsymbol{K}_i$).

We shall complete here on some points the investigations of [9] concerning the interrelations of the above properties.

(α) *If* X *is complete with respect to the norm* $\| \ \|_0$ *(whence in particular if* $X_s(\omega)$ *is a Saks space) then* (A, $\boldsymbol{K}_2$) *implies* (B, $\boldsymbol{K}_1$).

(β) *Every space* $X_s(\omega)$ *having the property* (B, $\boldsymbol{K}_3$) *has the property* (A, $\boldsymbol{K}_2$).

Ad (α). Let $X_s(\omega)$ have the property (A, $\boldsymbol{K}_2$) and let ξ_n be a sequence of functionals of $\Xi_s(\omega)$ convergent in the entire space X. By 1.33 (α)

$$(*) \qquad |\xi_n(x)| \leqslant k\|x\|_0 \quad \text{when} \quad x \in X.$$

[2]) A set H_n of functionals linear over B-normed space Y is called *fundamental* if $\|\eta\| \leqslant C < \infty$ for $\eta \in H_n$ and $\sup_{\eta \in H_n} |\eta(y)| \geqslant c\|y\|$ for every $y \in Y$, the constant $c > 0$ being independent of y.

Let $x_n \in X_s$, $x_n \xrightarrow{\omega} 0$; choose an increasing sequence of indices so that $\xi_{k_{n-1}}(x_{k_n}) \to 0$ as $n \to \infty$. Then we define an operation V from X to the space of the sequences convergent to 0 setting

$$V(x) = \{\xi_{k_n}(x) - \xi_{k_{n-1}}(x)\}.$$

Since every functional η linear over the space of the sequences converging to 0 is of the form $\eta(y) = \sum_1^\infty a_n t_n$ where $\sum_1^\infty |a_n| < \infty$, $y = \{t_n\}$, we obtain

$$\eta(V(x)) = \sum_{n=1}^\infty a_n(\xi_{k_n}(x) - \xi_{k_{n-1}}(x)).$$

From $\xi_{k_n} \in \Xi_s(\omega)$ and $(*)$ it follows that $\eta(V(x))$ is a linear functional on $X_s(\omega)$, whence by (A, $\boldsymbol{K}_2$) the operation $V(x)$ is continuous in $X_s(\omega)$. Hence $\xi_{k_n}(x_{k_n}) - \xi_{k_{n-1}}(x_{k_n}) \to 0$, which implies

$$\xi_{k_n}(x_{k_n}) = \xi_{k_n}(x_{k_n}) - \xi_{k_{n-1}}(x_{k_n}) + \xi_{k_{n-1}}(x_{k_n}) \to 0.$$

Since a similar argument may be applied to every subsequence ξ_{l_n}, we have proved that every partial sequence $\xi_{l_n}(x_{l_n})$ contains a subsequence convergent to 0; therefore $\xi_n(x_n) \to 0$, which implies that ξ_n are equicontinuous in $X_s(\omega)$.

Now let U_n be $(X_s(\omega), Y)$-linear operations, let $U_n(x) \to U(x)$ for $x \in X$, and let $x_n \in X_s$, $x_n \xrightarrow{\omega} 0$. Let us choose, as before, an increasing sequence of indices so that $U_{k_{n-1}}(x_{k_n}) \to 0$, and then let us choose functionals η_n linear on Y such that

$$(*) \qquad \|\eta_n\| = 1, \qquad \eta_n(U_{k_n}(x_{k_n}) - U_{k_{n-1}}(x_{k_n})) = \|U_{k_n}(x_{k_n}) - U_{k_{n-1}}(x_{k_n})\|.$$

Since

$$|\eta_n(U_{k_n}(x) - U_{k_{n-1}}(x))| \leqslant \|\eta_n\| \, \|U_{k_n}(x) - U_{k_{n-1}}(x)\| \to 0,$$

we see from what it has already been proved that the functionals $\eta_n(U_{k_n}(x) - U_{k_{n-1}}(x))$ are equicontinuous in $X_s(\omega)$, whence $\|U_{k_n}(x_{k_n}) - U_{k_{n-1}}(x_{k_n})\| \to 0$. As for functionals, it follows that $U_{k_n}(x_{k_n}) \to 0$, and in consequence $U_n(x_n) \to 0$ as $n \to \infty$.

Ad (β). The proof is identical to that of [9], p. 61, since the argument applied there does not use the completeness of $X_s(\omega)$. The same reasoning shows that (B, $\boldsymbol{K}_3$) implies the property (A′, $\boldsymbol{K}_2$).

By the foregoing considerations and since (A′, $\boldsymbol{K}_2$) implies (A, $\boldsymbol{K}_2$) it follows that

(γ) *If $X_s(\omega)$ is a Saks space, then the properties* (A, $\boldsymbol{K}_2$), (A′, $\boldsymbol{K}_2$) *and* (B, $\boldsymbol{K}_2$) *are equivalent.*

(δ) *If the space $X_s(\omega)$ is separable,* (A, $\boldsymbol{K}_2$) *implies* (A, $\boldsymbol{K}_1$).

Let the sequence $x_1, x_2, \ldots$ be dense in $X_s(\omega)$. If $\eta(U(x)) \in \Xi_s(\omega)$ for every functional η linear on Y, then by the well-known argument, $U(x)$ may be approximated by linear combinations of the elements $U(x_i)$, whence the range of the operation U lies in the separable linear closed space spanned upon the elements $U(x_i)$.

In virtue of the property (A, $\boldsymbol{K}_2$) the operation U is $(X_s(\omega), Y)$-linear, whence the property (A, $\boldsymbol{K}_1$) is fulfilled.

1.41. The space $X_s(\omega)$ may have the property (A, $\boldsymbol{K}_1$) without being complete. This is the case, for example, when $X_s(\omega)$ is equivalent to a normed space; in this case $\Xi_s(\omega) = \Xi$. Suppose that U is a distributive operation from X to a normed space Y, and let $\eta(U(x)) \in \Xi_s(\omega)$ for every functional linear on Y. The operation U is continuous with respect to the norm $\| \;\|$. It is sufficient to prove this for $x = 0$. In the contrary case there must exist a sequence x_n such that $\|x_n\| \to 0$, $\|U(x_n)\| \to \infty$. In the space H conjugate to Y let us define a sequence $\varrho_n(\eta) = \eta(U(x_n))$ of linear functionals. Since $\eta(U(x_n)) \to 0$ as $\eta \in H$, we infer that $|\eta(U(x_n))| \leqslant C$ in the sphere $\|\eta\| \leqslant 1$, whence $\|U(x_n)\| \leqslant C$, which is contradictory. In the case we are considering now the property (B, $\boldsymbol{K}_1$) is not fulfilled in general, although, on the other hand, there exist $X_s(\omega)$ equivalent to normed spaces, non-complete, for which (B, $\boldsymbol{K}_1$) is satisfied.

For applications of Saks spaces it would be interesting to establish, in terms of metric properties of the space $X_s(\omega)$, the sufficient and necessary conditions for $X_s(\omega)$ to have the property (A, $\boldsymbol{K}_i$) or (B, $\boldsymbol{K}_i$). Some sufficient conditions, called the conditions (Σ_1) and (Σ_2), have been given in [5] (see also [9]); we shall use them in this paper without further references.

1.5. Let Y be a normed space. Let us denote by

$$T(Y) = Y \times Y \times \ldots$$

the space of all sequences $u = \{y_n\}$ with terms from Y, with the usual definitions of addition and multiplication by scalars. By $T_b(Y)$, $T_c(Y)$ or $T_0(Y)$ we shall denote the subspace of $T(Y)$ composed of sequences that are bounded, convergent or convergent to 0, respectively. Let $\|u\| = \sup \|y_n\|$; with this norm $T_b(Y)$, $T_c(Y)$, and $T_0(Y)$ are normed spaces; if Y is a Banach space, so are $T_b(Y)$, $T_c(Y)$ and $T_0(Y)$; the spaces $T_c(Y)$, $T_0(Y)$ are separable if and only if the space Y has this property.

1.51. *Let the Saks space $X_s(\omega)$ have the property* (A, $\boldsymbol{K}_2$). *Let U_n be $(X_s(\omega), Y)$-linear operations. Suppose that for every $x \in X$ there exists a representation*

$$(*) \qquad x = x' + x'',$$

such that $\{U_n(x')\} \in T \subset T_b(Y)$, *$T$ being a separable subspace*, $\{U_n(x'')\} \in T_c(Y)$. *Under these hypotheses the operations are equicontinuous in* $X_s(\omega)$.

Let $x_n \in X_s$, $x_n \xrightarrow{\omega} 0$; let us choose, as in 1.4, an increasing sequence of indices k_n so that $U_{k_{n-1}}(x_{k_n}) \to 0$ as $n \to \infty$; then let us choose functionals η_n linear on Y satisfying the conditions 1.4 (*). Let us write $V_n(x) = U_{k_n}(x) - U_{k_{n-1}}(x)$. Let $u_1, u_2, \ldots$ where $u_k = \{y_{nk}\} \in T$ for $k=1,2,\ldots$ be a sequence dense in T.

Now we select an increasing sequence l_n of indices so that the sequence $\eta_{l_n}(y_{k_{l_n}k})$, $\eta_{l_n}(y_{k_{l_{n-1}}k})$ converges for $k=1,2,\ldots$

If $u = \{y_{k_n}\}$ is a sequence extracted from a sequence $\{y_n\} \in T$, the sequence $\eta_{l_n}(y_{k_{l_n}})$, $\eta_{l_n}(y_{k_{l_{n-1}}})$ converges; this follows from the inequality

$$|\eta_m(y_{k_n}) - \eta_m(y_{k_nk})| \leqslant \|y_{k_n} - y_{k_nk}\| \leqslant \sup_n \|y_n - y_{nk}\|$$

and from the fact that u_k lie dense in T. Given an $x \in X$, let x', x'' be elements of the representation (*). Since $\{V_n(x')\}$ is the difference of two sequences extracted from a sequence belonging to T, the sequence $\eta_{l_n}(V_{l_n}(x'))$ is convergent. Since $\{V_n(x'')\} \in T_0(Y)$, inequality

$$\left|\eta_{l_n}(V_{l_n}(x''))\right| \leqslant \|\eta_{l_n}\| \, \|V_{l_n}(x'')\|$$

implies $\eta_{l_n}(V_{l_n}(x'')) \to 0$. Therefore the sequence $\eta_{l_n}(V_{l_n}(x))$ converges in the whole of x. By 1.4 (α) the functionals $\eta_{l_n}(V_{l_n})$ are equicontinuous in $X_s(\omega)$. By 1.4 (*)

$$V_{l_n}(x_{k_{l_n}}) = U_{k_{l_n}}(x_{k_{l_n}}) - U_{k_{l_n-1}}(x_{k_{l_n}}) \to 0$$

and since $U_{k_{l_n-1}}(x_{k_{l_n}}) \to 0$, we infer that $U_{k_{l_n}}(x_{k_{l_n}}) \to 0$ as $n \to \infty$. The hypotheses of our theorem are also valid when we replace the sequence U_n by any of its subsequences. Thus we have proved that every subsequence U_{n_i} contains a partial one, $U_{n_i'}$, such that $U_{n_i'}(x_{n_i'}) \to 0$, consequently U_n are equicontinuous in $X_s(\omega)$.

1.52. *Let $X_s(\omega)$ be a Saks space with property* (A, $\boldsymbol{K}_2$); *let U_n be $(X_s(\omega), Y)$-linear operations, such that the sequence $U_n(x)$ converges in a set dense in $X_s(\omega)$, but diverges for at least one x. Suppose moreover, that the sequence $U_n(x)$ is bounded everywhere. Under these conditions for every positive integer r there exist elements $x_1, x_2, \ldots, x_r$ in $X_s(\omega)$ such that the sequence $U_n(y)$ diverges for every $y = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r$ where* $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_r| > 0$.

Let X_c be the set of convergence of the sequence $U_n(x)$. Suppose that the quotient space X/X_c is finite, say s-dimensional. Therefore $u_i \in X/X_c$ exist for $i=1,2,\ldots,s$ which are linearly independent and such that if $u \in X/X_c$, then

$$u = \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_s u_s. \tag{*}$$

Let us fix an x_i from the class u_i for $i=1,2,\ldots,s$ and let u be the class in which there is a given element $x \in X$. By (*) there exist elements y, $y_i \in X_c$ such that

$$x + y = \sum_{i=1}^{s} \lambda_i x_i + \sum_{i=1}^{s} \lambda_i y_i, \quad \text{whence} \quad x = \sum_{i=1}^{s} \lambda_i x_i + z \quad \text{with} \quad z \in X_c.$$

Let T be the linear subspace of $T_b(Y)$ spanned upon the elements $\{U_n(x_i)\}$ with $i=1,2,\ldots,s$. Let us write $x' = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_s x_s$, $x'' = z$. Then the hypotheses of 1.51 are satisfied, whence the operations are equicontinuous in $X_s(\omega)$, and since the sequence $U_n(x)$ converges in a set dense in $X_s(\omega)$, it must converge everywhere, contradicting the hypothesis that it diverges for at least one x. Thus we have proved that the space X/X_c is infinitely dimensional. To prove the theorem it suffices, with any given r, to choose linearly independent classes $u_1, u_2, \ldots, u_r$ of X/X_c and then select $x_i \in u_i$.

1.53. *Let $X_s(\omega)$ and U_n satisfy all the hypotheses of* 1.52. *Then the set of all sequences $\{U_n(x)\}$, $x \in X$, is not separable in* $T_b(Y)$.

This results from 1.51.

1.54. *Let A and B be matrix methods of summability corresponding to the matrices (a_{in}) and (b_{in}) respectively. Let A be permanent for null sequences, let B be conservative (= convergence preserving) for null sequence and such that there exists a bounded sequence A-summable to* 0 *but not B-summable.*

Under these hypotheses:

(a) *the set of all B-transforms $\{B_i(x)\}$, where x is a bounded sequence, A-summable to* 0, *is non-separable in the space of bounded sequences* (compare [3], [8]);

(b) *for every positive integer r there exist bounded sequences $x_1, x_2, \ldots, x_r$, A-summable to* 0, *such that any sequence $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r$, $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_r| > 0$ is not B-summable* (compare [1]) [3].

Let X be the space of bounded sequences, A-summable to 0, and, given $x = \{t_n\} \in X$, write

$$\|x\| = \sup_n |t_n|, \qquad \|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |t_n| + \sup_n |A_n(x)|,$$

where $A_i(x) = \sum_{n=1}^{\infty} a_{in} t_n$. With these norms, $X_s(\omega)$ is a Saks space satisfying the condition (Σ_1), whence having the property (A, $\boldsymbol{K}_2$) (com-

[3]) In this section we use the terminology and the notation of [3].

pare [5]). Moreover the sequences x converging to 0 and satisfying $\|x\|\leqslant 1$ lie dense in $X_s(\omega)$. It is sufficient to apply 1.53 and 1.52.

By aid of theorems 1.52 and 1.53 one can obtain analogous theorems for matrix methods of the summability of multiple sequences, for methods of summability consisting in the transformation of sequences into functions, or functions into functions. Similar results may be obtained also for matrix methods of summability (the matrices being numerical) transforming sequences of elements of normed spaces, etc.

1.6. (α) *Let starred norms* $\|\ \|_1^*$, *and* $\|\ \|_2^*$ *be defined in* X, *giving rise to spaces* $X_s(\omega_1)$ *and* $X_s(\omega_2)$. *If* $X_s(\omega_2)$ *has the property* (A, $\boldsymbol{K}_1$) *and* $\Xi_s(\omega_1)\subset\Xi_s(\omega_2)$ *then the norm* $\|\ \|_2^*$ *is non-weaker in* X_s *than the norm* $\|\ \|_1^*$.

Let U be the identical transformation from $X_s(\omega_2)$ to X provided with the norm $\|\ \|_1^*$. Let Ξ_1^* be the space conjugate to X, $\|\ \|_1^*$. Since $\Xi_1^*\subset\Xi_s(\omega_1)\subset\Xi_s(\omega_2)$, $\eta\big(U(x)\big)$ is a continuous functional if $\eta\in\Xi_1^*$. The property (A, $\boldsymbol{K}_1$) of $X_s(\omega_2)$ implies that U is linear in $X_s(\omega_2)$, whence $\|x_n\|\leqslant 1$, $\|x_n\|_2^*\to 0$ implies $\|U(x_n)\|_1^*=\|x_n\|_1^*\to 0$ as $n\to\infty$.

(β) *Let* $\overline{\Xi}$ *be an arbitrary set of functionals distributive on* X; *then there exists in* X *at most one starred norm, unique in the sense of the equivalence of norms in* X_s, *having the property* (A, $\boldsymbol{K}_1$) *and such that* $\Xi_s(\omega)=\overline{\Xi}$.

This above follows directly from (α).

Let us notice that in these theorems we may replace (A, $\boldsymbol{K}_1$) by (A, $\boldsymbol{K}_2$) adding in (α) the hypothesis that one of spaces $X_s(\omega_1)$ or $X_s(\omega_2)$ is separable and in (β) the hypothesis that both are so.

1.61. (α) *The necessary and sufficient condition in order that every linear functional in* $X_s(\omega)$ *be uniformly bounded in* X_s *is that the fundamental norm be non-weaker in* X *than the starred one.*

The sufficiency being trivial, we prove only the necessity. Let us introduce, besides the starred norm $\|\ \|^*$ corresponding to the convergence ω, a second starred norm, $\|x\|_1^*=\|x\|$, for $x\in X$. By 1.41 $X_s(\omega_1)$ has the property (A, $\boldsymbol{K}_1$). By hypothesis $\Xi_s(\omega)\subset\Xi_s(\omega_1)$, whence by 1.6 ($\alpha$) the norm $\|\ \|$ is non-weaker than $\|\ \|^*$ in X_s, and hence in X.

Let us notice that if the fundamental norm is non-weaker in X than the starred one, then every $\big(X_s(\omega), Y\big)$-linear operation is uniformly bounded in X_s.

(β) *If* $X_s(\omega)$ *has the property* (A, $\boldsymbol{K}_1$) *and every functional distributive on* X *and uniformly bounded in* X_s *is linear in* $X_s(\omega)$, *then the starred norm is non-weaker in* X *than the fundamental norm.*

Let us define the norm $\|\ \|_1^*$ as in the proof of (α). By hypothesis $\Xi=\Xi_s(\omega_1)\subset\Xi_s(\omega)$. Let $\|x_n\|^*\to 0$ as $n\to\infty$. Suppose that $\|x_n\|\geqslant\varepsilon>0$; write $y=x_n/\|x_n\|$. Since $\|y_n\|^*\to 0$, $\|y_n\|=1$, we see by 1.6 (α) that $\|y_n\|_1^*=\|y_n\|\to 0$, which is contradictory. Thus $\|x_n\|\to 0$.

Let us notice that without the hypothesis that $X_s(\omega)$ has the property (A, $\boldsymbol{K}_1$) the proposition 1.61 (β) may be false. Indeed, let X be the space of functions integrable in (a,b) with the exponent $\alpha>1$. Let

$$\|x\|=\Big(\int_a^b|x(t)|^\alpha dt\Big)^{1/\alpha},\qquad \|x\|^*=\sup_{a\leqslant t\leqslant b}\Big|\int_a^t x(\tau)\,d\tau\Big|.$$

Then $\Xi_s(\omega)=\Xi$ (compare [9]) in spite of the fact that the convergence generated by the starred norm does not imply that generated by the fundamental one.

1.62. *Let the space* X *be separable with respect to the fundamental norm, and let* Ξ_n *be a fundamental set of linear functionals on* X (compare [2])). *If* $X_s(\omega)$ *is a Saks space having the property* (A, $\boldsymbol{K}_2$) *and* $\Xi_n\subset\Xi_s(\omega)$ *then the starred norm is non-weaker in* X *than the fundamental norm.*

The separability of X implies that every sequence $\xi_k\in\Xi_n$ contains a subsequence $\xi_{k_i}(x)$ converging for every $x\in X$. Since, by 1.4 (α), $X_s(\omega)$ has the property (B, $\boldsymbol{K}_1$), the functionals ξ_{k_i} are equicontinuous. In consequence the functionals of the set Ξ_n are equicontinuous. Let $\|x_n\|\leqslant 1$, $\|x_n\|^*\to 0$; since for $n=1,2,\ldots$ $\sup_{\xi\in\Xi_n}|\xi(x_n)|\geqslant c\|x_n\|$ with a universal constant $c>0$, and since equicontinuity implies $\sup_{\xi\in\Xi_n}|\xi(x_n)|\to 0$, we get $\|x_n\|\to 0$. Finally, we can free ourselves from the hypothesis $\|x_n\|\leqslant 1$ as in the proof of 1.61 (β).

Let us call the starred norm *useful* if: 1° the corresponding $X_s(\omega)$ is a Saks space, 2° $X_s(\omega)$ has the property (A, $\boldsymbol{K}_2$), 3° a class of functionals $\overline{\Xi}$ belongs to $\Xi_s(\omega)$.

The meaning of 3° is not precisely formulated. The set $\overline{\Xi}$ may be taken in various manners depending on X. $\overline{\Xi}$ is only required to contain linear functionals of a simple structure.

The last theorem enables us to prove that in the case of typical separable Banach spaces it is impossible to introduce a useful norm essentially different (in the sense of the equivalence of norms in X) from $\|\ \|$.

If X, $\|\ \|$ is a Banach space, then, by 1.2, 1° implies that the fundamental norm is non-weaker in X than the starred one. Let us suppose that $\overline{\Xi}$ is such that a certain set of linear combinations of functionals of $\overline{\Xi}$ is fundamental. If, moreover, 2° is to be satisfied, then the starred norm is non-weaker in X than the fundamental one. Hence the fundamental norm is equivalent to the starred one in X.

In the following Banach spaces there exists no useful starred norm, non-equivalent to the fundamental one.

1. Let $C(\Delta)$ stand for the space of functions continuous in a closed interval Δ, $\|x\|=\sup_{\Delta}|x(t)|$; let Ξ be the set of functionals of the form $\xi(x)=x(\tau)$ where $\tau\in\Delta$, or the set of the functionals of the form

$$(*)\qquad \xi(x)=\int_{\tau_1}^{\tau_2}x(t)\,dt$$

where $\tau_1,\tau_2\in\Delta$. Let us observe that if the interval Δ is open (finite or not), then it is possible to introduce a useful norm non-equivalent to the norm $\|\ \|$ (see, for instance, [9]).

2. Let $L^a(a,b)$ be the space of the functions integrable in (a,b) with the exponent $a\geqslant 1$, $\|x\|=\Big(\int_a^b|x(t)|^a dt\Big)^{1/a}$, let Ξ be the set of the functionals $(*)$ where $\tau_1,\tau_2\in(a,b)$.

3. Let H^a be the space of convergent series with the exponent $a\geqslant 1$, $\|x\|=\Big(\sum_1^\infty|t_n|^a\Big)^{1/a}$, and let Ξ be the set of the functionals of the form $\xi(x)=t_m$ where $x=\{t_n\}$.

2. For $a\geqslant 1$ we shall denote by M^a the space of measurable functions, bounded almost everywhere in $(-\infty,\infty)$ and satisfying the condition

$$\int_{-\infty}^{\infty}|x(t)|^a dt<\infty$$

(the addition of elements, etc. being defined as usual). By a' we denote the exponent conjugate to $a>1$, *i. e.*, such that $1/a+1/a'=1$.

As the fundamental norm in M^a we define

$$\|x\|=\sup_{(-\infty,\infty)}{}^{*}|x(t)|\ ^{4}),$$

as the starred one:

$$\|x\|_a^*=\Big(\int_{-\infty}^{\infty}|x(t)|^a dt\Big)^{1/a}.$$

By L^a or M we always denote in this section the Banach space of functions integrable in $(-\infty,\infty)$ with the exponent $a\geqslant 1$, with the norm $\|\ \|_a^*$ or the space of measurable bounded functions in $(-\infty,\infty)$ with the norm $\|\ \|$.

[4]) $\sup_{\Delta}^* y(t)$ denotes the essential supremum in Δ of the function $y(t)$.

Let us set

$$M_s^a=\mathop{E}_{x}\{x\in M^a,\ \|x\|\leqslant 1\};$$

M^a is not complete either with respect to the fundamental norm, or with respect to the starred one; M_s^a is, however, a Saks space.

Let $1\leqslant a_1<a_2$; if x is a measurable function in $(-\infty,\infty)$ and $\|x\|\leqslant 1$, then

$$\int_{-\infty}^{\infty}|x(t)|^{a_2}dt\leqslant\int_{-\infty}^{\infty}|x(t)|^{a_1}dt,$$

whence (in the sense of set-theoretical inclusion)

$$M_s^{a_1}\subset M_s^{a_2}\quad\text{as}\quad 1\leqslant a_1<a_2.$$

The converse inclusion is not true. To see this let us consider the following functions:

Let $1\leqslant a_1<a_2$ and set

$$x(t)=\frac{1}{n^{1/a_1}}\quad\text{for}\quad t\in\langle n-1,n)\cup\langle -n,-n+1)$$

where $n=1,2,\ldots$,

$$x_m(t)=\begin{cases}\dfrac{1}{(\log m)^{1/2a_1}}\cdot\dfrac{1}{n^{1/a_1}} & \text{for } t\in\langle n-1,n)\cup\langle -n,-n+1),\ n=1,2,\ldots,m,\\ 0 & \text{elsewhere,}\end{cases}$$

m being an arbitrary positive integer. Then $x\in M_s^{a_2}$ and $x\notin M_s^{a_1}$, moreover $x_m\in M_s^{a_1}\subset M_s^{a_2}$, $\|x_m\|_{a_2}^*\to 0$, $\|x_m\|_{a_1}^*\to\infty$.

2.1. (α) *The space M_s^a is separable.*

(β) *The functions of M_s^β, where β is an arbitrary exponent $\geqslant 1$, lie dense in the space M_s^a.*

Ad (α). The set of all linear combinations with rational coefficients of characteristic functions of rational intervals belonging to M_s^a is dense in M_s^a.

Ad (β). It suffices to take bounded functions vanishing outside a finite interval.

2.2. *The space M_s^a satisfies the condition (Σ_1) and (Σ_2).*

We shall prove only that (Σ_1) is fulfilled. Let $K(x_0,\varrho)$ be the sphere with centre x_0 and radius ϱ in M_s^a. We are free to suppose that $\|x_0\|<1$. Let $k=1-\|x_0\|$, $\|x\|\leqslant 1$, $\|x\|_a^*\leqslant\delta$,

$$A=\mathop{E}_{t}\{|x(t)|\geqslant k\},\qquad B=(-\infty,\infty)-A.$$

By the definition of A we have $|A|\leqslant(\delta/k)^{\alpha}$. Let us set if $\|x\|\leqslant 1$, $\|x\|_{\alpha}^{*}<\delta$,

$$x_1(t)=\begin{cases} 0 & \text{as} \quad t\in A,\\ x(t)+x_0(t) & \text{as} \quad t\in B;\end{cases}$$

$$x_2(t)=\begin{cases} -x(t) & \text{as} \quad t\in A,\\ x_0(t) & \text{as} \quad t\in B.\end{cases}$$

Obviously $x=x_1-x_2$ and the following inequalities are satisfied: $\|x_1\|\leqslant 1$, $\|x_2\|\leqslant 1$,

$$\|x_1-x_0\|_{\alpha}^{*}=\Big(\int_A |x_0(t)|^{\alpha}dt+\int_B |x(t)|^{\alpha}dt\Big)^{1/\alpha}<[|A|+(\|x\|_{\alpha}^{*})^{\alpha}]^{\alpha}\leqslant\delta/k+\delta,$$

$$\|x_2-x_0\|_{\alpha}^{*}=\Big(\int_A |-x(t)-x_0(t)|^{\alpha}dt\Big)^{1/\alpha}\leqslant 2|A|^{1/\alpha}\leqslant 2\delta/k.$$

Taking δ sufficiently small we obtain

$$\|x_1-x_0\|_{\alpha}^{*}<\varrho,\qquad \|x_2-x_0\|_{\alpha}^{*}<\varrho.$$

2.3. A. *Every linear functional in M_s^{α} with $\alpha>1$ may be written in the form*

$$(*)\qquad \xi(x)=\int_{-\infty}^{\infty}x(t)y(t)\,dt,\qquad x\in M_s^{\alpha},$$

where

$$(**)\qquad y(t)=y_1(t)+y_2(t),\qquad y_1\in L^1,\qquad y_2\in L^{\alpha'}.$$

The functions of $(**)$ *may always be chosen so that*

(a) $\|\xi\|_0=\|y_1\|_1^{*}+\|y_2\|_{\alpha'}^{*}$,

or

(b) $y_1\in L^1$, $y_1(t)\neq 0$ *in a set of finite measure*, $y_2\in M_s^{\alpha'}$, $\|y_1\|_1^{*}\leqslant\|\xi\|_0+(\|\xi\|_0)^{\alpha'}$, $\|y_2\|_{\alpha'}^{*}\leqslant\|\xi\|_0$,

$$(\overset{*}{**})\qquad \|\xi\|_0\leqslant\|y_1\|_1^{*}+\|y_2\|_{\alpha'}^{*}.$$

B. *Every functional of the form* $(*)$ *with y satisfying* $(**)$ *is linear in M_s^{α}. Its norm satisfies the inequality* $(\overset{*}{**})$.

Ad A. Let $M_s^{n\alpha}$ denote the Saks-subspace of M_s^{α} composed of the functions vanishing outside the interval $(-n,n)$. By a known theorem of Fichtenholz

$$\xi(x)=\int_{-n}^{n}x(t)y_n(t)\,dt\qquad\text{for}\qquad x\in M_s^{n\alpha},$$

where y_n is integrable in $(-n,n)$. It is easily seen that $y_m(t)=y_n(t)$ for $t\in(-n,n)$, $m>n$. Let us set

$$y(t)=\lim_{n\to\infty}y_n(t)\qquad\text{for}\qquad t\in(-\infty,\infty).$$

Let $x\in M_s^{\alpha}$ and let

$$x_n(t)=\begin{cases} x(t) & \text{for} \quad t\in(-n,n),\\ 0 & \text{elsewhere},\end{cases}$$

$$z_n(t)=\operatorname{sign}x(t)\operatorname{sign}y(t)x_n(t),$$

$$z(t)=\operatorname{sign}x(t)\operatorname{sign}y(t)x(t).$$

Then

$$\xi(z_n)=\int_{-n}^{n}|x(t)||y(t)|\,dt,\qquad \xi(x_n)=\int_{-n}^{n}x(t)y(t)\,dt,$$

and since $\|z_n\|\leqslant 1$, $\|z_n-z\|_{\alpha}^{*}\to 0$ as $n\to\infty$, we see that $\xi(z_n)\to\xi(z)$, whence $x(t)y(t)$ is integrable in $(-\infty,\infty)$. Since $\|x_n\|\leqslant 1$, $\|x_n-x\|_{\alpha}^{*}\to 0$, we infer that $\xi(x_n)\to\xi(x)$, whence ξ is representable in the form $(*)$. By 1.32, $\xi(x)$ may be represented in the form $\xi(x)=\xi_1(x)+\xi_2(x)$ for $x\in M_s^{\alpha}$ where ξ_1 is a continuous functional in M, and ξ_2 is continuous in L^{α}, moreover 1.32 $(**)$ is satisfied. Now

$$\xi_2(x)=\int_{-\infty}^{\infty}x(t)y_2(t)\,dt\qquad\text{for}\qquad x\in L^{\alpha},$$

where $y_2\in L^{\alpha'}$, $\|\xi_2\|^{*}=\|y_2\|_{\alpha'}^{*}$ and since ξ is of the form $(*)$, we get

$$\xi_1(x)=\int_{-\infty}^{\infty}x(t)y_1(t)\,dt\qquad\text{for}\qquad x\in M_s^{\alpha}.$$

Obviously $|\xi_1(x)|\leqslant\|\xi_1\|$ for $x\in M_s^{\alpha}$, whence

$$\int_{-\infty}^{\infty}|y_1(t)|\,dt\leqslant\|\xi_1\|.$$

Thus we have proved $(**)$ and the inequality

$$\|y_1\|_1^{*}+\|y_2\|_{\alpha'}^{*}\leqslant\|\xi\|_0.$$

On the other hand, if the functional $(*)$ is written in the form $(**)$, then

$$|\xi(x)|\leqslant\|y_1\|_1^{*}\|x\|+\|y_2\|_{\alpha'}^{*}\|x\|_{\alpha}^{*}\leqslant(\|y_1\|_1^{*}+\|y_2\|_{\alpha'}^{*})\|x\|_0,$$

whence

$$\|\xi\|_0\leqslant\|y_1\|_1^{*}+\|y_2\|_{\alpha'}^{*};$$

thus (a) is valid.

To prove (b) let us write

$$A=\underset{t}{E}\{|y(t)|\leqslant\|\xi\|_0\},\qquad A_n=A\cap(-n,n),$$

$$B=(-\infty,\infty)-A,\qquad B_n=B\cap(-n,n).$$

Let us define

$$z(t)=\begin{cases}\Big(\int\limits_{A_n}|y(t)|^{\alpha'}dt\Big)^{-1/\alpha}|y(t)|^{\alpha'-1}\operatorname{sign} y(t) & \text{for} \quad t\in A_n,\\ 0 & \text{elsewhere.}\end{cases}$$

It is easy to verify that $\|z\|_\alpha^*\leqslant 1$; let

$$r=\|\xi\|_0^{\alpha'/\alpha}\Big(\int\limits_{A_n}|y(t)|^{\alpha'}dt\Big)^{-1/\alpha}.$$

Then $\|z\|\leqslant r$, and since

$$|\xi(z)|=\Big|\int\limits_{A_n}z(t)y(t)\,dt\Big|=\int\limits_{A_n}|y(t)|^{\alpha'}dt\Big(\int\limits_{A_n}|y(t)|^{\alpha'}dt\Big)^{-1/\alpha}$$

$$\leqslant\|\xi\|_0\|z\|_0\leqslant\|\xi\|_0\max(1,r),$$

we see that

$$\Big(\int\limits_{A_n}|y(t)|^{\alpha'}dt\Big)^{1/\alpha}\leqslant\|\xi\|_0^{\alpha'/\alpha}[\max(1,r)]^{\alpha'/\alpha},$$

which is possible only when $r\geqslant 1$. Therefore

$$\Big(\int\limits_{A_n}|y(t)|^{\alpha'}dt\Big)^{1/\alpha'}\leqslant\|\xi\|_0 \quad \text{for} \quad n=1,2,\ldots,$$

whence

$$\text{(o)} \qquad \Big(\int\limits_{A}|y(t)|^{\alpha'}dt\Big)^{1/\alpha'}\leqslant\|\xi\|_0 .$$

Let us now consider the function

$$\bar z(t)=\begin{cases}|B_n|^{-1/\alpha}\operatorname{sign} y(t) & \text{for} \quad t\in B_n,\\ 0 & \text{elsewhere.}\end{cases}$$

Then, supposing $|B_n|>0$, we have

$$\|\xi\|_0\frac{|B_n|}{|B_n|^{1/\alpha}}<|\xi(\bar z)|=\frac{1}{|B_n|^{1/\alpha}}\int\limits_{B_n}|y(t)|\,dt\leqslant\|\xi\|_0\|\bar z\|_0\leqslant\|\xi\|_0\max\Big(1,\frac{1}{|B_n|^{1/\alpha}}\Big),$$

which implies $|B_n|^{1/\alpha'}<\max(1,1/|B_n|^{1/\alpha})$, whence $|B_n|\leqslant 1$ for $n=1,2,\ldots$ and $|B|\leqslant 1$. Let us notice further that

$$\text{(oo)} \qquad \int\limits_E|y(t)|\,dt\leqslant\|\xi\|_0 \quad \text{if} \quad |E|\leqslant 1 .$$

To prove this it suffices to set

$$\bar{\bar z}(t)=\begin{cases}\operatorname{sign} y(t) & \text{for} \quad t\in E,\\ 0 & \text{elsewhere,}\end{cases}$$

and to observe that $\|\bar{\bar z}\|_0\leqslant 1$, $|\xi(\bar{\bar z})|\leqslant\|\xi\|_0$.

Suppose now that $\|\xi\|_0\leqslant 1$ and consider the functions

$$y_1(t)=\begin{cases}0 & \text{for} \quad t\in A,\\ y(t) & \text{elsewhere,}\end{cases} \qquad y_2(t)=\begin{cases}y(t) & \text{for} \quad t\in A,\\ 0 & \text{elsewhere.}\end{cases}$$

By (o) and (oo) there follows $\|y_1\|^*\leqslant\|\xi\|_0$, $\|y_2\|_{\alpha'}^*\leqslant\|\xi\|_0$. Let us suppose that $\|\xi\|_0>1$, and let $C=\underset{t}{E}\{|y(t)|\leqslant 1\}$; we define the functions y_1 and y_2 as above only replacing the set A by C. By (o):

$$\|y_2\|_{\alpha'}^*\leqslant\|\xi\|_0,$$

$$|A-C|\leqslant\int\limits_{A-C}|y(t)|\,dt\leqslant\int\limits_{A-C}|y(t)|^{\alpha'}dt\leqslant(\|\xi\|_0)^{\alpha'}.$$

This inequality together with $|B|\leqslant 1$ and (oo) implies $\|y_1\|_1^*\leqslant\|\xi\|_0+(\|\xi\|_0)^{\alpha'}$; moreover the set $(-\infty,\infty)-C=B\cup A-C$ is of finite measure, $y=y_1+y_2$. The inequality $(\overset{*}{*})$ is, in virtue of Hölder's inequality, true for every decomposition $(**)$, whence in particular for the one defined above.

Ad B. If $y_2\in L^{\alpha'}$, then the functional

$$\xi_2(x)=\int\limits_{-\infty}^{\infty}x(t)y_2(t)\,dt$$

is continuous with respect to the norm $\|\ \|_\alpha^*$ since $|\xi_2(x)|\leqslant\|y_2\|_{\alpha'}^*\|x\|_\alpha^*$; consequently $\xi_2\in\Xi_s(\omega)$. Let $y_1\in L^1$. For every $\varepsilon>0$ there exists a decomposition $y_1(t)=z_1(t)+z_2(t)$ where $z_1\in L^1$, $\|z_1\|_1^*<\varepsilon$, $z_2\in L^1$ and $z_2(t)$ is bounded in $(-\infty,\infty)$. If

$$\zeta_1(x)=\int\limits_{-\infty}^{\infty}x(t)z_1(t)\,dt \quad \text{for} \quad x\in M, \qquad \zeta_2(x)=\int\limits_{-\infty}^{\infty}x(t)z_2(t)\,dt \quad \text{for} \quad x\in L^\alpha,$$

then ζ_1 and ζ_2 are linear functionals on M or L^α respectively, $\|\zeta_1\|=\|z_1\|_1^*<\varepsilon$. By 1.32 ($\beta'$) it follows that $\xi_1=\zeta_1+\zeta_2$ is a linear functional on M_s^α.

2.31. (α) *Theorem* 2.3 *remains true for* $\alpha=1$ *if we replace* $L^{\alpha'}$ *by* M, $M_s^{\alpha'}$ *by* M, $\|y_2\|_{\alpha'}^*$ *by* $\|y_2\|$ *and* $\|\xi\|_0+(\|\xi\|_0)^{\alpha'}$ *in* (b) *by* $\|\xi\|_0$.

The proof is similar.

Let $\mathfrak{E}$ be the ring of all measurable sets (bounded or not) of finite measure.

(β) *In order that the functional* ξ *be linear in* M_s^1 *it is necessary and sufficient that* ξ *be representable in the form* 2.3 $(*)$ *with* y *having the following property*:

(a) *The set-function* $\varphi(E)=\int\limits_E|y(t)|\,dt$ *is absolutely continuous in* $\mathfrak{E}$, *i. e., for every* $\varepsilon>0$ *there is a* $\delta>0$ *such that* $|E|<\delta$, $E\in\mathfrak{E}$ *implies* $|\varphi(E)|<\varepsilon$.

Necessity. In virtue of 2.31 (α) every linear functional on M_s^1 may be represented in the form 2.3 ($*$) where $y=y_1+y_2$, $y_1 \in L^1$, $y_2 \in M$. Obviously

$$\varphi_i(E)=\int_E |y_i(t)|\,dt, \quad i=1,2,$$

and consequently $\varphi(E)$ are absolutely continuous in $\mathfrak{E}$.

Sufficiency. It is easily seen that the absolute continuity of $\varphi(E)$ in $\mathfrak{E}$ implies

$$\sup_{\substack{|E|\leqslant 1\\ E\in\mathfrak{E}}} \int_E |y(t)|\,dt = K < \infty .$$

Hence for every set $E \in \mathfrak{E}$ of measure $\geqslant 1$

$$\int_E |y(t)|\,dt \leqslant 2K|E| .$$

Write

$$A=\mathop{E}_t\{|y(t)|>2K\}, \quad A_{nr}=\mathop{E}_t\{t\in(-n,n),\ |y(t)|\geqslant 2K+1/r\} .$$

By the last inequality $|A_{nr}|\leqslant 1$ for $r,n=1,2,\dots$, whence $|A|\leqslant 1$.

Let us set

$$y_1(t)=\begin{cases} y(t) & \text{for} \quad t\in A,\\ 0 & \text{elsewhere,}\end{cases} \qquad y_2(t)=y(t)-y_1(t);$$

we then obtain for y the decomposition 2.3 ($**$) with $y_1 \in L^1$, $y_2 \in M$. Thus $\xi \in \Xi_s(\omega)$ by (α).

2.4. A. *Let $a>1$; the following conditions are necessary and sufficient that the sequence*

$$(*) \qquad \xi_n(x)=\int_{-\infty}^{\infty} x(t)\,y_n(t)\,dt$$

be convergent for every $x \in M_s^a$:

(a) *for $n=1,2,\dots$ there exist representations*

$$y_n(t)=y_{1n}(t)+y_{2n}(t)$$

such that $y_{1n}\in L^1$, $y_{2n}\in L^{a'}$ and

$$\|y_{1n}\|_1^* \leqslant K, \qquad \|y_{2n}\|_{a'}^* \leqslant K,$$

K being a constant,

(b) *the set-functions*

$$\varphi_n(E)=\int_E |y_n(t)|\,dt, \quad E\in\mathfrak{E},$$

are equi-absolutely continuous in $\mathfrak{E}$, i. e., the condition of 2.31 (a) is satisfied uniformly in n,

(c) *there exists a function y representable in the form 2.3 ($**$) such that for every τ*

$$\lim_{n\to\infty}\int_0^\tau y_n(t)\,dt=\int_0^\tau y(t)\,dt .$$

B. *If the conditions* (a)-(c) *are satisfied, then the sequence ξ_n converges to a functional of form 2.3 ($*$), where the function y of* (c) *is taken as y.*

Ad A. As M_s^a satisfies the condition (Σ_1), the limit functional is also linear in M_s^a ([6], p. 10). Taking as x in ($*$) the characteristic function of the interval $\langle 0,\tau\rangle$ we obtain (c). The condition (a) results by the application of 1.33 (α) and 2.3 A (a). For a given set E of finite measure, let

$$x_n(t)=\begin{cases}\operatorname{sign} y_n(t) & \text{for} \quad t\in E,\\ 0 & \text{elsewhere.}\end{cases}$$

Since $\|x_n\|_a^*\leqslant |E|^{1/a}$ and the condition (Σ_1) implies equicontinuity of ξ_n in M_s^a, we infer that $|E|<\delta$ with δ sufficiently small implies $|\xi_i(x_i)|\leqslant\varepsilon$ for $i=1,2,\dots$, whence

$$\xi_n(x_n)=\int_E |y_n(t)|\,dt<\varepsilon \quad \text{for} \quad n=1,2,\dots$$

Ad B. Let

$$\xi_{in}(x)=\int_{-\infty}^{\infty} x(t)\,y_{in}(t)\,dt \quad \text{where} \quad i=1,2,\ n=1,2,\dots$$

By (a) $|\xi_{2n}(x)|\leqslant\|y_{2n}\|_{a'}^*\|x\|_a^*\leqslant K\|x\|_a^*$, whence the set-functions

$$\int_E |y_{2n}(t)|\,dt$$

are equi-absolutely continuous. Therefore by (b) the functions

$$\int_E |y_{1n}(t)|\,dt$$

are also equi-absolutely continuous.

Let $A_n=\mathop{E}_t\{|y_{1n}(t)|\geqslant K/\delta\}$, $B_n=(-\infty,\infty)-A_n$. From $\|y_{1n}\|_1^*\leqslant K$ follows $|A_n|\leqslant\delta$, and thus, δ being sufficiently small,

$$\int_{A_n} |y_{1n}(t)|\,dt<\varepsilon \quad \text{for} \quad n=1,2,\dots$$

If $x \in M_s^a$, this leads to the inequality

$$|\xi_{1n}(x)| \leqslant \int_{A_n} |x(t)|\,|y_{1n}(t)|\,dt + \int_{B_n} |x(t)|\,|y_{1n}(t)|\,dt$$

$$\leqslant \sup_{(-\infty,\infty)}{}^{*} |x(t)| \int_{A_n} |y_{1n}(t)|\,dt + \int_{B_n} \frac{K}{\delta}|x(t)|\,|y_{1n}(t)|\left(\frac{K}{\delta}\right)^{-1} dt$$

$$\leqslant \varepsilon\|x\| + \frac{K}{\delta}\|x\|_a^* \delta^{1/a'} \leqslant \varepsilon + K\delta^{-1/a}\|x\|_a^*,$$

since

$$\left(\int_{B_n}\left[|y_{1n}(t)|\left(\frac{K}{\delta}\right)^{-1}\right]^{a'} dt\right)^{1/a'} \leqslant \left(\frac{K}{\delta}\right)^{-1/a'}\left(\int_{B_n}|y_{1n}(t)|\,dt\right)^{1/a'} \leqslant \delta^{1/a'}.$$

As the linear combinations of characteristic functions of intervals lie dense in M_s^a, the sequence $\xi_n(x)$ converges in a set $\overline{M}_s^a$ dense in M_s^a. Choose $x_0 \in \overline{M}_s^a$ so that $\|x-x_0\|_a^* < \varepsilon/K$, $\|x-x_0\|_a^* < \varepsilon\delta^{1/a}/K$; then $|\xi_{in}(x)-\xi_{in}(x_0)| < 2\varepsilon$ for $i=1,2$, $n=1,2,\ldots$; consequently the sequence $\xi_n(x)$ converges. By (c) it follows that the limit-functional, which must be representable in the form 2.3 (*), is defined by the function $y(t)$ described by (c).

Analogously we can prove

2.41. *The sequence of linear functionals in M_s^1, of form 2.4 (*) converges for $x \in M_s^1$ if and only if the conditions 2.4 (b), (c) are satisfied.*

3. *Let $X_s(\omega)$ be a Saks space, and let U be a distributive operation from X to a normed space Y, such that*

$$(*) \qquad \varliminf_{n\to\infty} \|U(x_n)\| \geqslant \|U(x_0)\|,$$

as $x_n \in X_s$, $x_n \xrightarrow{\omega} x_0$. Then U maps every bounded set $X_0 \subset X_s(\omega)$ into the bounded set $U(X_0)$.

Distributivity and (*) imply that U is continuous in the B-space X, $\|\ \|_0$, where $\|\ \|_0$ is defined by 1.1 (*). It follows that $\|U(x)\| \leqslant K$ when $\|x\| \leqslant 1$, $\|x\|^* \leqslant 1$. If $\|x\|^* \leqslant k$ $(k>1)$, as $x \in X_0$, then $\|U(x)\| \leqslant Kk$ for $x \in X_s$.

Remark. The theorem remains true if Y is F-normed.

3.1. Let S be the linear space composed of measurable functions in Δ (the interval Δ may be infinite) with the usual definitions of addition, etc. A sequence $y_n \in S$ is called *almost convergent in measure to y* if it converges in measure in every bounded interval $\overline{\Delta} \subset \Delta$. The almost convergence in measure of series of elements is defined similarly. It is possible to define an F-norm in S so that almost convergence in measure be equivalent to convergence in norm; for example we can put for $x \in S$

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n|\Delta_n|}\int_{\Delta_n} \frac{|x(t)|}{1+|x(t)|}\,dt,$$

where the intervals Δ_n are bounded, $\Delta_1 \subset \Delta_2 \subset \ldots$ and $\bigcup_1^\infty \Delta_n = \Delta$.

In the sequel Y will denote a normed space whose elements are measurable functions in a (finite or infinite) interval Δ. The space Y will be said to have the *property* (W) if the following condition is satisfied: if $y_n \in Y$, $y \in Y$ and y_n almost converge in measure to y, then

$$\varliminf_{n\to\infty} \|y_n\| \geqslant \|y\|.$$

3.11. The following Banach spaces (norm, addition, etc. being defined as usual) have the property (W):

(I) $C(\Delta)$ — the space of bounded continuous functions in Δ,

(I′) $M(\Delta)$ — the space of bounded measurable functions in Δ,

(II) $L^a(\Delta)$, $a \geqslant 1$ — the space of functions integrable in Δ with the exponent a,

(III) $VC(\Delta)$ — the space of continuous functions of finite variation in Δ,

(IV) $V^*(\Delta)$ — the space of functions equivalent to functions of finite variation in Δ.

3.2. *Let $X_s(\omega)$ be a Saks space having the property* (A, $\mathbf{K}_2$), *let Y be one of the following spaces of* 3.11: *the space* (I) *where Δ is a finite closed interval; a linear separable subspace of $M(\Delta)$; the space* (II) *with $a>1$. Let U be a distributive operation from X to Y such that:*

(*) *$x_n \in X_s$, $x_n \xrightarrow{\omega} x_0$ implies in Δ the almost convergence in measure of $U_n(x)$ to $U(x_0)$.*

Then U is $(X_s(\omega), Y)$-linear (compare [4]).

As every space Y under consideration has the property (W), lemma 3 implies the existence of a constant $K>0$ such that

$$(**) \qquad \|U(x)\| \leqslant K \quad \text{if} \quad \|x\| \leqslant 1,\ \|x\|^* \leqslant 1.$$

Let $\|x_n\| \leqslant 1$, $\|x_n\|^* \to 0$ and set $U(x_n) = y_n$; each of the spaces Y is separable; therefore it is sufficient to define in each case a fundamental set H_n of functionals such that $\eta(U(x)) \in \Xi_s(\omega)$ for $\eta \in H_n$.

Let $Y = C(\Delta)$, Δ being finite and closed; as H_n we choose the set of the functionals of the form

$$\eta(y) = \pm\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} y(t)\,dt \quad \text{where} \quad \tau_1<\tau_2,\ \tau_1,\tau_2\in\Delta\,.$$

By hypothesis and (**) $y_n(t)$ converges in measure to 0 in Δ, $\sup_{\Delta}^{*}|y_n(t)|\leqslant K$, whence $\eta(y_n')\to 0$ for $\eta\in H_n$.

A similar argument may be applied when Y is a linear separable subspace of $M(\Delta)$.

Now let $Y = L^{\alpha}(\Delta)$, $\alpha>1$. As H_n we choose the set of functionals

$$\eta(y) = \int_{\Delta} z(t)\,y(t)\,dt\,,$$

where z is an arbitrary measurable function, vanishing outside a finite interval contained in Δ and $\sup^{*}|z(t)|\leqslant 1$. By (**)

$$\Big(\int_{\Delta} |y_n(t)|^{\alpha}dt\Big)^{1/\alpha}\leqslant K \quad \text{for} \quad n=1,2,\ldots,$$

therefore

$$\int_{E} |y_n(t)|\,dt\leqslant K|E|^{1/\alpha'} \quad \text{for} \quad n=1,2,\ldots$$

Let $z(t)=0$ as $t\notin\bar{\Delta}$, where $\bar{\Delta}$ is a finite interval $\subset\Delta$; then

$$\big|\eta\big(U(x_n)\big)\big| = \Big|\int_{\bar{\Delta}} z(t)\,y_n(t)\,dt\Big|\leqslant \sup^{*}|z(t)|\int_{\bar{\Delta}}|y_n(t)|\,dt\to 0\,,$$

whence $\eta\big(U(x_n)\big)\to 0$ as $\eta\in H_n$.

3.21. *Let $X_s(\omega)$ be a Saks space having the property (Σ_2), let Y be one of the spaces of* 3.11: *the space* (II) *with $\alpha=1$, the space* (III) *or* (IV).

If U is a distributive operation from X to Y with property 3.2 (*), *then U is $\big(X_s(\omega),Y\big)$-linear.*

Each of the spaces quoted in the hypothesis has the following property: if for every sequence λ_n composed of 0's or 1's the sums

(*) $$\Big\|\sum_{n=1}^{i}\lambda_n y_n\Big\| \quad \text{with} \quad i=1,2,\ldots$$

are commonly bounded, then the series $\sum_{1}^{\infty} y_n$ converges in Y. It suffices to make use of this property, of the property (W) and to apply theorem 3 of [5], p. 271.

3.3. *Let Y be one of the spaces of* 3.2 *and* 3.21. *If the series*

(*) $$\sum_{n=1}^{\infty} t_n y_n$$

almost converges in measure in Δ to a function $y\in Y$, $\{t_n\}$ being an arbitrary bounded sequence, then every series

(**) $$\sum_{n=1}^{\infty}\lambda_n y_n \quad \textit{where} \quad \lambda_n=0 \ \textit{or}\ =1$$

converges in Y.

Let X be the space of bounded sequences in which the fundamental norm $\|x\| = \sup_n |t_n|$ and the starred norm $\|x\|^{*} = \sum_{1}^{\infty}|t_n|/2^n$ are defined; the corresponding $X_s(\omega)$ is a Saks space satisfying the condition (Σ_2). The operations

(***) $$U_m(x) = \sum_{n=1}^{m} t_n y_n, \quad m=1,2,\ldots,$$

are $\big(X_s(\omega),S\big)$-linear and converge in S to $U=\sum_{1}^{\infty}t_n y_n$, whence (compare [6]) U is $\big(X_s(\omega),S\big)$-linear, and by 3.2 and 3.21 $\big(X_s(\omega),Y\big)$-linear. The sequence x_{pq}: $0,\ldots,0,\lambda_p,\lambda_{p+1},\ldots,\lambda_q,0,\ldots$ converges with respect to the starred norm to 0 as $p,q\to\infty$, and this implies the convergence of the series (**).

Remark 1. The convergence of any of the series (**) is equivalent to the commutative convergence of the series $\sum_{1}^{\infty} y_n$.

Remark 2. The theorem remains true if we replace the bounded sequences in (*) by those composed of 0's and 1's (compare [7]).

3.31. *Let A be a matrix method of summability permanent for null sequences, corresponding to the matrix (a_{in}), and such that $\sup_i |a_{in}|\to 0$ as $n\to\infty$. Let Y be one of the spaces of* 3.2 *and* 3.21.

If the series 3.3 (*) *almost converges in measure in Δ to $y\in Y$ for every $\{t_n\}$ A-summable to 0, then there exists a constant K such that for $\lambda_n=0$ or $=1$ the sums* 3.21 (*) *are commonly bounded by constant K and $y_n\to 0$.*

Let X be the space of bounded sequences A-summable to 0 in which the norms $\|\;\|$ and $\|\;\|^{*}$ are defined as in 1.54; then $X_s(\omega)$ satisfies the condition (Σ_2) (compare [5]), whence, as in 3.3, we can show that the operation $U(x)=\sum_{1}^{\infty}t_n y_n$ is $\big(X_s(\omega),Y\big)$-linear. As the fundamental norm is non-weaker than the starred one, U is bounded in X_s. Hence, writing x: $\lambda_1, \lambda_2, \ldots, \lambda_i, 0, \ldots$ we infer that the sums 3.21 (*) are bounded. Since the sequence x_n: $0, \ldots, 0, \lambda_n, 0, \ldots$, where $\lambda_n = 1$, tends to 0 according to the starred norm, for $|A_i(x_n)| \leqslant |a_{in}|$, we get $U(x_n) = y_n \to 0$.

References

[1] А. Л. Брудно, *Суммирование ограниченных последовательностей матрицами*, Мат. Сборн. 16 (58) (1945), p. 191-247.

[2] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (II)*, Stud. Math. 13 (1953), p. 137-179.

[3] — — *On linear methods of summability*, Stud. Math. 14 (1954), p. 129-160.

[4] W. Orlicz, *Sur les opérations linéaires dans l'espace des fonctions bornées*, Stud. Math. 10 (1948), p. 60-89.

[5] — *Linear operations in Saks spaces (I)*, Stud. Math. 11 (1950), p. 237-272.

[6] — *Linear operations in Saks spaces (II)*, Stud. Math. 15 (1955), p. 1-25.

[7] — *O szeregach doskonale zbieżnych w pewnych przestrzeniach funkcyjnych*, Roczn. PTM, Ser. I, Prace Mat. 1 (1955), p. 393-414.

[8] — *On the continuity of linear operations in Saks spaces with an application to the theory of summability*, Stud. Math. 16 (1957), p. 69-73.

[9] — and V. Pták, *Some remarks on Saks spaces*, Stud. Math. 16 (1957), p. 56-68.

Reçu par la Rédaction le 12.6.1956

On the method of category in analytic manifolds

by

S. Balcerzyk (Toruń) and **Jan Mycielski** (Wrocław)

1. The following hypothesis may be considered as classical (see W. Sierpiński [4] and F. Hausdorff [3]):

(H) *The sum of fewer than $2^{\aleph_0}$ nowhere dense subsets of a complete metric space is a border set in this space.*

Clearly, if the continuum hypothesis is supposed, a positive answer follows by the theorem of Baire on the sets of the first category. Then, by the results of Gödel, (H) is consistent with the present mathematical knowledge, but it has not been proved even in the case where the space is the real line.

It is the purpose of this paper to prove (H) for some special classes of nowhere dense sets in analytic manifolds.

Several applications of theorem 1 of this paper will be given in other works. This theorem is a refinement of a lemma of J. de Groot and T. Dekker [1].

2. Analytic surfaces in analytic manifolds. All manifolds considered are supposed to be connected, real and analytic.

For two manifolds M and A, a mapping f: $M \to A$ is called *analytic* if the local coordinates of $f(p)$ in A are analytic functions of the local coordinates of p in M.

LEMMA. *Let f_1 and f_2 be two analytic mappings of a manifold M into a manifold A. If the set $S = \{p\colon p \in M,\ f_1(p) = f_2(p)\}$ has an interior point, then $S = M$.*

Indeed it can easily be proved that then the set S is open and closed and non-empty. Hence $S = M$ since M is connected.

Definition. A set S is called an *analytic surface* in a manifold M if there exist an open connected set $C \subset M$, a manifold A, and two analytic mappings f_1, f_2: $C \to A$ such that $f_1(p_0) \neq f_2(p_0)$ for some $p_0 \in C$, and

$$S \subset \{p\colon p \in C,\ f_1(p) = f_2(p)\}.$$

THEOREM 1. *The sum of fewer than $2^{\aleph_0}$ analytic surfaces in an analytic manifold M is a border set in M.*