On the extension of equalities in connected topological groups

by

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1. A function on a group of the form

$$\sigma(x_1, \ldots, x_n) = x_{s_1}^{k_1} x_{s_2}^{k_2} \ldots x_{s_m}^{k_m}$$

where $s_i \in \{1, \ldots, n\}$, $k_i = \pm 1$, $x_{s_i}^{k_i} \neq x_{s_{i+1}}^{-k_{i+1}}$ is called a *word*.

A *word with constants on a group* G is a function on this group of the form

$$\varphi(x_1, \ldots, x_n) = a_0\sigma_1 a_1 \sigma_2 a_2 \ldots \sigma_m a_m$$

where $a_i \in G$ and $\sigma_i = \sigma_i(x_1, \ldots, x_n)$ are words.

We consider the following property of a topological group G:

(A) *Let be given any two words with constants* $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ *on* G. *If*

$$(1) \qquad \varphi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n)$$

for every $x_1 \in V_1, \ldots, x_n \in V_n$, *where* $V_1, \ldots, V_n$ *are open sets in* G, *then* (1) *holds for every* $x_1, \ldots, x_n \in G$.

This property has been introduced by S. Balcerzyk as a natural hypothesis in some theorems (cf. S. Balcerzyk and Jan Mycielski [2]). The problem whether every connected group has property (A) remains open [1]). It is the purpose of this paper to prove this for locally compact groups and for Abelian groups.

2. We shall consider functions of the form

$$(2) \qquad \chi(x) = x^{k_1} a_1 x^{k_2} a_2 \ldots x^{k_m} a_m$$

where $a_i \in G$, and k_i are integers $\neq 0$, and the following property:

(A′) *For every function* χ *on* G *if* $\chi(V) = 1$ [2]), *for any set* V *open in* G, *then* $\chi(G) = \{1\}$.

[1]) N. Aronszajn [1] puts forward a weaker hypothesis.

[2]) 1 denotes the unity of G.

Lemma 1. *The properties* (A) *and* (A′) *are equivalent.*

Proof. (A)→(A′) trivially.

(A′)→(A). We put (1) into the form $a_0^{-1}\varphi\psi^{-1}a_0 = 1$; then (A′) implies (A) in the case $n = 1$. An induction with respect to n completes the proof.

We need also the following lemma:

Lemma 2. *Every connected Lie group* G *has property* (A).

Proof. We consider the elements $x, y, xy^{-1} \in G$ in any local systems of analytic coordinates. Then, as is well known (see [3], p. 316), the coordinates of xy^{-1} are analytic functions of the coordinates of x and y. By an easy induction we verify that the coordinates of $\chi(x)$ are analytic functions of the coordinates of x in any local systems of analytic coordinates. Then, by well known properties of analytic functions, (A′) holds since G is connected.

3. Now we can prove the following

Theorem 1. *Every locally compact connected group has property* (A).

Proof. It is a fundamental result of H. Yamabe [4] that

(3) *Every neighbourhood* U *of the unity of a locally compact connected group* G *contains such a closed normal subgroup* N *that* G/N *is a Lie group.*

Let us take the natural mappings $h\colon G \to G/N$, and put

$$\chi^*(x) = x^{k_1} h(a_1) x^{k_2} h(a_2) \ldots x^{k_m} h(a) \quad \text{for} \quad x \in G/N.$$

We suppose that $\chi(V) = \{1\}$ for an open $V \subset G$, then

$$(4) \qquad \chi^*\big(h(V)\big) = \{h(1)\}.$$

The mapping h being open, $h(V)$ is open in G/N. Then, G/N being a Lie group, by Lemma 2 and (4)

$$\chi^*(G/N) = \{h(1)\}, \quad i.\ e., \quad \chi(G) \subset N \subset U.$$

Then, by (3), $\chi(G) \subset \bigcap_U U$ where U runs over all neighbourhoods of the unity of G. Hence $\chi(G) = \{1\}$, q. e. d.

Theorem 2. *Every connected Abelian group has property* (A).

Proof. Let G be a connected Abelian group with the additive notation. Then (A′) states that if

$$(5) \qquad nV = \{a\} \quad (a \in G) \quad \text{for an open} \quad V \subset G$$

then

$$nG=\{a\}.$$

G being connected, V generates G and by (5)

(6) $$nG\subset\{0,\pm a,\pm 2a,\ldots\}.$$

But G being connected, (6) implies $nG=\{0\}$ (and $a=0$), q. e. d.

Remark. It can be proved for each of the equations

$$ax=xa,\quad ax=x^{-1}a,\quad ax=x^2a$$

that if it is satisfied in a neighbourhood of the unity of a connected group G ($a\in G$) then it is satisfied in all G. But for the equations

$$x^3=1,\quad ax^2=x^2a$$

the same problem remains open.

References

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[4] H. Yamabe, *A generalization of a theorem of Gleason*, Ann. of Math. 58 (1953), p. 351-365.

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On the existence of free subgroups in topological groups

by

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1. In this paper we prove the existence of free algebraic subgroups in some topological groups. Our chief results (theorems 2 and 3) state the existence of a free subgroup of the rank $2^{\aleph_0}$ in every compact, connected, non-Abelian group and in every locally compact, connected non-solvable group [1]).

These problems arise in connection with some special constructions on the sphere (see *e. g.* [12], [13]). W. Sierpiński [17] has proved that the group of rotations of the sphere contains a free subgroup of the rank $2^{\aleph_0}$. His proof is effective (*i. e.*, does not use the axiom of choice). This effective method is further developed by J. de Groot [5]. M. Kuranishi [10] proves the existence of a free subgroup of rank 2 in every semi-simple connected Lie group. Clearly our results generalizes all these theorems, but we are using the axiom of choice and the result of Kuranishi. A part of our reasoning (the proofs of Theorem 1 and Lemma 1) is analogous to a proof of the above mentioned theorem of Sierpiński given by J. de Groot and T. Dekker [6].

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2. Notions and notations. The *rank* of a free group F [of an Abelian free group F] is the cardinal number of a set of free generators of F and is, as we know, uniquely determined by F.

A set of free generators of a subgroup of a group G is called a *free set* in G.

The group generated by all the commutators in a group G is called the *commutant* of G.

The closure of the commutant of a topological group is called its *closed commutant* (it is normal because the closure of a normal subgroup is normal).

[1]) These results have been announced in [2].