# Algebraic characterization of abelian divisible groups which admit compact topologies

by

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In this note we shall give a complete characterization of abelian divisible groups which admit compact topologies.

This work is in part an answer to the following problem of S. Hartman: Let $\{K_\tau\}_{\tau \in T}$ denote the class of topological groups obtained from the group of rotations of the circle by different compact and undecomposable topologies. The question is whether all groups $K_\tau$ are topologically isomorphic?

We shall show that there exists a class $T^* \subset T$ of the cardinal continuum of non-isomorphic groups $K_\tau$, $\tau \in T^*$.

## 1. Terms and notation

All groups are additively written abelian groups. A group is called *divisible* if for each integer $n$ and each $g \in G$ the equation $nx = g$ is solvable in $G$. We know [3] that a divisible group is of the form

$$G = \sum_\iota R_\iota^+ + \sum_p C_{p^\infty}^{\lambda_p} \tag{1}$$

where the groups $R^+$ are isomorphic to the group of rational numbers and the group $C_{p^\infty}^{\lambda_p}$ is the direct sum of $\lambda_p$ quasicyclic groups $C_{p^\infty}$. Denote the cyclic group of order $p$ by $C_p$.

The topology $\tau$ of the group $G$ is decomposable if the topological group $G_\tau$ with the topology $\tau$ is the direct sum of its two closed subgroups. If any topology of the group $G$ is one-dimensional, connected and compact, then that topology is undecomposable ([4], p. 263).

Two topological groups $G'$ and $G''$ are topologically isomorphic (symbolically $G' \simeq G''$) if there exists a bicontinuous isomorphism of $G'$ and $G''$.

The character group (topologized in the known manner ([4], p. 242) of the topological group $G$ is denoted by $\hat{G}$.



Let $G$ be an arbitrary torsion-free group, $g \neq 0$ an element of $G$ and $p_1, p_2, \ldots$ the increasing sequence of all prime numbers. We denote by $k_n(g)$ the greatest integer $k$ for which the equation $p_n^k x = g$ is solvable in $G$; if there is no maximal exponent of this property we write $k_n(g) = \infty$. The sequence

$$\big(k_n(g)\big) = \big(k_1(g), k_2(g), \ldots\big)$$

thus defined is called the *characteristic* of the element $g \in G$. Consider two characteristics, $\big(k_n'(g)\big)$ and $\big(k_n''(h)\big)$, as equivalent if $k_n'(g) = k_n''(h)$ for all but a finite number of indices $n$ with $k_n'(g) \neq \infty$ and $k_n''(h) \neq \infty$.

## 2. Lemmas

The following lemmas are well known:

1. *If a topological group $G$ is discrete, then $\hat{G}$ is compact* ([4], p. 242).

2. *If a topological group $G$ is discrete or compact, then $G \simeq \hat{\hat{G}}$* ([4], p. 256).

3. *If $G$ is discrete and $\overline{\overline{G}} = \aleph$, then $\overline{\overline{\hat{G}}} = 2^{\aleph}$* ([4], p. 261).

4. *The character group of a compact torsion-free topological group is divisible* ([2], p. 55).

5. *The character group of a divisible compact topological group is torsion-free* ([2], p. 55).

6. *The (compact) character group of a discrete topological group of rank $n$ has the dimension $n$* ([4], p. 263).

7. *A compact divisible topological group is connected and, vice versa, a compact connected topological group is divisible* ([2], p. 55).

8. *The characteristics of non-zero elements of a torsion-free group of rank 1 are equivalent* ([3], p. 191).

9. *The class of the characteristics equivalent to the sequence $\big(k_n(g)\big)$ defines up to an isomorphism the group $G$ of rank 1* ([3], p. 191).

Each characteristic of this class is called a *characteristic of the group $G$*.

10. *To each sequence of positive numbers and the symbols $\infty$ corresponds a torsion-free group of rank 1, having that sequence for its characteristic* ([3], p. 192).

## 3. Theorems

Let $G$ be any compact connected topological group. The group $G$, as a divisible group, is of the form (1). By Lemma 5, the character group $\hat{G}$ is a discrete torsion-free topological group. Let the sequence $\big(k_n(g)\big)$ be the characteristic of an element $g \in \hat{G}$.

THEOREM 1. *If $k_n(g)=\infty$ for each $g\in\hat{G}$, then the group $C_{p_n^\infty}$ does not occur in the sum* (1).

Proof. If the above statement were not true, there would exist such a non-zero character $\chi$ of the group $\hat{G}$, that $p_n\chi$ would be a zero character, or $p_n\chi(\hat{G})=0$; hence $\chi(p_n\hat{G})=0$. But according to the hypothesis each element of $\hat{G}$ is divisible through $p_n$, which means that $p_n\hat{G}=\hat{G}$, whence $\chi(\hat{G})=0$, which contradicts the assumption.

THEOREM 2. *If there exists such an element $g\in\hat{G}$ that $k_n(g)\neq\infty$, then the group $C_{p_n^\infty}$ appears in the sum* (1).

Proof. The equation $p_n^{k_n(g)+1}x=g$ is not solvable in $G$, whence $p_n^{k_n(g)+1}\hat{G}\subsetneqq\hat{G}$. The elements of the group $\hat{G}/p_n^{k_n(g)+1}\hat{G}$ are at most of order $p_n^{k_n(g)+1}$. This group, as a discrete topological group, has a non-zero character $\chi_1$. The superposition of the natural homomorphism $\hat{G}$ onto $\hat{G}/p_n^{k_n(g)+1}\hat{G}$ and of the character $\chi_1$ is a character of order at most $p_n^{k_n(g)+1}$ of the discrete topological group $\hat{G}$. In view of the divisibility of $G$ we conclude from the existence of such element in $G$ that the whole group $C_{p_n^\infty}$ appears in the sum (1).

THEOREM 3. *If $\hat{G}/p_n\hat{G}$ is a cyclic group, then in the sum* (1) *appears exactly one direct summand $C_{p_n^\infty}$.*

Proof. Each character of the order $p_n$ of the group $\hat{G}$ is a character of the group $\hat{G}/p_n\hat{G}$, and vice versa. Hence there are exactly $p_n$ elements of order $p_n$ in $G$, which completes the proof.

THEOREM 4. *If $\hat{G}$ is of rank 1 then $\hat{G}/p_n\hat{G}$ is cyclic.*

Proof. Since all elements of the group $\hat{G}/p_n\hat{G}$ have bounded orders, the group $\hat{G}/p_n\hat{G}$ is the direct sum of cyclic groups. Let $a_1+p_n\hat{G}$, $a_2+p_n\hat{G},\ldots$ be generators of those groups. Let $a_1\neq a_2$; since $\hat{G}$ is of rank 1, we have $ka_1=ma_2$ and $(k,m=1)$; then $ka_1+p_nG=ma_2+p_nG$, but thus $ka_1=ma_2\in p_nG$, whence $p_n|k$ and $p_n|m$, which gives a contradiction.

## 4. Corollaries

Let us take an arbitrary sequence of integers and symbols $\infty$ $(k_1,k_2,\ldots)$. This sequence defines (up to an isomorphism) the torsion-free group $\hat{G}$ of rank 1 for which the sequence $(k_1,k_2,\ldots)$ is the characteristic. If we consider the group $\hat{G}$ as a discrete topological group, its character group $G$ is compact, connected and one-dimensional. Let the group $G$ correspond to the characteristic $(k_1,k_2,\ldots)$ of the group $\hat{G}$. Isomorphic, discrete, topological groups have topologically isomorphic character groups, and vice versa.



COROLLARY 1. *Equivalent characteristics correspond to topologically isomorphic topological groups, and vice versa: topologically isomorphic groups correspond to equivalent characteristics.*

By theorems 1-4 we have

COROLLARY 2. *The characteristic $(k_1,k_2,\ldots)$ of the group $\hat{G}$ defines the algebraical form of the group $G$.*

Indeed: if $k_n=\infty$, then the direct summand $C_{p_n^\infty}$ does not appear in $G$; if $k_n\neq\infty$, then there is exactly one direct summand $C_{p_n^\infty}$ in $G$.

COROLLARY 3. *The group $H$ of the form*

$$\sum_{\iota\in\Omega}R_\iota+\sum_{i}C_{p_{n_i}^\infty}$$

*where $\overline{\overline{\Omega}}=c$ has a compact topology.*

Proof. As follows from Corollary 2 and lemma 3, the group $H$ can be the character group of the group $\hat{H}$ of rank 1 and in the characteristic $\{k_n\}$ of $H$ only $k_{n_i}\neq\infty$.

The characteristic $(k_1,k_2,\ldots)$ for which $k_n=\infty$ for all but a finite number of indices $n=n_1,n_2,\ldots,n_m$ corresponds in our correspondence to a one-dimensional, connected and compact topological group $G$ of the algebraic form $G=\sum_\iota R_\iota^+ + C_{p_{n_1}^\infty}+\ldots+C_{p_{n_m}^\infty}$. From the fact that all characteristics whose element $=\infty$ for all but a given finite number of indices $n=n_1,n_2,\ldots,n_m$ are equivalent follows

COROLLARY 4. *All one-dimensional, compact, connected topological groups of the algebraic form $G=\sum_\iota R_\iota+C_{p_{n_1}^\infty}+\ldots+C_{p_{n_m}^\infty}$ are topologically isomorphic (with the sequence $p_{n_1},\ldots,p_{n_m}$ fixed).*

Let $(k_1,k_2,\ldots)$ be a characteristic for which there exists an infinite sequence of indices $\{n_i\}$ such that $k_{n_i}\neq\infty$. This characteristic corresponds to the compact, connected, one dimensional topological group $G$ of the form $G=\sum_\iota R_\iota^+ + \sum_i C_{p_{n_i}^\infty}$.

The class of non-equivalent characteristics with $k_{n_i}\neq\infty$ $(i=1,2,\ldots)$ has the cardinal continuum.

COROLLARY 5. *The class of topologically non-isomorphic, one-dimensional, connected and compact topological groups of the algebraic form $\sum_\iota R_\iota+\sum_i C_{p_{n_i}^\infty}$ has the cardinal continuum.*

This answers the problem concerning the group of rotations of the circle raised by S. Hartman.

## 5. The main result

THEOREM 5. *A divisible abelian group $G$ admits a compact topology if and only if it is of the form*

$$G = \sum_{\iota \in T} R_\iota^+ + \sum_p C_{p^\infty}^{\lambda_p} \tag{2}$$

*where* $1^\circ$ $\overline{\overline{T}} = 2^{\mathfrak{m}}$, $\mathfrak{m} \geqslant \aleph_0$, $2^\circ$ *$\lambda_p$ is finite or* $\lambda_p = 2^{\aleph}$, $\aleph \geqslant \aleph_0$, $3^\circ$ *for each $p$* $\overline{\overline{T}} \geqslant \lambda_p$.

Proof. Sufficiency. It can easily be seen that the group $G$ of the form (2) with conditions $1^\circ$-$3^\circ$ can be expressed as a complete direct sum of groups having the form

$$\sum_{\iota \in \Omega} R_\iota^+ + \sum_i C_{p_{n_i}^\infty} \tag{3}$$

where $\overline{\overline{\Omega}} = \mathfrak{c}$ and $\{n_i\}$ is a finite or infinite sequence of integers. Each group of the form (3) has a compact topology, as was stated in Corollary 2. The group $G$ has then a compact (product) topology.

Necessity. Let $G$ be an arbitrary divisible group that has a compact topology. By lemma 3 $\overline{\overline{G}} = 2^{\mathfrak{n}}$ where $\mathfrak{n} = \overline{\overline{\hat{G}}}$. The following simple reasoning shows that the torsion-free part of the group $G$ has also the cardinal $2^{\mathfrak{n}}$; that implies that $G$ fulfils $1^\circ$ and $3^\circ$. Let $a_1, \ldots, a_\mu, \ldots$ be a maximal set of linearly independent elements of the torsion-free group $\hat{G}$, and $x_1, x_2$ two linearly independent elements of the torsion-free part of the group of rotations of the circle $K$. Let $\chi$ be an arbitrary function on $a_1, \ldots, a_\mu, \ldots$ to $x_1, x_2$. This function can be prolonged to the character of the discrete topological group $\hat{G}$. Indeed: let $a \in \hat{G}$; we have $na = n_{\mu_1} a_{\mu_1} + \ldots + n_{\mu_m} a_{\mu_m}$. Let $\chi(na) = n_{\mu_1} \chi(a_{\mu_1}) + \ldots + n_{\mu_m} \chi(a_{\mu_m})$ and $\chi(a) = z$, where $z$ belongs to the torsion-free part of $K$ and is a solution of the equation $nx = n_{\mu_1} \chi(a_{\mu_1}) + \ldots + n_{\mu_m} \chi(a_{\mu_m})$. Then $\chi$ is the character of the infinite order of the discrete topological groups $\hat{G}$, whence the torsion-free part of the group $G$ has the cardinal $2^{\mathfrak{n}}$ and $\mathfrak{n} \geqslant 0_0$.

To get condition $2^\circ$ we consider the subgroup $G_p$ of all elements of order $p$ of the group $G$, and observe that $G_p$ is the character group of the discrete topological group $\hat{G}/p\hat{G}$. If $\hat{G}/p\hat{G}$ is infinite and has the cardinal $\aleph$ by lemma 3 $G_p$ has the cardinal $2^{\aleph}$ [1]).

## 6. Example

From the existence of torsion-free undecomposable groups of rank greater than 1 follows the existence of compact connected topological groups of dimension greater than 1 which are not direct sums of their

[1]) The fact that the number of $C_{p^\infty}$ summands in the compact topological group either is finite or is an infinite cardinal of the form $2^{\aleph}$ is known (see [2], p. 55).



closed subgroups. Therefore the product topologies of complete direct sums of one-dimensional, compact, topological groups do not exhaust the class of all compact topologies of divisible groups. For example: any pure subgroup $C$ of the group of $p$-adic integers is undecomposable (cf. [3]) and $C/pC \simeq C_p$. Hence the character group of the discrete topological group $C$ is by theorems 1-4 of the algebraical form $C = \sum_\iota R_\iota + C_{p^\infty}$. If $C$ has rank $a$, finite or not, then $\dim \hat{C} = a$, and, in view of the undecomposability of $C$, $\hat{C}$ is topologically undecomposable.

### References


[1] S. Hartman und C. Ryll-Nardzewski, *Zur Theorie der lokal-kompakten abelschen Gruppen*, Coll. Math. 4 (1956), p. 157-188.

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