On theories categorical in power

by

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The aim of this paper is to give some applications of theorems on automorphisms of models [1] to the study of theories categorical in power [1]. The main result is contained in theorem 1 which states, roughly speaking, that no antisymmetric and connected relation is definable in a theory categorical in power $2^{\mathfrak{n}}$ where $\mathfrak{n} \geqslant \aleph_0$. As a corollary we find that no ordering relation can be defined in any such theory. The final theorems deal with the existence of mutually indiscernible elements in each model of a theory categorical in power $2^{\mathfrak{n}}$ as well as with the existence of universal models of such theories.

The terminology and notation used in this paper are the same as in [1]. For more detailed information concerning theories categorical in power and examples of such theories the reader is referred to papers [3] and [5].

Definitions. An n-ary relation $R(\xi_1, \ldots, \xi_n)$ is *antisymmetric* in the set A if, for arbitrary $x_1, \ldots, x_n \in A$,

$$\neq(x_1, \ldots, x_n) \quad \text{implies} \quad \sum_{\pi \in S_n} \sim R(x_{\pi(1)}, \ldots, x_{\pi(n)})\ ^{2)}$$

where S_n is the set of all permutations of the set $\{1, \ldots, n\}$.

The relation $R(\xi_1, \ldots, \xi_n)$ is *connected* in the set A if, for arbitrary $x_1, \ldots, x_n \in A$,

$$\neq(x_1, \ldots, x_n) \quad \text{implies} \quad \sum_{\pi \in S_n} R(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

Let α be an order-type and P a subset of S_n. We shall say that an n-ary relation R defined in a set A *belongs* to the set $K(\alpha, P, A_1)$ where A_1 is a subset of A if there is an ordering relation $\prec$ of the order-type α in the set A_1, such that, for arbitrary $x_1, \ldots, x_n$ in A_1,

$$x_1 \prec \ldots \prec x_n \quad \text{implies} \quad [R(x_{\pi(1)}, \ldots, x_{\pi(n)}) \text{ if and only if } \pi \in P].$$

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[1]) A theory T is *categorical in power* $\mathfrak{m}$ if any two of its models of power $\mathfrak{m}$ are isomorphic.

[2]) $\neq(x_1, \ldots, x_n)$ is the conjunction of inequalities $x_i \neq x_j$ where $1 \leqslant i < j \leqslant n$. The letter $\sum$ stands for "there is" and the symbol $\sim$ stands for negation.

THEOREM 1. *Let* $\mathfrak{n}$ *be an arbitrary infinite cardinal number and* T *a theory with the following property: there is an n-ary predicate* ϱ *defined in* T *and a model* M *of* T *over a set* $|M|$ [3]) *such that* M_ϱ *is antisymmetric and connected in an infinite set* $A_1 \subset |M|$. *Under these assumptions* T *is not categorical in the power* $\mathfrak{m} = 2^{\mathfrak{n}}$.

LEMMA 1. *If* $\mathfrak{m} = 2^{\mathfrak{n}}$, *then there is a set* X *and an ordering relation* $\precsim$ *in the set* X *such that*

(1) *the ordering relation is homogeneous (i. e., each two intervals are similar),*

(2) $\overline{\overline{X}} = \mathfrak{m}$,

(3) *there is a subset* $X_1 \subset X$ *dense in* X *such that* $\overline{\overline{X}}_1 = \mathfrak{n}$.

An outline of the proof of this lemma is given in the appendix. Here we remark only that a set X satisfying (1)-(3) can be obtained by applying the well-known Dedekind procedure of completing the cuts to a set X_1 of power $\mathfrak{n}$ ordered in the type η_ξ (see [2]).

LEMMA 2. *Under the assumptions of theorem* 1, *there is a set* P_0, $\emptyset \subsetneqq P_0 \subsetneqq S_n$ *and a countable set* $A_0 \subset A_1$ *such that*

$$M_\varrho \in K(\omega, P_0, A_0).$$

Proof. Let A' be an arbitrary countable subset of A_1 and $\precsim$ an ordering relation of order type ω in A'. Further let $\mathfrak{R}$ be the class of the subsets of A' having exactly n elements. We define a division into $2^{n!}$ subclasses K_P, each subclass being defined by a subset P of S_n. The definition of K_P is as follows:

$$Z \in K_P \quad \text{if and only if} \quad [M_\varrho(x_{\pi(1)}, \ldots, x_{\pi(n)}) \text{ is equivalent to } \pi \in P],$$

where $x_1, \ldots, x_n$ are distinct elements of Z, and $x_1 \precsim \ldots \precsim x_n$. It is obvious that subclasses K_P and $K_{P'}$ are either equal or disjoint.

Since M_ϱ is antisymmetric and connected in the set A', classes K_{S_n} and $K_\emptyset$ are empty. By Ramsey's theorem [4] it follows, that there is a class K_{P_0} and infinite subset $A_0 \subset A'$ such that conditions $Z \subset A_0$ and $Z \in \mathfrak{R}$ imply $Z \in K_{P_0}$. Lemma 2 is thus proved.

To prove theorem 1 we now construct two non-isomorphic models M^1, M^2 of the theory T, both having the power $\mathfrak{m}$.

In order to define the first model we denote by T^* an open theory which is an inessential extension of T (cf. [1], p. 52) and by X a set (ordered by a relation $\precsim$) whose existence is stated in lemma 1. Further, let β be the order type of the relation $\precsim$ in X. Applying theorem 6.1 of [1], we obtain a model $M^1 = M(X)$ of T^* (and hence of T) which satisfies the conditions:

[3]) $|M|$ denotes the set of individuals of the model M.

(4) $M^1_\varrho \in K(\beta, P_0, X)$,

(5) each one-one mapping f of X onto itself which preserves the relation $\precsim$ can be extended to an automorphism of M^1.

The essential property of the model M^1 is given in the following

LEMMA 3. *For no set* $B \subset |M^1|$

$$M_\varrho \in K(\omega(\mathfrak{m}), P_0, B) \tag{6}$$

where $\omega(\mathfrak{m})$ *is the least ordinal number of power* $\mathfrak{m}$.

The proof of this lemma, which constitutes the central point of the whole paper, will be divided into several stages:

I. Suppose that the lemma is false, *i. e.*, that there is a set $B \subset |M^1|$ and an ordering relation $\preccurlyeq$ of the order-type $\omega(\mathfrak{m})$ in the set B such that for arbitrary $x_1, \ldots, x_n$ in B the formula $x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_n$ implies that

$$M_\varrho(x_{\pi(1)}, \ldots, x_{\pi(n)}) \quad \text{is equivalent to} \quad \pi \in P_0.$$

II. From the construction of the model $M^1 = M(X)$ it easily follows that each element x in $|M^1|$ and hence each element x in B is representable in the form $\mathrm{val}_{M^1}\varphi(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k \in X$ and φ is a term of T^*.

Since the number of terms of T^* is $\aleph_0$, there is a term φ_0 such that $\mathfrak{m}$ elements of B are representable in the form

$$x = \mathrm{val}_{M^1}\varphi_0(x_1, \ldots, x_k), \qquad x_1, \ldots, x_k \in X. \tag{7}$$

We denote by B' the set of these elements.

III. We shall select from B' a narrower set B'' of power $\mathfrak{m}$ with the following property: there is a term ψ of T^* with $p+q$ free variables $(p \geqslant 0, q \geqslant 0)$, q elements $y_1, y_2, \ldots, y_q$ of X and a subset X'' of X such that each element $u \in B''$ is representable in the form

$$u = \mathrm{val}_{M^1}\psi(x_1, \ldots, x_p, y_1, \ldots, y_q), \qquad x_j \in X'', \quad x_i \neq x_j \text{ for } i \neq j;$$

moreover if

$$u' = \mathrm{val}_{M^1}\psi(x_1', \ldots, x_p', y_1, \ldots, y_q), \qquad u'' = \mathrm{val}_{M^1}(x_1'', \ldots, x_p'', y_1, \ldots, y_q)$$

$(x_j', x_j'' \in X''$ for $j = 1, 2, \ldots, p)$ and $u' \neq u''$, then $x_i' \neq x_j''$ for $i, j = 1, 2, \ldots, p$.

It will be sufficient to indicate the construction of B'' and X'' in the particular case $k = 2$.

For each $u \in B'$ we select a pair $z_1 = z_1(u)$, $z_2 = z_2(u)$ of element of X such that

$$u = \mathrm{val}_{M^1}\varphi_0(z_1, z_2)$$

and regard $z_1(u)$, $z_2(u)$ as projections of u on the coordinate axes x and y.

If the projection of B' on the x-axis has the power $<\mathfrak{m}$ there is a $y_1 \in X$ such that $\mathfrak{m}$ elements of B' have the form $\mathrm{val}_{M^1}\varphi_0(y_1,z_2)$. Reversing the order of variables in φ_0 we represent $\mathfrak{m}$ elements of B' in the form $\mathrm{val}_{M^1}\psi(x_1,y_1)$ where $\psi(x_1,y_1)=\varphi_0(y_1,x_1)$.

It is now sufficient to put

$$X''=\underset{z_2(u)}{E}[u \in B' \text{ and } z_1(u)=y_1]$$

and to take as B'' the set of all elements $\mathrm{val}_{M^1}\psi(x,y_1)$ where $x \in X''$.

If the projection of B' on the x-axis has the power $\mathfrak{m}$ we select from B' a subset such that no two elements of the subset have the same abscissa. We call this subset B^*.

If B^* has $\mathfrak{m}$ elements in common with the diagonal set (consisting of elements with identical coordinates), then we omit all the other elements and call the remaining set B''. The elements of B'' are representable in the form $\mathrm{val}_{M^1}\psi(x)$ where $\psi(x)=\varphi_0(x,x)$. It is now sufficient to take as X'' the projection of B'' on the x-axis.

It remains to consider the case when B^* has less than $\mathfrak{m}$ elements in common with the diagonal set. Then we omit these elements, call the remaining set again B^* and consider the projection of B^* on the y-axis. If its power is $<\mathfrak{m}$, then we proceed as in the first case, considered above. If its power is $\mathfrak{m}$, then we select a subset B'' having at most one element on each parallel to the x-axis and take as X'' the sum of the projection of B'' on the x-axis and the projection of B'' on the y-axis.

IV. The relation $\lll$ orders the set B'' in the type $\omega(\mathfrak{m})$; we can therefore enumerate the elements of B'' by means of ordinals $<\omega(\mathfrak{m})$. Let

$$u_0 \lll u_1 \lll \ldots \lll u_\xi \lll \ldots, \qquad \xi<\omega(\mathfrak{m}),$$

be all the elements of B''. We divide this sequence into the n-tuples

$$(u_0,\, u_1,\, \ldots,\, u_{n-1}),\; \ldots,\; (u_\xi,\, u_{\xi+1},\, \ldots,\, u_{\xi+n-1}),\ldots \tag{8}$$

For each of those n-tuples $(u_\xi,\, u_{\xi+1},\, \ldots,\, u_{\xi+n-1})$ we represent $u_{\xi+j}$ in the form (see III)

$$u_{\xi+j}=\mathrm{val}_{M^1}\psi(x_{\xi+j,1},\ldots,x_{\xi+j,p},y_1,\ldots,y_q)$$

where the "coordinates" $x_{\xi+j,1},\ldots,x_{\xi+j,p}$ belong to the set X''. We further establish a correspondence between the considered n-tuple and a system $\mathfrak{A}_\xi$ of np disjoint intervals of the set X

$$\mathfrak{A}_\xi=(\langle a_{\xi 1},b_{\xi 1}\rangle,\ldots,\langle a_{\xi p},b_{\xi p}\rangle,\ldots,\langle a_{\xi+n-1,1},b_{\xi+n-1,1}\rangle,\ldots,\langle a_{\xi+n-1,p},b_{\xi+n-1,p}\rangle)$$

such that for $i=1,2,\ldots,p$, $j=0,1,\ldots,n-1$ the following relations hold:

$$a_{\xi+j,i} \prec x_{\xi+j,i} \prec b_{\xi+j,i}, \qquad a_{\xi+j,i}, b_{\xi+j,i} \in X_1.$$

The existence of such intervals follows from the density of X_1 in X.

Since $\overline{\overline{X}}_1=\mathfrak{n}$, the number of different systems is at most $\mathfrak{n}$, hence at least one system, say $\mathfrak{A}$, is correlated with n different n-tuples (8). Thus there are ordinals $\xi_1,\xi_2,\ldots,\xi_n$ such that

$$\mathfrak{A}=\mathfrak{A}_{\xi_1}=\mathfrak{A}_{\xi_2}=\ldots=\mathfrak{A}_{\xi_n}, \qquad \xi_1<\xi_2<\ldots<\xi_n. \tag{9}$$

V. Now let α,β be two arbitrary permutations belonging to S_n. We consider the finite sequence

$$u_{\xi_{\alpha(1)}+\beta(1)-1},\ldots,u_{\xi_{\alpha(n)}+\beta(n)-1}.$$

The ith coordinate $x_{\xi_{\alpha(j)}+\beta(j)-1,i}$ of the jth term of this sequence lies in the interval $(a_{\xi_{\alpha(j)}+\beta(j)-1,i},\, b_{\xi_{\alpha(j)}+\beta(j)-1,i})$ which in view of (9) is identic with $\Delta_{ji}=(a_{\xi_1+\beta(j)-1,i},\, b_{\xi_1+\beta(j),i})$ and thus is independent of α. In view of the homogeneity of X (lemma 1) there exists a similarity transformation of Δ_{ji} onto itself which maps $x_{\xi_{\alpha(j)}+\beta(j)-1,i}$ onto $x_{\xi_1+\beta(j)-1,i}$. Since the different intervals, as members of one and the same system $\mathfrak{A}$, are disjoint from each other these similarity transformations can be extended to a transformation of X onto itself which preserves the ordering relation $\prec$. By (5) we can extend this transformation to an automorphism of M^1. We have thus obtained an automorphism of M^1 which maps $x_{\xi_{\alpha(j)}+\beta(j)-1,i}$ onto $x_{\xi_1+\beta(j)-1,i}$ for $i=1,\ldots,p$, $j=1,\ldots,n$, and hence (see III) maps $u_{\xi_{\alpha(j)}+\beta(j)-1}$ onto $u_{\xi_1+\beta(j)-1}$, $j=1,\ldots,n$.

It follows that

$$M^1_\varrho(u_{\xi_{\alpha(1)}+\beta(1)-1},\ldots,u_{\xi_{\alpha(n)}+\beta(n)-1}) \quad \text{if and only if} \quad M^1_\varrho(u_{\xi_1+\beta(1)-1},\ldots,u_{\xi_1+\beta(n)-1}). \tag{10}$$

VI. We now conclude the proof of lemma 3. Since

$$u_{\xi_1+l_1} \lll u_{\xi_2+l_2} \lll \ldots \lll u_{\xi_n+l_n}$$

for arbitrary non negative $l_1,\ldots,l_n<n$ and since

$$u_{\xi_1} \lll u_{\xi_1+1} \lll \ldots \lll u_{\xi_1+n-1},$$

we obtain from I the equivalences

$$M_\varrho(u_{\xi_{\alpha(1)}+\beta(1)-1},\ldots,u_{\xi_{\alpha(n)}+\beta(n)-1}) \quad \text{if and only if} \quad \alpha \in P_0,$$

$$M_\varrho(u_{\xi_1+\beta(1)-1},\ldots,u_{\xi_1+\beta(n)-1}) \quad \text{if and only if} \quad \beta \in P_0.$$

Using (10) we infer that $\alpha \in P_0$ if and only if $\beta \in P_0$ whence $P_0=\emptyset$ or $P_0=S_n$, which contradicts lemma 2.

Lemma 3 is thus proved.

We now define a second model, M^2. To this end we add to the theory T a set $Y=\{y_\xi\}_{\xi<\omega(\mathfrak{m})}$ of new constants and a set of $\mathfrak{m}$ axioms of the following forms:

$$\varrho(y_{\xi_{\alpha(1)}},\dots,y_{\xi_{\alpha(n)}}) \quad \text{if} \quad \alpha\in P_0,$$
$$\sim\varrho(y_{\xi_{\alpha(1)}},\dots,y_{\xi_{\alpha(n)}}) \quad \text{if} \quad \alpha\notin P_0,$$

where $\xi_1<\dots<\xi_n$ are arbitrary ordinals.

The extended theory obviously has the following property:

for each model M^2 of it such that $\overline{\overline{|M^2|}}=2^{\mathfrak{n}}$

$$M^2_\varrho\in K\big(\omega(\mathfrak{m}),P_0,\underset{M^2_y}{E}[y\in Y]\big). \tag{11}$$

Let M^2 be an arbitrary model of T satisfying (11). In view of lemma 3 formula (11) proves that models M^1 and M^2 are non-isomorphic, which completes the proof of theorem 1.

From theorem 1 we immediately obtain the following

COROLLARY. *No ordering relation can be defined in a theory which is categorical in power* $2^{\mathfrak{n}}$ *with* $\mathfrak{n}\geqslant\aleph_0$.

THEOREM 2. *If a theory T is categorical in power $2^{\mathfrak{n}}$ and $|\overline{\overline{M}}|=2^{\mathfrak{n}}$, then there is a set $A_0\subset|M|$ such that $\overline{\overline{A}}_0=2^{\mathfrak{n}}$ and every one-one transformation of A_0 onto itself can be extended to an automorphism of the model M.*

Proof. From theorem 1 it follows that for any predicate ϱ of T and for any infinite set $B\subset|M|$ there is an infinite subset $B'\subset B$ whose elements are indiscernible with respect to M_ϱ, *i. e.* are such that either $M_\varrho(x_1,\dots,x_n)$ for an arbitrary n-tuple $x_1,\dots,x_n$ of distinct elements of B' or $\sim M_\varrho(x_1,\dots,x_n)$ for such an n-tuple. It easily follows (see [1], p. 54) that the theory T^* remains consistent upon adjunction of a set X of power $2^{\mathfrak{n}}$ of new constants and of a set of axioms of the form

$$\varrho(y_1,\dots,y_n)\supset\varrho(x_1,\dots,x_n) \tag{12}$$

where $(y_1,\dots,y_n)$ and $(x_1,\dots,x_n)$ are arbitrary n-tuples of distinct constants of X and ϱ is an arbitrary predicate of T^*. Indeed, any finite set of the axioms of the form (12) in which predicates $\varrho_1,\dots\varrho_s$ occur can be satisfied in M by interpreting the constants occurring in the new axioms as elements of M which are indiscernible with respect to relations $M_{\varrho_1},\dots,M_{\varrho_s}$.

Let $M'(X)$ be a model of the extended theory built in the way indicated in [1], p. 35-57. Each one-one transformation of X onto itself can obviously be extended to an automorphism of $M'(X)$ (see [1], p. 63). Since T^* is categorical in power $2^{\mathfrak{n}}$, it follows that the models M and $M'(X)$ are isomorphic; the desired subset A_0 of $|M|$ is thus the image of X under this isomorphism.

THEOREM 3. *If a theory T is categorical in power $2^{\mathfrak{n}}$, then for any $\mathfrak{p}<2^{\mathfrak{n}}$ it has a model universal for the power $\mathfrak{p}$ (i. e., a model M^0 of power $\mathfrak{p}$ such that each model M of power $\mathfrak{p}$ is isomorphic with a submodel of M_0).*

Proof. Let $M'(X)$ be a model of T of power $2^{\mathfrak{n}}$ such that elements of X are indiscernible with respect to each M_ϱ and further that every element of $M'(X)$ is a value of a term of the theory T^* for arguments in X. The existence of such a model easily follows from theorem 2. Any model M of power $\mathfrak{p}<2^{\mathfrak{n}}$ is obviously isomorphic with a submodel of a model of power $2^{\mathfrak{n}}$. Hence by the categoricity of T, M is isomorphic with a submodel of $M'(X)$ whence it easily follows that there is a subset X_0 of X of power $\mathfrak{p}$ such that M is isomorphic with a submodel of the least submodel $M'(X_0)$ of $M'(X)$ which contains X_0.

But, by theorem 2, for any two sets $X_0,X_1\subset X$ such that $\overline{\overline{X}}_0=\overline{\overline{X}}_1$, there is an automorphism of $M'(X)$ which maps X_0 onto X_1. Hence $M'(X_0)$ and $M'(X_1)$ are isomorphic and thus $M'(X_0)$ is the required universal model.

Appendix. Proof of lemma 1

A set X_1 of order type η_ξ of power $\mathfrak{n}$ is by definition the set of sequences $a=\{a_\alpha\}_{\alpha<\omega(\mathfrak{n})}$ with $a_\alpha=0$ or $a_\alpha=1$ for each $\alpha<\omega(\mathfrak{n})$ which satisfy the following condition: there is an ordinal $\alpha<\omega(\mathfrak{n})$ such that $a_\beta=0$ for $\beta>\alpha$. The ordering of X_1 is lexicographical. Let X be the set obtained from X_1 by the completion of all cuts of X_1. The elements of X can be represented uniquely by sequences $\{a_\alpha\}_{\alpha<\omega(\mathfrak{n})}$; with $a_\alpha=0$ or $a_\alpha=1$ such that for every $\alpha<\omega(\mathfrak{n})$ there is a $\beta>\alpha$ with $a_\beta=0$. Clearly the conditions (2) and (3) are satisfied.

In order to prove (1) it is sufficient to show that each interval (a,b) of X is similar to X.

Let $x\in(a,b)$, $x=\{x_\alpha\}$, $a=\{a_\alpha\}$, $b=\{b_\alpha\}$.

We establish a correspondence between x and an element $y\in X$ by the transfinite induction:

1° $y_0=x_{(\mu\gamma)[a_\gamma\neq b_\gamma]}$ [4].

2° If y_α are defined for $\alpha<\alpha_1$ and depend only on X_β with $\beta<\beta_1$, then

$$y_{\alpha_1}=x_{(\mu\gamma)[\sum_{x'}(x'_\beta=x_\beta \text{ for } \beta<\beta_1, \text{ and } x'_\gamma\neq x_\gamma)]}\,.$$

[4]) $(\mu\gamma)[\dots]$ denotes the least ordinal γ satisfying $[\dots]$.

Thus y_{α_1} is defined and depend only on x_α with

$$\alpha \leqslant (\mu\gamma)\Big[\sum_{x'}(x'_\beta = x_\beta) \text{ for } \beta < \beta_1 \text{ and } x'_\gamma \neq x_\gamma)\Big].$$

One can easily verify that the correspondence $x \leftrightarrow y$ defined above is a similarity mapping of (a,b) onto X.

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