On the ε-theorems *

by

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The purpose of this paper is to provide a simple proof [1] of the two well-known *ε-theorems* (see Hilbert and Bernays [5]) for elementary classical theories and to demonstrate that these theorems hold also for some non-classical theories considered in my paper [10], in particular those based on the *intuitionistic* logic (see Heyting [4]). The second ε-theorem for non-classical theories proved below, permits the elimination of quantifiers only in the case of theories whose axioms are all in the prenex normal form (*i. e.* all quantifiers occur before a formula without quantifiers).

To achieve the above-mentioned aim, we shall use the algebraic method which has been successfully applied to simplify the proofs of Gödel's theorem (see *e. g.* Rasiowa and Sikorski [11], of the Gödel-Skolem-Löwenheim theorem (see *e. g.* Rasiowa and Sikorski [12]) and of Herbrand's theorem (see Łoś, Mostowski and Rasiowa [7]). The notion of the algebraic model investigated in my paper [10] plays the essential part in the proof of ε-theorems. Thus this paper is a continuation of [10]. In order to enable those readers who are not interested in non-classical theories to read this paper independently, the proof will be given only for classical theories. The knowledge of [10] makes it possible to pass without any difficulty to the case of non-classical theories considered in that paper.

It is worth noticing that the proof given below holds for non-enumerable theories. Thus the Gödel-Malcev theorem (see Gödel [1], Malcev [9] and *e. g.* Henkin [3], Robinson [14]) can easily be proved by the use of Łoś's method [2].

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[1]) Another proof of the first ε-theorem was given by Łoś [6]. For a proof of the second ε-theorem cf. Hasenjaeger [2].

[2]) Indeed, if a theory is consistent, then by the second ε-theorem the theory arising from it by Skolem's elimination of quantifiers is also consistent. The proof that this theory has a model presents no difficulties (cf. Łoś [6], Rasiowa and Sikorski [11]). But this model is also the model of the previous theory.

1. Elementary formalized theories. Let $\mathfrak{T}(\mathfrak{A})$ be an axiomatic theory (with a finite or infinite set $\mathfrak{A}$ of axioms) formalized on the basis of the first order functional calculus. In particular, $\mathfrak{T}(\emptyset)$ [3]) is the system of functional calculus on the basis of which the theory $\mathfrak{T}(\mathfrak{A})$ is formalized.

We assume that the following symbols are the primitive signs of $\mathfrak{T}(\mathfrak{A})$:

(1) *individual variables* x_ι, where $\iota \in I$, $\overline{\overline{I}} \geqslant \aleph_0$ (it is convenient to assume that the set I contains the set of all positive integers);

(2) *individual constants* a_η, where $\eta \in E$, $\overline{\overline{E}} \geqslant 0$;

(3) *functors* f_μ (*i. e.* symbols for functions from individuals to individuals) where $\mu \in M$, $\overline{\overline{M}} \geqslant 0$;

(4) *predicates* F_ν (*i. e.* symbols for relations), where $\nu \in N$, $\overline{\overline{N}} \geqslant 1$ [4]); among these signs the sign of equality may occur;

(5) *logical connectives* $-, +, \cdot, \rightarrow$ and *quantifiers* $\prod\limits_{x_\iota}$, $\sum\limits_{x_\iota}$, $\iota \in I$.

We shall denote by $m(\mu)$ and $n(\nu)$ the number of arguments of f_μ and of F_ν, respectively.

Among the expressions constructed from these signs we distinguish *terms* and *formulas*.

The set J_0 of all terms is the least set such that

(i) $x_\iota \in J_0$ for $\iota \in I$ and $a_\eta \in J_0$ for $\eta \in E$,

(ii) if $\xi_1, \ldots, \xi_{m(\mu)} \in J_0$ then $f_\mu(\xi_1, \ldots, \xi_{m(\mu)}) \in J_0$ for $\mu \in M$.

The set of all formulas is the least set fulfilling the following conditions:

(i) if $\xi_1, \ldots, \xi_{n(\nu)} \in J_0$ then $F_\nu(\xi_1, \ldots, \xi_{n(\nu)})$ is a formula,

(ii) if α, β are formulas, then so are $-\alpha$, $\alpha+\beta$, $\alpha\cdot\beta$, $\alpha\rightarrow\beta$, $\prod\limits_{x_\iota}\alpha$, $\sum\limits_{x_\iota}\alpha$ $(\iota \in I)$.

We shall write for brevity $\alpha \equiv \beta$ instead of $(\alpha\rightarrow\beta)\cdot(\beta\rightarrow\alpha)$.

We regard as known the distinction of *free* and *bound* variable in a formula. A formula without quantifiers is said to be *open*. A formula without free occurrences of variables is called a *closed* formula. Sometimes we shall find it useful to write $\alpha(x_{\iota_1}, \ldots, x_{\iota_n})$ if $x_{\iota_1}, \ldots, x_{\iota_n}$ are all free variables in α. The closed formula $\prod\limits_{x_{\iota_1}} \ldots \prod\limits_{x_{\iota_n}} \alpha\,(x_{\iota_1}, \ldots, x_{\iota_n})$ is said to be the *closure* of $\alpha(x_{\iota_1}, \ldots, x_{\iota_n})$.

We shall denote by $\alpha\binom{\xi}{x_\iota}$ a formula which results from α by the substitution of the term ξ for x_ι assuming that the necessary changes

[3]) $\emptyset$ always denotes the empty set.

[4]) The cardinal numbers of the sets I, E, M, N can be arbitrarily large.

in the bound variables of a were performed before the operation of substitution.

The set $Cn(\mathfrak{A})$ of all *theorems* (or of all *provable* formulas) of $\mathcal{T}(\mathfrak{A})$ is the least set such that

1° $Cn(\mathfrak{A})$ contains all axioms,

2° $Cn(\mathfrak{A})$ contains all substitutions of tautologies of the sentential calculus,

3° if $a \in Cn(\mathfrak{A})$ and β is obtained from a by the admissible substitution of a term ξ for x_ι $(\iota \in I)$, then $\beta \in Cn(\mathfrak{A})$,

4° if $a \in Cn(\mathfrak{A})$, $a \to \beta \in Cn(\mathfrak{A})$, then $\beta \in Cn(\mathfrak{A})$,

5° if $a \to \prod_{x_\iota} \beta \in Cn(\mathfrak{A})$ then $a \to \beta \in Cn(\mathfrak{A})$,

if $\sum_{x_\iota} a \to \beta \in Cn(\mathfrak{A})$ then $a \to \beta \in Cn(\mathfrak{A})$.

6° if there is no free occurrence of x_ι in a (in β), $\iota \in I$, and if $a \to \beta \in Cn(\mathfrak{A})$, then $a \to \prod_{x_\iota} \beta \in Cn(\mathfrak{A})$ (then $\sum_{x_\iota} a \to \beta \in Cn(\mathfrak{A})$).

A theory $\mathcal{T}(\mathfrak{A})$ is *consistent*, if there exists a formula a of that theory such that $a \notin Cn(\mathfrak{A})$.

The following theorem is well-known:

1.1. THE DEDUCTION THEOREM. *If $\beta \in Cn(\mathfrak{A} \cup \{a\})$, then $a \to \beta \in Cn(\mathfrak{A})$, for any set $\mathfrak{A}$ of closed formulas, any closed formula a and any formula β.*

If the set $\mathfrak{A}$ contains only open formulas, we shall distinguish the set $Cn_0(\mathfrak{A})$ of theorems formally proved without using the rules of inference for the quantifiers. More precisely: the set $Cn_0(\mathfrak{A})$ is the least set of formulas of $\mathcal{T}(\mathfrak{A})$ fulfilling the conditions 1°-4°. In the considered case we shall denote by $\mathcal{T}_0(\mathfrak{A})$ the theory with the set $\mathfrak{A}$ of axioms and with the set $Cn_0(\mathfrak{A})$ of theorems.

In the rest of this section let $\mathcal{T}(\mathfrak{A})$ be a theory whose axioms $a_\varrho \in \mathfrak{A}$ $(\varrho \in R)$ are all closed formulas in the prenex normal form. Using Skolem's known method (see *e. g.* Hilbert and Bernays [5]) of the elimintaion of quantifiers it is possible to pass from $\mathcal{T}(\mathfrak{A})$ to a theory $\mathcal{T}^*(\mathfrak{A}^*)$ with the set $\mathfrak{A}^*$ of open formulas and with new primitive signs of individual constants and functors. We shall restrict the description of Skolem's method to a few examples.

It

$$a_\varrho = \sum_{x_1} \cdots \sum_{x_n} \beta_\varrho(x_1, \ldots, x_n),$$

then we add the new individual constants $b_1^\varrho, \ldots, b_n^\varrho$ and instead of the axiom a_ϱ we take as an axiom the formula $a_\varrho^* = \beta_\varrho(b_1^\varrho, \ldots, b_n^\varrho)$ arising from a_ϱ by the rejection of quantifiers and by the substitution of b_j^ϱ for x_j.

If

$$a_\varrho = \prod_{x_1} \cdots \prod_{x_n} \beta_\varrho(x_1, \ldots, x_n),$$

then instead of a_ϱ we take the axiom $a_\varrho^* = \beta_\varrho(x_1, \ldots, x_n)$.

In the case of

$$a_\varrho = \prod_{x_1} \sum_{x_2} \sum_{x_3} \prod_{x_4} \sum_{x_5} \beta_\varrho(x_1, \ldots, x_5)$$

we introduce the new signs of functors g_1^ϱ, g_2^ϱ, g_3^ϱ and instead of a_ϱ we take as an axiom $a_\varrho^* = \beta_\varrho\big(x_1, g_1^\varrho(x_1), g_2^\varrho(x_1), x_4, g_3^\varrho(x_1, x_4)\big)$.

It is important that by this elimination we always add new symbols of individual constants and of functors.

According to the notation previously introduced, $\mathcal{T}(\emptyset)$ denotes the functional calculus on the basis of which the theory $\mathcal{T}(\mathfrak{A})$ is formalized, and $\mathcal{T}^*(\emptyset)$ — the functional calculus establishing the basis for the formalization of $\mathcal{T}^*(\mathfrak{A}^*)$.

Given $a_\varrho \in \mathfrak{A}$, we shall denote by $a_\varrho^* \in \mathfrak{A}^*$ the formula arising from a_ϱ in the process of eliminating quantifiers and by γ_ϱ — the closure of a_ϱ^*. The set of all γ_ϱ for $\varrho \in R$ will be denoted by $\overline{\mathfrak{A}}^*$.

Obviously

1.2. *a is a theorem of $\mathcal{T}^*(\overline{\mathfrak{A}}^*)$ if and only if it is a theorem of $\mathcal{T}^*(\mathfrak{A}^*)$ for any formula a of $\mathcal{T}^*(\mathfrak{A}^*)$.*

1.3. *If $a \in \mathfrak{A}$, then a is a theorem of $\mathcal{T}^*(\overline{\mathfrak{A}}^*)$.*

2. Algebraic treatment of elementary theories. Given a theory $\mathcal{T}(\mathfrak{A})$, we shall treat each formula a of this theory as an algebraic functional $(J,B)\Phi_a$ (see *e. g.* Rasiowa [10], p. 297) defined in a domain $J \neq \emptyset$ of individuals with values belonging to a fixed complete Boolean algebra B, by regarding:

(a) individual variables as variables running over J;

(b) individual constants as fixed elements of J;

(c) functors f_μ $(\mu \in M)$ as $m(\mu)$-argument functions defined on J with values in J; the set of all such functions will be denoted by $\boldsymbol{f}(J)$;

(d) predicates F_ν $(\nu \in N)$ as $n(\nu)$-argument functions defined on J with values in B; the set of all such functions will be denoted by $\boldsymbol{F}(J,B)$;

(e) the logical connectives $-$, $+$, $\cdot$, $\to$ as the Boolean operations of the complement, join (sum), meet (product) and the operation $a \to b = -a+b$, respectively;

(f) the logical quantifiers $\sum_{x_\iota}$, $\prod_{x_\iota}$ as the signs of infinite sums and products $(B)\sum_{x_\iota \in J}$, $(B)\prod_{x_\iota \in J}$ in the algebra B, respectively.

Sometimes, it is convenient to write $(J,B)\Phi(a)$ instead of $(J,B)\Phi_a$. Let

(i) $a_\eta = j_\eta \epsilon J \quad (\eta \epsilon E)$,

(ii) $f_\mu = \tau_\mu \epsilon \boldsymbol{f}(J) \quad (\mu \epsilon M)$,

(iii) $F_\nu = \varphi_\nu \epsilon \boldsymbol{F}(J,B) \quad (\nu \epsilon N)$,

be an arbitrary but fixed system of valuations of all individual constants, functors and predicates of $\mathfrak{T}(\mathfrak{A})$ according to the algebraical interpretation described above. This system will also be denoted by $\mathfrak{M} = [\{j_\eta\}, \{\tau_\mu\}, \{\varphi_\nu\}]$ and will be called an *algebraic pseudomodel* of $\mathfrak{T}(\mathfrak{A})$ in the domain J and algebra B. The symbol $(J,B,\mathfrak{M})\Phi_a(\{i_\iota\})$ or $(J,B,\mathfrak{M})\Phi(a;\{i_\iota\})$ will denote the value of the functional $(J,B)\Phi_a$ for the values of its arguments fixed above by (i), (ii), (iii) and the values of $x_\iota = i_\iota$ $(i_\iota \epsilon J$, for $\iota \epsilon I)$. We shall write $(J,B,\mathfrak{M})\Phi_a = c$ if $c \epsilon B$ and $(J,B,\mathfrak{M})\Phi_a(\{i_\iota\})$ is equal to c identically (*i. e.* for every system $x_\iota = i_\iota$ $(\iota \epsilon I)$).

An algebraic pseudomodel $\mathfrak{M}$ is said to be a *generalized algebraic model* (see Rasiowa [10], p. 298) of $\mathfrak{T}(\mathfrak{A})$ in the domain J and the algebra B if for every $a \epsilon \mathfrak{A}$, $(J,B,\mathfrak{M})\Phi_a = 1$, *i. e.* the unit of B.

Instead of $(J,B,\mathfrak{M})\Phi_a$ we shall also write briefly $(\mathfrak{M})\Phi_a$ provided that it does not lead to a mistake.

The following known theorem (cf. Rasiowa [10], p. 299) is easy to prove:

2.1. *If* $a \epsilon Cn(\mathfrak{A})$, *then* $(J,B,\mathfrak{M})\Phi_a = 1$ *in every generalized algebraic model* $\mathfrak{M}$ *in every domain* $J \neq \emptyset$ *and every complete Boolean algebra* B.

Given a theory $\mathfrak{T}(\mathfrak{A})$ $(\mathfrak{T}_0(\mathfrak{A}))$, let L (L_0) be the *Lindenbaum algebra* (see *e. g.* Rasiowa [10], § 2) of this system. The construction of L (L_0) being known, we shall only give an outline of the description of L (L_0). Given a formula a of $\mathfrak{T}(\mathfrak{A})$ $(\mathfrak{T}_0(\mathfrak{A}))$, let $|a|$ denote the class of all formulas β of this theory such that $a \equiv \beta \epsilon Cn(\mathfrak{A})$ $(a \equiv \beta \epsilon Cn_0(\mathfrak{A}))$. Then L (L_0) is the algebra of all cosets $|a|$ with the operations

$$|a|+|\beta| = |a+\beta|, \qquad |a|\cdot|\beta| = |a\cdot\beta|,$$
$$-|a| = |-a|, \qquad |a| \to |\beta| = -|a|+|\beta| = |a\to\beta|.$$

It is known (see for instance Rasiowa [10], 2.1, 2.3, and Rasiowa and Sikorski [13]) that:

2.2. (i) *L (L_0) is a Boolean algebra with the unit element* $1 = |a|$, *where* $a \epsilon Cn(\mathfrak{A})$ $(a \epsilon Cn_0(\mathfrak{A}))$,

(ii) $|a| \subset |\beta|$ *if and only if* $a \to \beta \epsilon Cn(\mathfrak{A})$ $(a\to\beta \epsilon Cn_0(\mathfrak{A}))$,

(iii) $\displaystyle\sum_{\xi \epsilon J_0} \left|a\binom{\xi}{x_\iota}\right| = \left|\sum_{x_\iota} a\right|, \qquad \prod_{\xi \epsilon J_0} \left|a\binom{\xi}{x_\iota}\right| = \left|\prod_{x_\iota} a\right|.$

Let $\bar{L}$ $(\bar{L}_0)$ be a complete Boolean algebra, constituting an extension of L preserving all sums and products (iii) of 2.2 [5]) (constituting an extension of L_0). We shall construct the following pseudomodel of $\mathfrak{T}(\mathfrak{A})$ $(\mathfrak{T}_0(\mathfrak{A}))$ in the domain J_0 of all terms of this theory and in the algebra $\bar{L}$ $(\bar{L}_0)$:

$$\mathfrak{N} = [\{a_\eta\}, \{f_\mu\}, \{\psi_\nu\}] \qquad (\mathfrak{N}_0 = [\{a_\eta\}, \{f_\mu\}, \{\psi_\nu\}])$$

where $\psi_\nu(\xi_1,\dots,\xi_{n(\nu)}) = |F_\nu(\xi_1,\dots,\xi_{n(\nu)})| \epsilon \bar{L}$ $(\bar{L}_0)$ for any $\xi_1,\dots,\xi_{n(\nu)} \epsilon J_0$.

It is known (see Rasiowa [10], 3.6, p. 299) that

2.3. *If* $\mathfrak{T}(\mathfrak{A})$ $(\mathfrak{T}_0(\mathfrak{A}))$ *is consistent, then for every formula a of this theory*

$$(J_0,\bar{L},\mathfrak{N})\Phi_a(\{\xi_\iota\}) = \left|a\binom{\xi_\iota}{x_\iota}\right| \qquad \left((J_0,\bar{L}_0,\mathfrak{N}_0)\Phi_a(\{\xi_\iota\}) = \left|a\binom{\xi_\iota}{x_\iota}\right|\right).$$

Consequently, $\mathfrak{N}$ $(\mathfrak{N}_0)$ *is the generalized algebraic model of* $\mathfrak{T}(\mathfrak{A})$ $(\mathfrak{T}_0(\mathfrak{A}))$. *Moreover,* $a \epsilon Cn(\mathfrak{A})$ $(a \epsilon Cn_0(\mathfrak{A}))$ *if and only if*

$$(J_0,\bar{L},\mathfrak{N})\Phi_a(\{x_\iota\}) = 1 \qquad ((J_0,\bar{L}_0,\mathfrak{N}_0)\Phi_a(\{x_\iota\}) = 1).$$

Let $\mathfrak{T}'(\mathfrak{A})$ be the theory arising from $\mathfrak{T}(\mathfrak{A})$ by the addition of new symbols $b_\varkappa$ $(\varkappa \epsilon K)$ of individual constants and new symbols g_ζ $(\zeta \epsilon Z)$ of functors to the primitive signs of $\mathfrak{T}(\mathfrak{A})$.

The following theorem is well-known and easy to prove.

2.4. *If a is a formula of the theory* $\mathfrak{T}(\mathfrak{A})$, *then a is provable in* $\mathfrak{T}'(\mathfrak{A})$ *if and only if it is provable in* $\mathfrak{T}(\mathfrak{A})$.

3. The first ε-theorem. In this section let $\mathfrak{T}(\mathfrak{A})$ denote a theory whose axioms are all open formulas.

3.1. THE FIRST ε-THEOREM. *Given an open formula a of the theory* $\mathfrak{T}(\mathfrak{A})$, *if* $a \epsilon Cn(\mathfrak{A})$ *then* $a \epsilon Cn_0(\mathfrak{A})$.

Indeed, suppose that $a \bar{\epsilon} Cn_0(\mathfrak{A})$. Then by 2.3 $(J_0,\bar{L}_0,\mathfrak{N}_0)\Phi_a(\{x_\iota\}) \neq 1$. It follows from the definition of the generalized algebraic model that $\mathfrak{N}_0$ is a generalized algebraic model of $\mathfrak{T}(\mathfrak{A})$. Hence, on account of 2.1, $a \epsilon Cn(\mathfrak{A})$, which completes the proof.

Quite a similar method makes it possible to prove the first ε-theorem for all non-classical theories described in my paper [10], in particular for the theories based on the *positive, minimal, intuitionistic* and *modal* $(\mathcal{S}_4)$ logic.

4. Lemma. In this section and in the next, let $\mathfrak{T}(\mathfrak{A})$ be a consistent theory whose axioms $a_\varrho \epsilon \mathfrak{A}$ $(\varrho \epsilon R)$ are all closed formulas in the prenex normal form and let $\mathfrak{T}^*(\mathfrak{A}^*)$ be the theory arising from $\mathfrak{T}(\mathfrak{A})$ by the

[5]) *E. g.* MacNeille's minimal extension, cf. [8].

elimination of quantifiers with the aid Skolem's method (cf. section 1, p. 158). Let $\gamma_{\varrho_1},\dots,\gamma_{\varrho_r},\gamma_{\varrho_{r+1}}$ $(0\leqslant r)$, be the closures of $\alpha^*_{\varrho_1},\dots,\alpha^*_{\varrho_r},\alpha^*_{\varrho_{r+1}}\in\mathfrak{A}^*$ and let α be an arbitrary closed formula of $\mathcal{T}(\mathfrak{A})$.

We shall consider the theory $\mathcal{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$ formalized on the basis of the system of the functional calculus $\mathcal{T}^*(\emptyset)$ and the theory $\mathcal{T}'(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$, whose language is richer than the language of $\mathcal{T}(\mathfrak{A})$ only by the primitive signs occuring in $\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}$.

Let us suppose that $\mathcal{T}'(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$ is consistent and let $\mathfrak{M}'$ be a generalized algebraic model of this theory in a domain J and in algebra B. Fixing a valuation of all annexed signs so as to obtain the language of $\mathcal{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$ we can extend $\mathfrak{M}'$ to a generalized algebraic model $\mathfrak{M}^*$ of $\mathcal{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$.

4.1. *If in a generalized algebraic model $\mathfrak{M}'$ of $\mathcal{T}'(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$, we have*

(*) $$(\mathfrak{M}')\Phi(\alpha_{\varrho_{r+1}})\not\subset(\mathfrak{M}')\Phi(\alpha),$$

then there exists an extension of $\mathfrak{M}'$ to a generalized algebraic model $\mathfrak{M}^*$ f $\mathcal{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$ such that

(**) $$(\mathfrak{M}^*)\Phi(\gamma_{\varrho_{r+1}})\not\subset(\mathfrak{M}^*)\Phi(\alpha).$$

Indeed, if for $\varrho=\varrho_{r+1}$,

$$\alpha_\varrho=\sum_{x_1}\dots\sum_{x_n}\beta_\varrho\ (x_1,\dots,x_n)$$

then, on account of (*), there exists a system of valuations $x_\iota=i_\iota\in J$ $(\iota\in I)$ of individual variables, such that $(\mathfrak{M}')\Phi(\beta_\varrho;\ \{i_\iota\})\not\subset(\mathfrak{M}')\Phi(\alpha)$. In the case in question, $\gamma_\varrho=\beta_\varrho(b_1^\varrho,\dots,b_n^\varrho)$. Thus it is sufficient to put $b_1^\varrho=i_1,\dots,b_n^\varrho=i_n$ in the $\mathfrak{M}^*$ in order to obtain (**).

As the introduced signs $b_1^\varrho,\dots,b_n^\varrho$ appear only in $\gamma_{\varrho_{r+1}}$ that valuation can always be performed.

If

$$\alpha_{\varrho_{r+1}}=\prod_{x_1}\dots\prod_{x_n}\beta_{\varrho_{r+1}}\ (x_1,\dots,x_n),$$

then $\gamma_{\varrho_{r+1}}$ is identical with $\alpha_{\varrho_{r+1}}$. Thus in this case there is no need to put any conditions upon $\mathfrak{M}^*$ in order to obtain (**).

If for $\varrho=\varrho_{r+1}$, both the existential quantifiers and the universal quantifiers appear in α_ϱ *e. g.*

$$\alpha_\varrho=\prod_{x_1}\sum_{x_2}\sum_{x_3}\prod_{x_4}\sum_{x_5}\beta_\varrho\ (x_1,\dots,x_5),$$

then from (*) it follows that

$$\prod_{i_1\in J}\sum_{i_2\in J}\sum_{i_3\in J}\prod_{i_4\in J}\sum_{i_5\in J}(\mathfrak{M}')\Phi_{\beta_\varrho}(i_1,i_2,i_3,i_4,i_5)\not\subset(\mathfrak{M}')\Phi_\alpha.$$

Consequently, for every $i\in J$

$$\sum_{i_2\in J}\sum_{i_3\in J}\prod_{i_4\in J}\sum_{i_5\in J}(\mathfrak{M}')\Phi_{\beta_\varrho}(i,i_2,i_3,i_4,i_5)\not\subset(\mathfrak{M}')\Phi_\alpha.$$

Hence, it follows that there exist two functions $\sigma_1^\varrho(i)$ and $\sigma_2^\varrho(i)$ defined on J with values in J such that for every $i\in J$

$$\prod_{i_4\in J}\sum_{i_5\in J}(\mathfrak{M}')\Phi_{\beta_\varrho}(i,\sigma_1^\varrho(i),\sigma_2^\varrho(i),i_4,i_5)\not\subset(\mathfrak{M}')\Phi_\alpha.$$

Thus we have for every $i,j\in J$

$$\sum_{i_5\in J}(\mathfrak{M}')\Phi_{\beta_\varrho}(i,\sigma_1^\varrho(i),\sigma_2^\varrho(i),j,i_5)\not\subset(\mathfrak{M}')\Phi_\alpha.$$

Consequently, there exists a function $\sigma_3^\varrho(i,j)$ defined on J with values in J such that for any $i,j\in J$

$$(\mathfrak{M}')\Phi_{\beta_\varrho}(i,\sigma_1^\varrho(i),\sigma_2^\varrho(i),j,\sigma_3^\varrho(i,j))\not\subset(\mathfrak{M}')\Phi_\alpha.$$

Since

$$\gamma_\varrho=\prod_{x_1}\prod_{x_4}\beta_\varrho(x_1,g_1^\varrho(x_1),g_2^\varrho(x_1),x_4,g_3^\varrho(x_1,x_4)),$$

it is sufficient to choose an extension $\mathfrak{M}^*$ of $\mathfrak{M}'$ in which $g_1^\varrho=\sigma_1^\varrho$, $g_2^\varrho=\sigma_2^\varrho$, $g_3^\varrho=\sigma_3^\varrho$ in order to obtain (**).

Since the introduced signs $g_1^\varrho,g_2^\varrho,g_3^\varrho$ appear only in $\gamma_{\varrho_{r+1}}$, the conditions on $\mathfrak{M}^*$ stated above do not lead to any ambiguity and can always be performed. It is an easy task to generalize our proof to a fairly general formula $\alpha_{\varrho_{r+1}}$ in the prenex normal form.

It follows from our considerations that it is always possibly to choose such an extension $\mathfrak{M}^*$ of $\mathfrak{M}'$ that we have (**).

5. The second ε-theorem. In this section we shall use the notation introduced in section 4 (cf. p. 161).

5.1. THE SECOND ε-THEOREM. *Given a formula α of a consistent theory $\mathcal{T}(\mathfrak{A})$, if α is provable in $\mathcal{T}^*(\mathfrak{A}^*)$, then it is also provable in $\mathcal{T}(\mathfrak{A})$.*

It follows from 1.2 that 5.1 is equivalent to the following theorem

5.2. *Given a closed formula α of a consistent theory $\mathcal{T}(\mathfrak{A})$ if α is provable in $\mathcal{T}^*(\overline{\mathfrak{A}}^*)$ then it is also provable in $\mathcal{T}(\mathfrak{A})$.*

To prove 5.2, let us suppose that α is provable in $\mathfrak{T}^*(\overline{\mathfrak{A}}^*)$ but not provable in $\mathfrak{T}(\mathfrak{A})$. By 1.3, the first condition is equivalent to the condition that α is provable in $\mathfrak{T}^*(\mathfrak{A}\cup\overline{\mathfrak{A}}^*)$. Hence, there exist $\gamma_{\varrho_1},\dots,\gamma_{\varrho_{r+1}}$ $(r\geqslant 0)$, such that α is provable in $\mathfrak{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_{r+1}}\})$ but not provable in $\mathfrak{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$. Thus this last theory is consistent. By 1.1 the formula $\gamma_{\varrho_{r+1}}\to\alpha$ is provable in $\mathfrak{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$. Consequently, by 2.2 in every generalized algebraic model $\mathfrak{M}^*$ of this theory we have

$$(\mathfrak{M}^*)\Phi(\gamma_{\varrho_{r+1}})\subset(\mathfrak{M}^*)\Phi(\alpha)\,. \tag{1}$$

On the other hand, α being not provable in $\mathfrak{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$, by 2.4 it is not provable in $\mathfrak{T}'(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$. Consequently, the formula $\alpha_{\varrho_{r+1}}\to\alpha$ is not provable in $\mathfrak{T}'(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$. Thus it is consistent and by 2.3 there exists a generalized algebraic model $\mathfrak{M}'$ of this theory such that $(\mathfrak{M}')\Phi(\alpha_{\varrho_{r+1}}\to\alpha)\neq 1$. By 2.2 (ii), $(\mathfrak{M}')\Phi(\alpha_{\varrho_{r+1}})\not\subset(\mathfrak{M}')\Phi(\alpha)$. Hence, by 4.1 there exists a generalized algebraic model $\mathfrak{M}^*$ of $\mathfrak{T}^*(\mathfrak{A}\cup\{\gamma_{\varrho_1},\dots,\gamma_{\varrho_r}\})$ such that $(\mathfrak{M}^*)\Phi(\gamma_{\varrho_{r+1}})\not\subset(\mathfrak{M}^*)\Phi(\alpha)$, which contradicts (1).

In an analogical way one can prove that the second ε-theorem holds also for some non-classical theories described in my paper [10] [6]), in particular for theories based on the *positive*, *minimal* and *intuitionistic* logic.

Obviously, 5.2 implies the following theorem 5.3 which holds for all the theories cited above [7]):

5.3. *The consistency of the theory $\mathfrak{T}(\mathfrak{A})$ implies the consistency of $\mathfrak{T}^*(\mathfrak{A}^*)$.*

References

[1] K. Gödel, Die *Vollständigkeit der Axiome der logischen Funktionenkalküle*, Monatsh. für Math. und Phys. 37 (1930), p. 349-360.

[2] G. Hasenjaeger, *Eine Bemerkung zu Henkin's Beweis für die Vollständigkeit des Prädikatenkalküls der ersten Stufe*, J. Symbolic Logic 18 (1953), p. 42-48.

[3] L. Henkin, *The completeness of the first order functional calculus*, J. Symbolic Logic 14 (1949), p. 159-166.

[4] A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, Sitzungber. der Preuss. Akademie der Wiss. Phys. Mathem. Klasse (1930), p. 42-56.

[6]) Namely it holds for theories based on the system $\mathcal{S}$ of logic having the property E (see [10], p. 293) and such that for every set of closed formulas the deduction theorem holds.

[7]) The theorem 5.3 for intuitionistic theories follows also from the following unpublished result of Łoś: if all axioms of a consistent theory based on the intuitionistic functional calculus are in the prenex normal form, then the theory has a semantic model.

[5] D. Hilbert und P. Bernays, Grundlagen der Mathematik, Band II, Berlin 1939.

[6] J. Łoś, *Algebraical treatment of the methodology of elementary deductive systems*, Stud. Log. 2 (1955), p. 151-212.

[7] — A. Mostowski and H. Rasiowa, *A proof of Herbrand's theorem*, J. Math. Pures Appl. (to appear).

[8] H. MacNeille, *Partially ordered sets*, Trans. Amer. Math. Soc. 42 (1937), p. 416-460.

[9] A. Malcev, *Untersuchungen aus dem Gebiete der mathematischen Logic*, Récueil Mathématique, n. s. 1 (1936), p. 323-336.

[10] H. Rasiowa, *Algebraic models of axiomatic theories*, Fund. Mathem. 41 (1955), p. 291-310.

[11] — and R. Sikorski, *A proof of the completeness theorem of Gödel*, Fund. Math. 37 (1950), p. 193-200.

[12] — — *A proof of the Skolem-Löwenheim theorem*, Fund. Math. 38 (1951), p. 230-232.

[13] — — *Algebraic treatment of the notion of satisfiability*, Fund. Math. 40 (1953), p. 62-95.

[14] A. Robinson, *On the metamathematics of algebra*, Amsterdam 1951.

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