Decompositions of a set into disjoint pairs

by

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In this note we shall discuss the possibility of the decomposition of a set of elements into disjoint pairs such that every set which is the union of pairs and which satisfies one specified property, also satisfies a second specified property. The principal result, Theorem 1.1, is a generalization of a lemma due to Sierpiński [3].

1. THEOREM 1.1. *Let $F=\{A_\xi|\xi<\tau\}$ be a family of abstract sets, each set being of power $\aleph_\gamma$. Furthermore, let F contain a coinitial subfamily $G=\{B_\xi|\xi<\omega_\gamma\}$, i. e., for each $\xi<\tau$, A_ξ contains as a subset, at least one element of G. Then there exists a decomposition of $A=\bigcup_{\xi<\tau}A_\xi$ into disjoint pairs of elements with the following property: For any set S which is the union of pairs, if S contains at least one set A_ξ, then S has $\aleph_\gamma$ elements in common with each element of F.*

Proof. The demonstration is a modification of the proof of Lemma 1 of [3].

In Theorem 1 of [2] Sierpiński stated a general result on families of sets. An inspection of the proof reveals that the following result was proved at the same time:

(*) *If $\{D_\xi\}_{\xi<\omega_\gamma}$ is a sequence of sets, each set being of power $\aleph_\gamma$, then there exists a family $P=\{N_\xi|\xi<\omega_\gamma\}$ of pairwise disjoint sets, each set of power $<\aleph_\gamma$, such that 1° $\bigcup_{\xi<\omega_\gamma}N_\xi=\bigcup_{\xi<\omega_\gamma}D_\xi$, and 2° any set Q which is the union of a family T of $\aleph_\gamma$ sets in P has at least one element in common with each D_ξ.*

Since T is the union of $\aleph_\gamma$ disjoint families of $\aleph_\xi$ sets each and since the N_ξ are pairwise disjoint, it follows that Q has $\aleph_\gamma$ elements in common with each D_ξ.

Turning to the proof of Theorem 1.1 let M be the family of those sets B in G for which the power of $A-B$ is $\aleph_\gamma$. Let $H=M\cup\{A-B\,|\,B\in M\}$.

A. Suppose that H is non-empty. Well order the elements of H into a sequence $\{D_\xi\}_{\xi<\omega_\gamma}$ where the D_ξ are not necessarily all distinct. Then $\bigcup_{\xi<\omega_\gamma}D_\xi=A$. Let $P=\{N_\xi|\xi<\omega_\gamma\}$ be the family of sets obtained by

applying (*) to our sequence $\{D_\xi\}_{\xi<\omega_\gamma}$. Decompose the set of ordinal numbers $W=\{\xi|\xi<\omega_\gamma\}$ into a family $\{W_\xi|\xi<\omega_\gamma\}$ of pairwise disjoint sets, each set being of power $\aleph_\gamma$. For $\delta<\omega_\gamma$ let $C_\delta=\bigcup_{v\in\omega_\delta}N_v$. Thus $\bigcup_{\delta<\omega_\gamma}C_\delta=A$, and for $\xi,\delta<\omega_\gamma$, $D_\xi\cap C_\delta$ is of power $\aleph_\gamma$. From the definition of H, for each ξ, $A-D_\xi$ is in H. Therefore the power of the set $C_\xi\cap(A-D_\xi)$ is $\aleph_\gamma$. Denote by $\{x^\xi_v\}_{v<\omega_\gamma}$ and by $\{y^\xi_v\}_{v<\omega_\gamma}$ the elements of $C_\xi\cap D_\xi$ and $C_\xi\cap(A-D_\xi)$ respectively. Thus

$$\bigcup_{v<\omega_\gamma}(x^\xi_v,y^\xi_v)=(D_\xi\cap C_\xi)\cup[(A-D_\xi)\cap C_\xi]=A\cap C_\xi=C_\xi.$$

Consider the set of pairs $K=\{(x^\xi_v,y^\xi_v)\,|\,\xi<\omega_\gamma,v<\omega_\gamma\}$. Since A is the union of the pairwise disjoint sets C_ξ, it follows that A is the union of the pairwise disjoint pairs in K.

Now let S be any set which is the union of pairs and which contains an element of F. By hypothesis, there exists an element B in G for which $B\subseteq S$.

(i) Suppose that B is in $G-M$. For each ξ, $D_\xi=(D_\xi\cap B)\cup[D_\xi\cap(A-B)]$. Since D_ξ is of power $\aleph_\gamma$ and $A-B$ is of power $<\aleph_\gamma$, $D_\xi\cap B$ is of power $\aleph_\gamma$. Since G is coinitial in F, it follows that $B\cap A_\xi$ is of power $\aleph_\gamma$ for each $\xi<\tau$. Thus $S\cap A_\xi$ is of power $\aleph_\gamma$ for $\xi<\tau$.

(ii) Suppose that B is in M, say $B=D_\xi$. Therefore $C_\xi\cap D_\xi\subseteq S$. Let x be any element of $(A-D_\xi)\cap C_\xi$. From the definition of the pairs there exists an element y in $D_\xi\cap C_\xi$ such that (x,y) is a pair. Since S is the union of pairwise disjoint pairs, and x is in S, it follows that y is also in S. Consequently $(A-D_\xi)\cap C_\xi\subseteq S$. Hence

$$[(A-D_\xi)\cup C_\xi]\cup(D_\xi\cap C_\xi)=C_\xi\subseteq S.$$

For each $v<\omega_\gamma$, $C_\xi\cap D_v$, therefore $S\cap D_v$, is of power $\aleph_\gamma$. For an element D of $G-M$, we have seen in (i) that $D\cap D_\xi$, thus $D\cap S$, is of power $\aleph_\gamma$. Thus $D_\xi\cap B_v$, for $v<\omega_\gamma$, is of power $\aleph_\gamma$. Therefore $D_\xi\cap A_\sigma$, for $\sigma<\tau$, is of power $\aleph_\gamma$.

B. Suppose that M is empty, i. e., there is no set B_ξ such that $A-B_\xi$ is of power $\aleph_\gamma$. Decompose A, in an arbitrary manner, into disjoint pairs. Using (i) above it is easily seen that this decomposition satisfies the conclusion of the theorem, q. e. d.

COROLLARY. *Let K be an infinite set of power $\aleph_\gamma$. The family A of all subsets of K, of power $\aleph_\gamma$ each, can be decomposed into disjoint pairs with the following property: If S is any subfamily of A, consisting of pairs, and if S contains some element Y of A together with all subsets of Y in A, then for each element Z in A, S contains an element U in A which is a subset of Z, i. e., S is a coinitial subfamily of the family A, ordered by set inclusion.*

Remarks. 1. Assume that the hypothesis of Theorem 1.1 is fulfilled. For a given decomposition of A into disjoint pairs it is possible that each set S, of power $\aleph_\gamma$, which is the union of pairs, has an intersection with each set in F of power $\aleph_\gamma$. For example, let A be an abstract set of power $\aleph_\gamma$, $A=\{x_\xi, y_\xi \mid \xi<\omega_\gamma\}$. For each ξ let (x_ξ, y_ξ) be a pair and let

$$A_\xi=\{x_v \mid v<\omega_\gamma\}\cup\{y_\xi\} \qquad \text{and} \qquad B_\xi=\{y_v \mid v\neq\xi\}.$$

Denote by F $(=G)$ the family of sets $\{A_\xi, B_\xi \mid \xi<\omega_\gamma\}$. The hypothesis of Theorem 1.1 is satisfied. Now each set S, of power $\aleph_\gamma$, which consists of pairs, contains $\aleph_\gamma$ of the x's and y's. For each $\xi<\omega_\gamma$, A_ξ and B_ξ contains all but a finite number of the x's and the y's respectively. Consequently each of the sets, $A_\xi\cap S$ and $B_\xi\cap S$, is of power $\aleph_\gamma$.

The situation is different if the family F contains at least three disjoint sets, say A_0, A_1, and A_2. For any decomposition of A into disjoint pairs, there always exists a set S, of power $\aleph_\gamma$, consisting of pairs, which has an empty intersection with some set in F, in particular, with either A_1 or A_2. To see this let Z be the set of those elements which are in pairs that are in A_0. Thus Z is a subset of A_0. If the power of Z is $\aleph_\gamma$, let $S=Z$. Since A_0 and A_1 are disjoint, S and A_1 are disjoint. If the power of Z is $<\aleph_\gamma$, then the set U, defined as the set A_0-Z, is of power $\aleph_\gamma$. This is so because each set in F is of power $\aleph_\gamma$. For each element u in U, let (u,v) be the pair which contains u, and let $V=\{v \mid v\in(u,v), u\in U\}$. If $\aleph_\gamma$ of the elements of V are in A_1 let S be the union of those pairs (u,v), where u is in U and v is in A_1. Since A_0, A_1, and A_2 are pairwise disjoint, $S\cap A_2$ is empty. If there are fewer than $\aleph_\gamma$ elements of V in A_1, let S be the union of those pairs (u,v), where u is in U and v is not in A_1. Here $S\cap A_1$ is empty. It is clear that in each case, the power of S is $\aleph_\gamma$.

2. From remark 1 and Theorem 1.1 we see that if F contains three disjoint sets, then there exists a set S, of power $\aleph_\gamma$, consisting of pairs, which contains no set in F. The conclusion of the previous statement holds even if F does not contain three disjoint sets. To be specific the following result will now be proved:

Let a set A, of power $\aleph_\gamma$, be decomposed into disjoint pairs. For any family of sets $F=\{A_\xi \mid \xi<\omega_\gamma\}$, of power $\aleph_\gamma$ each, there exists a set S, of power $\aleph_\gamma$, consisting of pairs, which contains as a subset no set in F. Furthermore, for each ξ, $S\cap A_\xi$ is non-empty.

In order to see this let the elements of A and the pairs in A each be well ordered into the two sequences, $\{p_\xi\}_{\xi<\omega_\gamma}$ and $\{a_\xi\}_{\xi<\omega_\gamma}$ respectively. Let x_0 be the first element in A_0 and let b_0 be the pair containing x_0. Let y_0 be the first element in the set A_0-b_0 and c_0 the pair containing y_0. We continue by transfinite induction. For each $\xi<\alpha<\omega_\gamma$ let x_ξ, y_ξ, b_ξ, and c_ξ be defined. Let x_α be the first element in the set $A_\alpha-[\bigcup_{\xi<\alpha}(b_\xi\cup c_\xi)]$ and let b_α be the pair which contains x_α. Let y_α be the first element in the set $A_\alpha-[\bigcup_{\xi<\alpha}(b_\xi\cup c_\xi)\cup b_\alpha]$ and let c_α be the pair which contains y_α. Clearly the elements x_ξ, x_v, y_σ, y_τ, $\xi\neq v$, $\sigma\neq\tau$ are in different pairs. Now define S as the set $\bigcup_{\xi<\omega_\gamma} b_\xi$. For each ξ, y_ξ is an element of A_ξ which is not in S. Consequently S contains no set in F.

2. We now restrict ourselves to decompositions of the plane. We denote the plane by E^2.

THEOREM 2.1. *There is no decomposition of the plane into disjoint pairs with the following property: S being any set which is the union of pairs, if S contains one line then it contains two lines.*

Proof. Suppose the contrary, *i. e.*, suppose that such a decomposition does exist. For each line L denote by $S(L)$ the union of the pairs (x,y), where x is in L. Let L_1 be any line. By assumption, $S(L_1)$ contains a second line, say L_2. Consider the set $S(L_2)$. $S(L_2)$ consists of the union of pairs, each pair containing an element in L_2 and an element in L_1. Suppose that L_1 contains a pair neither of whose elements is in L_2. Then $S(L_2)$ cannot contain L_1. Thus $S(L_2)$ contains but one line. From this contradiction we obtain the following:

(*) *For each line L_1 there corresponds a second line L_2 so that $S(L_1)=S(L_2)$. Under this correspondance L_1 corresponds to L_2. To each element x in L_1 there corresponds an element y in L_2 such that, for $x\neq y$, (x,y) is a pair. If $L_1\cap L_2$ is non-empty, say $L_1\cap L_2=\{x_0\}$, then the pair (x_0,y_0) which contains x_0, need not be a subset of $L_1\cup L_2$, i. e., y_0 need not be in either L_1 or L_2.*

Now let (p,q) be any pair and let L_1 be the line containing p and q. Let L_2 be the line which corresponds to L_1. By (*), L_2 contains either p or q, say p. Let $(r,s)\neq(p,q)$ be any pair with r in L_1 and s in L_2. Denote by L_3 the line which contains q and s. Clearly L_3 is neither L_1 nor L_2. Denote by L_4 the line corresponding to L_3. By (*), L_4 is neither L_1 nor L_2. Now the elements q and s belong to $S(L_3)$. From (*), if L_4 does not contain p, then L_4 contains q and r. Since L_1 contains q and r, L_4 is L_1. If L_4 does not contain q, then L_4 contains p. Now L_4 contains either r or s. If L_4 contains r, then L_4 is L_1. If L_4 contains s, then L_4 is L_2. In any case L_4 is either L_1 or L_2. This is a contradiction. Hence no such decomposition is possible, q. e. d.

If we only demand a decomposition such that each $S(L)$ contain a line segment not on L, then such a decomposition can be effected. To be precise we have

THEOREM 2.2. *There exists a decomposition of the plane into disjoint pairs with the following property: Every set which is the union of pairs and which contains one line*

1^0 *contains $2^{\aleph_0}$ disjoint, non-trivial closed intervals of lines in the plane,*

2^0 *has $2^{\aleph_0}$ elements in common with every line in the plane.*

Proof. Denote by K the set of points in the plane

$$K=\{(x,y)\,|\,-\infty<x<\infty,\,0\leqslant y\leqslant 1\}.$$

For each line L which is not entirely in K, let $L'=L-K$. Let θ be the smallest ordinal number whose power is $2^{\aleph_0}$. Let $F'=G'=\{B_\xi\,|\,\xi<\theta\}$ be the family of all such sets L'. Obviously each set L' contains $2^{\aleph_0}$ elements. On applying the same procedure to the sequence $\{B_\xi\}_{\xi<\theta}$ as in the proof of Theorem 1.1 we obtain a family of pairwise disjoint sets $\{C'_\xi\,|\,\xi<\theta\}$ whose union is E^2-K, and such that $C'_\xi\cap B_v$ is of power $2^{\aleph_0}$ for all $\xi,v<\theta$. For each real number x define D_x as the set $D_x=\{(x,y)\,|\,0\leqslant y\leqslant 1\}$. Since the power of the real numbers is $2^{\aleph_0}$, we may decompose the set of real numbers into a family $\{E_\xi\,|\,\xi<\theta\}$ of pairwise disjoint sets, each set being of power $2^{\aleph_0}$. For each ξ let $C_\xi=C'_\xi\cup\bigcup_{x\in E_\xi}D_x$. Note that each set C_ξ has $2^{\aleph_0}$ elements in common with each line in the plane. Let $F\,(=G)$ be the set of all lines in the plane. Repeating the proof of Theorem 1.1 on F and the C_ξ, we obtain a decomposition of E^2 into disjoint pairs. Any set S which is the union of pairs and which contains a line, contains some set C_ξ. Therefore S satisfies the conclusions of the theorem.

Finally we have

THEOREM 2.3. *There exists a decomposition of the plane into a family of disjoint pairs with the following property: For each line L in the plane, the set $S(L)$, which is the union of the pairs each of which is entirely in L, is exact* [1] *and has property A* [2]*. Furthermore, if $L\neq L'$ and if $T(L)$ and $T(L')$ are any subsets, of power $2^{\aleph_0}$ each, of $S(L)$ and $S(L')$ respectively, then $\overline{T(L)}$ and $\overline{T(L')}$ are incomparable order types* [3]*.*

Proof. From Theorem 3.4 of [1], it follows that there exists a family of linear sets $\{Y_\xi\,|\,\xi<\theta\}$, each of which is both exact and has property A. Furthermore, for any two finite, linear sets M and N, if $\xi\neq v$ and T_ξ and T_v are subsets of power $2^{\aleph_0}$ each of $Y_\xi\cup M$ and $Y_v\cup N$ respectively, then $\overline{T}_\xi$ and $\overline{T}_v$ are incomparable order types.

Now well order the set of lines in the plane into the sequence $\{L_\xi\}_{\xi<\theta}$. For each ξ decompose L_ξ into a family $\{B_v^\xi\,|\,v<\theta\}$ of pairwise disjoint sets, each B_v^ξ being similar to Y_ξ. By Theorem 2.2 of [1] such decompositions are possible. Denote by C_0 the set B_0^0. Suppose that for each $\xi<\alpha<\theta$, the set C_ξ has been defined so that 1^0 C_ξ is one of the sets B_v^ξ, and 2^0 the C_ξ are pairwise disjoint. Since each L_ξ is a line, for $\delta\neq\gamma$, the set $L_\gamma\cap L_\delta$ contains at most one element. The B_v^α being pairwise disjoint for fixed α, it follows that each set C_ξ has a non-empty intersection with at most one set B_v^α, say $B_{\mu(\xi)}^\alpha$. Since $\alpha<\theta$, there exists a set, call it $B_{v(\alpha)}^\alpha$, which has an empty intersection with each set C_ξ, $\xi<\alpha$. Define C_α to be the set $B_{v(\alpha)}^\alpha$.

Well order the elements of the set $E^2-\bigcup_{\xi<\theta}C_\xi$ into a sequence $\{x_\xi\}_{\xi<\tau\leqslant\theta}$. Let $L_{\gamma(0)}$ be the first line which contains x_0. Now suppose that $\gamma(\xi)$ has been defined for $\xi<\delta<\tau$. Let $L_{\gamma(\delta)}$ be the first line not one of the lines $L_{\gamma(\xi)}$, $\xi<\delta$, which contains x_δ. $L_{\gamma(\delta)}$ certainly exists since there are $2^{\aleph_0}$ different lines containing x_δ. If $v=\gamma(\xi)$ let $F_v=C_v\cup\{x_\xi\}$. If v is not a $\gamma(\xi)$, let $F_v=C_v$. Clearly $E^2=\bigcup_{v<\theta}F_v$. Now each set F_v is exact and has property A. From the selection of the sets Y_ξ it follows that if T_ξ and T_v are subsets, of power $2^{\aleph_0}$ each, of F_ξ and F_v respectively, then $\overline{T}_\xi$ and $\overline{T}_v$ are incomparable order types. Now decompose each set F_ξ into disjoint pairs. This yields a decomposition of E^2 into disjoint pairs, since E^2 is the union of the pairwise disjoint sets F_ξ. There is no trouble in verifying that this decomposition satisfies the conclusions of Theorem 2.3.

We conclude with the following questions:

Let $\alpha<\lambda$ be a linear order type of power $2^{\aleph_0}$. Does there exist a decomposition of the reals into disjoint pairs such that any set S which is the union of pairs and which contains, as a subset, a set of order type α, contains a subset of order type λ? If the answer is in the negative, then does there exist at least one such α?

[1] A simply ordered set is *exact* if the only similarity transformation f of E into E is the identity.

[2] A linear set E, of power $2^{\aleph_0}$, has property A if no two disjoint subsets of E, of power $2^{\aleph_0}$ each, are similar.

[3] Let C and D be two simply ordered sets. $\overline{C}$ and $\overline{D}$ are said to be *incomparable order types* if there is no similarity transformation of C into D and no similarity transformation of D into C.

References

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