[5] B. Knaster, *Quelques coupures singulières du plan*, Fund. Math. **7** (1925), p. 264-289.

[6] C. Kuratowski, *Sur les coupures irreductibles du plan*, Fund. Math. **6** (1924), p. 130-145.

[7] K. Borsuk et S. Mazurkiewicz, *Sur les rétractes absolus indécomposables*, Comptes Rendus de l'Ac. des Sc. **199** (1934), p. 110 and K. Borsuk, *On an Irreducible 2-dimensional Absolute Retract*, Fund. Math. **37** (1950), p. 137-160.

[8] C. Kuratowski, *Topologie II*, Warszawa-Wrocław 1950, p. 371 and p. 380.

[9] K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. **19** (1932), p. 220-242, in part. p. 224.

[10] K. Borsuk, *On the imbedding of systems of compacta in simplicial complexes*, Fund. Math. **35** (1948), p. 217-234. In particular p. 224 and 230.

A Solution of a Problem of R. Sikorski

By

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C. Kuratowski showed in his paper [1]) that there exist two 1-dimensional compact sets (on the plane) which are not homeomorphic to each other, although each of them is homeomorphic to a relatively open subset of the other. In this note we construct two 0-dimensional compact sets satisfying the same condition [2]), which give the answer to a problem of R. Sikorski [3]).

Let P be a given 0-dimensional perfect compact set and let p be a given point of P. Let Q_a be a countable compact set such that the a-th derivative $Q_a^{(a)}$ of Q_a consists of a single point q_a $(a<2\omega)$. Let $R_a = P\times q_a + p\times Q_a$ in the product space $P\times Q_a$.

Consider the set of points $p_n = 2^{(-2^{-n})}$, $n=0, \pm 1, \pm 2, \ldots$, and put $a_{mn} = (1/m, p_n)$, $m=\pm 1, \pm 2, \ldots$, on the plane.

We construct the sets $D_{mn} \ni a_{mn}$ on the plane as follows:

If m is positive and $m+n>0$, let D_{mn} be a topological image of R_{m+n}, where a_{mn} corresponds to (p, q_{m+n}), the diameter being less than

$$\frac{1}{2}\,\mathrm{Min}\left(\frac{1}{m}-\frac{1}{m+1},\, p_{|n|+1}-p_{|n|}\right).$$

If m is positive and $m+n\leqslant 0$, put $D_{mn}=a_{mn}$. If on the other hand m is negative and $|m|+n>0$, let D_{mn} be a topological image of $R_{\omega+|m|+n}$, where a_{mn} corresponds to $(p, q_{\omega+|m|+n})$, the diameter being less than

$$\frac{1}{2}\,\mathrm{Min}\left(\frac{1}{|m|}-\frac{1}{|m|+1},\, p_{|n|+1}-p_{|n|}\right).$$

If m is negative and $|m|+n\leqslant 0$, put $D_{mn}=a_{mn}$.

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[1]) C. Kuratowski, *On a Topological Problem Connected with the Cantor-Bernstein Theorem*, Fund. Math. **37** (1950), p. 213-216.

[2]) The construction of this example is essentially analogous to that of C. Kuratowski, which are 1-dimensional, for the proposition in his paper that two sets are not homeomorphic can be proved by the same method as in this paper.

[3]) Coll. Math. **1** (1947-48), p. 242. See also C. Kuratowski, loc. cit.

Put

$$A=\sum_{m,n}\left[D_{mn}+\underset{(x,y)}{E}(x=0)(y=p_n)+\underset{(x,y)}{E}\left(x=\frac{1}{m}\right)(y=0\ \text{ or }\ 1)\right]+(0,0)+(0,1),$$

$$B=A-\underset{D_{mn}}{E}(m=1)-(1,0)-(1,1),$$

$$C=B-\underset{D_{mn}}{E}(m=-1)-(-1,0)-(-1,1).$$

Then A, B and C are 0-dimensional compact sets and B and C are open subsets of A and B respectively. Clearly A is homeomorphic to C. Therefore each of A and B is homeomorphic to a relatively open subset of the other.

Now we shall prove that A is not homeomorphic to B. Suppose on the contrary that there exists a homeomorphism h which maps A onto B. It is easy to see that $h(0,1)=(0,1)$, $h(0,0)=(0,0)$.

If $x=(1/m,1)$, then there exists m' with $mm'>0$ such that $h(x)=(1/m',1)$;

if $x=(0,p_n)$, then there exists n' such that $h(x)=(0,p_{n'})$;

if $x=a_{mn}$, $m>0$ and $m+n>0$, then there exist $m'>0$ and n' with $m+n=m'+n'$ such that $h(x)=a_{m'n'}$;

and if $x=a_{mn}$, $m<0$ and $|m|+n>0$, then there exist $m'<0$ and n' with $|m|+n=|m'|+n'$ such that $h(x)=a_{m'n'}$.

For each natural number i and $k>i+1$, the homeomorphism h has the following properties:

(*) On the left-hand half of A, *i. e.* on the subset of A, the x-coordinates of whose points are negative, the number of points a_{mn} with $|m|+n=k$ such that

$$y[a_{mn}]>p_i,\qquad y[h(a_{mn})]\leqslant p_i\ ^{4)}$$

is equal to the number of points $a_{m'n'}$ with $|m'|+n'=k$ such that

$$y[a_{m'n'}]\leqslant p_i,\qquad y[h(a_{m'n'})]>p_i.$$

(**) On the right-hand half of A the number of points a_{mn} with $m+n=k$ such that

$$y[a_{mn}]>p_i,\qquad y[h(a_{mn})]\leqslant p_i$$

is larger than the number of points $a_{m'n'}$ with $m'+n'=k$ such that

$$y[a_{m'n'}]\leqslant p_i,\qquad y[h(a_{m'n'})]>p_i.$$

Put $t_n=(0,p_n)$. For each t_n there exists a natural number $u(t_n)$ such that $y[h(a_{mn})]=y[h(t_n)]$ for every m with $|m|\geqslant u(t_n)$. Put $T_i=\{t_0,t_{\pm1},\dots,t_{\pm i}\}$ $(i=1,2,\dots)$ and let $u[i]=\mathrm{Max}\{u(t_n)\}$, where $t_n\in T_i+h^{-1}(T_i)$. Then $u[i]\geqslant1$.

[4]) $y[a]$ denotes the y-coordinate of a.

Now let $v(i)=\mathrm{Max}\{y[t_n]\}$, where $t_n\in T_i+h^{-1}(T_i)$, and let $v[i]$ be the integer such that $v(i)=p_{v[i]}$. Then $v[i]\geqslant i$. Let $k_i>u[i]+v[i]$.

Then we have the following proposition:

(⁂) *There exists at least one $a_{m_in_i}$ which satisfies one of the following conditions*:

(1) $m_i>0$, $m_i+n_i=k_i$, $y[a_{m_in_i}]>p_i$ *and* $y[h(a_{m_in_i})]<p_{-i}$.

(2) $m_i<0$, $|m_i|+n_i=k_i$, $y[a_{m_in_i}]<p_{-i}$ *and* $y[h(a_{m_in_i})]>p_i$.

In fact, by the definition of k_i the number of points a_{mn} with $m+n=k_i$ (where $m>0$) such that

$$y[a_{mn}]>p_i,\qquad p_{-i}\leqslant y[h(a_{mn})]\leqslant p_i$$

is equal to the number of points $a_{m'n'}$ with $|m'|+n'=k_i$ (where $m'<0$) such that

$$y[a_{m'n'}]>p_i,\qquad p_{-i}\leqslant y[h(a_{m'n'})]\leqslant p_i.$$

This number will be denoted by w_1.

Similarly the number of points a_{mn} with $m+n=k_i$ (where $m>0$) such that

$$p_{-i}\leqslant y[a_{mn}]\leqslant p_i,\qquad y[h(a_{mn})]>p_i$$

is equal to the number of points $a_{m'n'}$ with $|m'|+n'=k_i$ (where $m'<0$) such that

$$p_{-i}\leqslant y[a_{m'n'}]\leqslant p_i,\qquad y[h(a_{m'n'})]>p_i.$$

This number will be denoted by w_2.

Now we assume that there is no point which satisfies (1) or (2). Then on the left-hand half of A we have $w_2\geqslant w_1$ by (*), since there is no point which satisfies (2). Similarly on the right-hand half of A we have $w_1>w_2$ by (**), since there is no point which satisfies (1). Then we have $w_1>w_2\geqslant w_1$, which is a contradiction, and this proves the proposition (⁂).

Now let $\{a_{m_in_i}\}$ be a sequence where $a_{m_in_i}$ satisfies one of the conditions of (⁂).

If there exist infinitely many $m_i>0$ in the sequence $\{a_{m_in_i}\}$, then there exists a subsequence $\{a_{m_{i_j}n_{i_j}}\}$ which converges to a point s such that $y[s]=1$. Since the sequence $\{h(a_{m_{i_j}n_{i_j}})\}$ must converge to $(0,0)$, we have $h(s)=(0,0)$, which is a contradiction.

If there exists infinitely many $m_i<0$ in the sequence $\{a_{m_in_i}\}$, then there exists a subsequence $\{a_{m_{i_j}n_{i_j}}\}$ which converges to $(0,0)$. Then the sequence $\{h(a_{m_{i_j}n_{i_j}})\}$ converges to $h(0,0)=(0,0)$, which is again a contradiction, and the proof is complete.

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