On The Third Symmetric Potency of S_1.

By

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(From a letter to K. Borsuk).

In your paper *On the third symmetric potency of the circumference*, Fund. Math. **36** (1949), p. 236-244, you assert that the third symmetric potency of $S_1^{(3)}$ of the circle S_1 is homeomorphic to the Cartesian product of S_1 and the two sphere S_2. In your proof of this statement two parts should be distinguished. In the first (p. 237-243) and main part you show that $S_1^{(3)}$ can be obtained by a suitable identification of the boundaries of two anchor rings. In the second part (p. 244) you assert that the identification on the boundaries of these anchor rings is such, that the obtained manifold is homeomorphic to $S_1 \times S_2$. But in fact the identification you have made is incorrect and in consequence your final conclusion that the obtained manifold is homeomorphic to $S_1 \times S_2$ is false. A quite simple and short argument shows that $S_1^{(3)}$ has a vanishing fundamental group whence by your first result $S_1^{(3)}$ is a simply connected lensespace [1], *i. e.* the three sphere S_3. Your final theorem should therefore be corrected to read:

Theorem. *The third symmetric potency $S_1^{(3)}$ of S_1 is homeomorphic with the three sphere S_3.*

Proof that the fundamental group of $S_1^{(3)}$ vanishes.

If X is a metric space, $X^{(3)}$ shall denote the space of sets of 3 or less points of X, topologized as in your paper [2].

Let I be the unit interval. We obtain a model of $I^{(3)}$ by considering triplets of real numbers (x,y,z) with $0 \leqslant x \leqslant y \leqslant z \leqslant 1$, where triplets of the type (a,a,b) are to be identified with triplets of the type (a,b,b). Hence the simplex of fig. 1 with the identifications

indicated represents $I^{(3)}$. $(0,1,3)$ is identified with $(0,2,3)$ by the simplicial map which sends

$$0 \to 0,$$
$$1 \to 2,$$
$$3 \to 3.$$

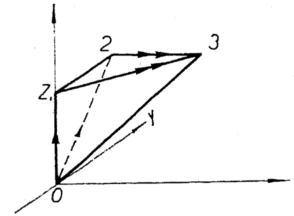


Fig. 1.

Clearly $I^{(3)}$ is a 3 cell.

S_1 is obtained from I by identifying the number 0 with the number 1. This induces an identification in our model of $I^{(3)}$ given by: $(0,a,b) \sim (a,b,1)$; or equivalently the face $(0,1,2)$ is identified with the face $(1,2,3)$ by the simplicial map which sends

$$0 \to 1,$$
$$1 \to 2,$$
$$2 \to 3.$$

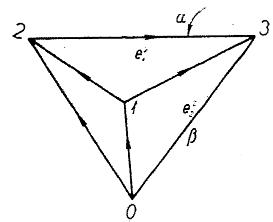


Fig. 2.

In fig. 2 we have indicated the model obtained after these additional indifications.

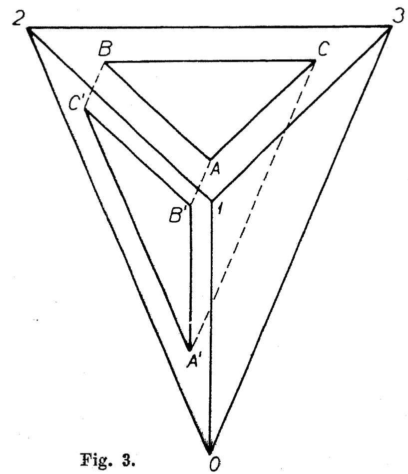


Fig. 3.

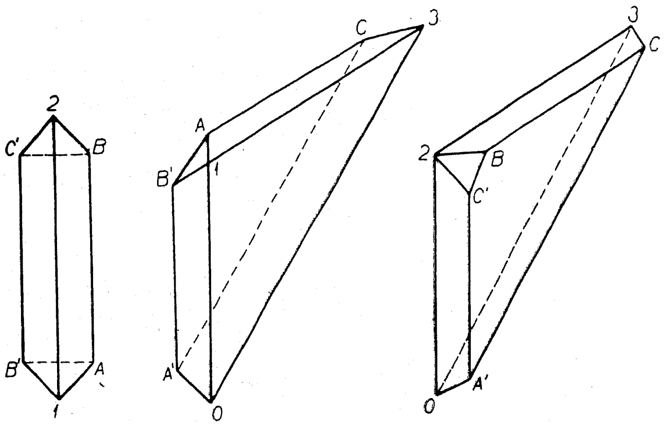
We observe that the resulting complex consists of one 3 cell e^3, two 2 cells e_1^2 and e_2^2, two 1 cells a and β, and a simple point p. Going around what used to be the face $(0,1,2)$ we see that $a = a^2 a^{-1} = I$ (represents the identity of the fundamental group). Hence β does too, ($\beta = a^2$!) and it follows that the fundamental group of $S_1^{(3)}$ is trivial.

You might be interested in the following proof that $S_1^{(3)}$ can be obtained by a suitable identification of the boundaries of two anchor rings, which is possibly a little simpler than yours [3].

Let K denote the simplex of fig. 1 without any identifications and let ABC (fig. 3) be the vertices of a triangle similar to the face 123 in the face (123). If A', B', C' are the images of A, B, C, under the simplicial map

$$\begin{aligned} 0 &\to 1, \\ 1 &\to 2, \\ 2 &\to 3, \end{aligned}$$

denote by U the convex hull of the points $[A, B, C, A', B', C']$ in K. It is then obvious that the image of U under f, the identification



a) b) c)

Fig. 4.

map which maps K onto $S_1^{(3)}$, is an anchor ring. To see that $f(\overline{K-U})$ is again an anchor ring we proceed as follows. We first cut $W=\overline{K-U}$ into the three cells shown in fig. 4. Fig. 4a shows the convex hull of the points $(1B'A\,C'B2)$, fig. 4b the convex hull of the points $(OA'B'\,1AC3)$, while 4c describes the convex hull of $(OA'C'B\,2C3)$. Under f the faces (013) and $(OA'C3)$ of fig. 4b are identified with the faces (023), and $(OA'C3)$ of fig. 4c respectively. If we perform these identifications in 3 space we obtain a 3 cell W' which after

an obvious isotopy can be represented as in fig. 5. (The faces $(OA'C'1)$ and $(2AC3)$ are ruled surfaces. $AB'A'C'BC$ are in a plane while 0,1,3 lye above that plane). In fig. 6 we subdivide W' into two wedge like pieces: (a) $(OA'B'1C')$ and (b) $(2AC3B)$; and the pyramid (c) $(1ABC'B')$. We next perform the identifications in-

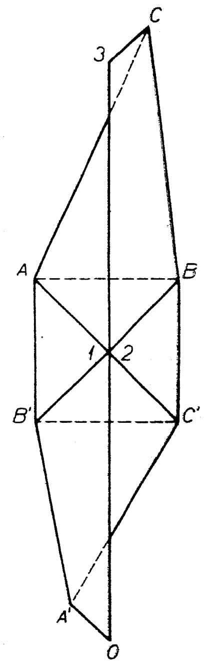
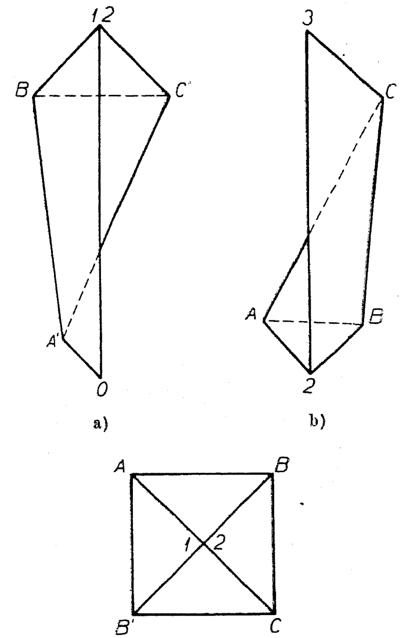


Fig. 5.



a) b) c)

Fig. 6.

dicated by f in the following order: first 6(a) is adjoined to 6(b), by identification of the face $(OA'C'2)$ with the face $(1AC3)$. The resulting triangular beam M is fitted to 4(a) by identifying $(1AB'C)$ and $(1BCB)$ of M with $(1AB2)$ and $(1B'C'2)$ of 4(a), respectively. The results is clearly an anchor ring from which the f image of our pyramid 6(c) has been drilled out. Adjoining 6(c) therefore completes our construction, for it is easily checked that all the identifications described by f have been carried out.

Bibliography.

[1] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig 1934, p. 215.

[2] K. Borsuk, *On the third symmetric potency of the circumference*, Fund. Math. **36** (1949), p. 236.

[3] l. c. pp. 237-243.

The Existence of Pseudoconjugates on Riemann Surfaces.

By

M. Morse and J. Jenkins.

§ 1. Introduction. Among the characteristics of a function U which is harmonic on a Riemann surface G^* are the topological interrelations of the level lines of U. One has merely to look at the level lines of $\mathcal{R}z$, $\mathcal{R}e^z$, $\mathcal{R}\log z$, etc. to sense both complexity and order. The dual level lines of a conjugate V of U add to this order and complexity. It seems likely that outstanding problems in Riemann surface theory, such as the type problem, the nature of essential singularities, the existence of functions on the Riemann surface with restricted properties cannot be thoroughly understood in the absence of a complete analysis of the topological characteristics of these level lines.

Such a topological study properly belongs to a somewhat larger study namely that of PH (pseudoharmonic) functions and their pseudoconjugates (defined in § 2). A first problem is that of the existence of PH functions U on G^* with prescribed level sets locally topologically like families of parallel straight lines except in the neighbourhood of points of a discrete set ω of points z_0. At a point z_0 the level curves of U may cross after the manner of the level curves of a harmonic function with a critical point at z_0.

Let E be the finite z-plane. In case $G^*=E$ and $\omega=0$, and excluding all recurrent level curves other than periodic curves, Kaplan has solved the above problem in [5]. More recently Boothby has extended Kaplan's result to the case of a general ω. As explained in [4] the *a priori* exclusion of recurrent level curves other than periodic curves does not seem justified. The writers of this paper have accordingly established the existence of PH functions without this hypothesis of non-recurrence [4].