A Characterization of Alephs.

By

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I_n will denote the set of the positive integers $1,\dots,n$. The class of all m-element subsets of I_n $(m\leqslant n)$ will be denoted by $\mathfrak{I}_{m,n}$. Clearly $\overline{\overline{\mathfrak{I}_{m,n}}}=\binom{n}{m}$.

X^n will denote the Cartesian product of n replicas of a set X, i. e. the set of all sequences $x=(x_1,\dots,x_n)$ where $x_i\in X$.

Let $\Lambda\in\mathfrak{I}_{m,n}$. Every set $P\subset X^n$ of the form $P=Y_1\times\dots\times Y_n$ where $Y_i=(a_i)$ [1] if $i\in\Lambda$, and $Y_i=X$ for i non $\in\Lambda$, is called a *Λ-set*. A Λ-set P is thus the „$(n-m)$-dimensional hyperplane" defined by the equations $x_i=a_i$ for $i\in\Lambda$.

Theorem. *Let m and k be positive integers and let X be a non-empty set. In order that $\overline{\overline{X}}<\aleph_{\tau+m}$, it is necessary and sufficient that X^{m+k} be the union of $\binom{m+k}{m}$ sets E_Λ $(\Lambda\in\mathfrak{I}_{m,m+k})$ such that $\overline{\overline{PE_\Lambda}}<\aleph_\tau$ for every Λ-set P.*

This Theorem is a generalization of theorems of Sierpiński [2]) and Kuratowski [3]).

Necessity. Let X_m be the set of all ordinals [4]) $\alpha<\omega_{\tau+m-1}$. It is sufficient to prove the existence of the required decomposition in the case $X=X_m$.

[1]) (a) denotes the set containing only one element a.

[2]) W. Sierpiński, *Sur quelques propositions concernant la puissance du continu*, this volume, pp. 1-13. See Theorems 1, 2, 3, 4, 8, 10 and the corollary at the end of the paper.

[3]) C. Kuratowski, *Sur une caractérisation des alephs*, this volume, pp. 14-17.

[4]) The cardinal of the set of all ordinals $<\alpha$ is denoted by $\overline{\alpha}$. The initial ordinal ω_ϱ is the least ordinal such that $\overline{\omega_\varrho}=\aleph_\varrho$.

For every ordinal α, let $\{\mu_{\beta}^{(\alpha)}\}_{\beta<\gamma}$ be a transfinite sequence containing every ordinal $\leqslant\alpha$ exactly once; γ is the least ordinal such that $\overline{\gamma}=\overline{\alpha}+1$.

By induction on m, we shall define some sets $E(i_1,\dots,i_m)$, where $i_1,\dots,i_m$ is any permutation of a set $\Lambda\in\mathfrak{I}_{m,m+k}$, as follows.

In the case $m=1$, if $i\in I_{1+k}$, then $E(i)$ is the set of all $(\alpha_1,\dots,\alpha_{k+1})\in X_1^{1+k}$ such that $\alpha_i\geqslant\alpha$ for $j=1,\dots,k+1$.

If $i_0,\dots,i_m$ is any permutation of a set $\Lambda\in\mathfrak{I}_{m+1,m+1+k}$ $(m\geqslant 1)$, then $E(i_0,\dots,i_m)$ is the set of all $(\alpha_1,\dots,\alpha_{m+1+k})\in X_{m+1}^{m+1+k}$ such that

(i) $\alpha_i=\mu_{\beta_{i'}}^{(\alpha_{i_0})}$ for $i\in I_{m+1+k}-(i_0)$, where

(ii) for each $i\in I_{m+1+k}-(i_0)$, $i'=i$ if $i<i_0$, and $i'=i-1$ if $i>i_0$; and

(iii) $(\beta_1,\dots,\beta_{m+k})\in E(i'_1,\dots,i'_m)$ [5]).

Clearly (i) implies that $\alpha_{i_0}=\max(\alpha_1,\dots,\alpha_{m+1+k})$ and $\beta_{i'}\in X_m$ since $\overline{\alpha}_{i_0}\leqslant\aleph_{\tau+m-1}$.

We shall prove by induction on m that

$(\mathbf{A}_m)$. $X_m^{m+k}=\sum E(i_1,\dots,i_m)$ where the sign $\sum$ is extended over all permutations $i_1,\dots,i_m$ of all sets $\Lambda\in\mathfrak{I}_{m,m+k}$.

$(\mathbf{B}_m)$. If P is a Λ-subset of X_m^{m+k}, $\Lambda\in\mathfrak{I}_{m,m+k}$, and $i_1,\dots,i_m$ is a permutation of Λ, then

$$\overline{\overline{P\cdot E(i_1,\dots,i_m)}}<\aleph_\tau.$$

The assertions $(\mathbf{A}_m)$ and $(\mathbf{B}_m)$ imply immediately the existence of the required decomposition of X_m^{m+k}. It is sufficient to put $E_\Lambda=$ the union of all sets $E(i_1,\dots,i_m)$ where $i_1,\dots,i_m$ is any permutation of $\Lambda\in\mathfrak{I}_{m,m+k}$.

$(\mathbf{A}_1)$. If $(\alpha_1,\dots,\alpha_{k+1})\in X_1^{1+k}$, let $\alpha_i=\max(\alpha_1,\dots,\alpha_{k+1})$. Consequently $(\alpha_1,\dots,\alpha_{k+1})\in E(i)$.

This proves that X_1^{1+k} is contained in the union of all sets $E(i)$. The converse inclusion is trivial.

$(\mathbf{A}_m)\to(\mathbf{A}_{m+1})$. If $(\alpha_1,\dots,\alpha_{m+1+k})\in X_{m+1}^{m+1+k}$, let $\alpha_{i_0}=\max(\alpha_1,\dots,\alpha_{m+1+k})$. We have $\alpha_i=\mu_{\beta_{i'}}^{(\alpha_{i_0})}$ for some $\beta_{i'}\in X_m$, $i\in I_{m+1+k}-(i_0)$, where i' is defined as in (ii). By induction $(\beta_1,\dots,\beta_{m+k})\in E(j_1,\dots,j_m)$ for a permutation $j_1,\dots,j_m$ of a set $\Lambda'\in\mathfrak{I}_{m,m+k}$. Let $j_1=i'_1,\dots,j_{m+k}=i'_{m+k}$. Then, by (i)-(iii), $(\alpha_1,\dots,\alpha_{m+1+k})\in E(i_0,i_1,\dots,i_m)$, which proves that X_{m+1}^{m+1+k} is contained in the union of all sets $E(i_0,\dots,i_m)$. The converse inclusion is trivial.

[5]) The ordinals $\beta_1,\dots,\beta_{m+k}$ are defined by (i) and (ii).

$(\mathbf{B}_1)$. Let $i \in I_{1+k}$, $\Lambda=(i) \in \mathfrak{I}_{1,1+k}$, and let P be a (i)-set, i. e. the set of all $(a_1,\ldots,a_{k+1}) \in X_1^{1+k}$ such that $x_i=a_0=\text{constant}$ $(a_0 \in X_1)$. Then $P\cdot E(i)$ is the set of all $(a_1,\ldots,a_{k+1})$ such that $a_i=a_0$ and $a_j \leqslant a_0$ for $j=1,\ldots,k+1$. Clearly, the power of this set is $<\aleph_\tau$ since $\overline{\overline{a}}_0<\aleph_\tau$.

$(\mathbf{B}_m)\rightarrow(\mathbf{B}_{m+1})$. Let $\Lambda \in \mathfrak{I}_{m+1,m+1+k}$ and let P be a Λ-set,

$$P=Y_1\times\ldots\times Y_{m+1+k},$$

where $Y_i=(a_i)$ if $i \in \Lambda$, and $Y_i=X_{m+1+k}$ if i non $\in \Lambda$.

The set $Q=P\cdot E(i_0,\ldots,i_m)$, where $i_0,\ldots,i_m$ is a permutation of Λ, is the set of all $(a_1,\ldots,a_{m+1+k}) \in X_{m+1}^{m+1+k}$ such that (i), (ii) and (iii) are satisfied and $a_i=a_i$ for $i \in \Lambda$. We recall that the ordinals $\beta_{i'}$ are uniquely determined by (i) since every ordinal $\leqslant\alpha$ appears in the sequence $\{\mu_\xi^\alpha\}$ exactly once. Consequently the ordinals $\beta_{i'}$ where $i \in \Lambda-(i_0)$ are uniquely determined by ordinals a_i $(i \in \Lambda)$ since $a_i=\mu_{\beta_{i'}}^{(a_{i_0})}$ for $i \in \Lambda-(i_0)$.

Let $P'=Y_1'\times\ldots\times Y_{m+k}'$, where $Y_{i'}'=(\beta_{i'})$ for $i \in \Lambda-(i_0)$, and $Y_{i'}'=X_{m+k}$ for $i \in I_{m+1+k}-\Lambda$. Clearly P' is a Λ'-set in X_m^{m+k}, where $\Lambda' \in \mathfrak{I}_{m,m+k}$ is the set of all integers i' such that $i \in \Lambda-(i_0)$.

Hence Q is the set of all elements $(a_1,\ldots,a_{m+1+k})$ such that (i) is satisfied, and $(\beta_1,\ldots,\beta_{m+k}) \in P'\cdot E(i_1',\ldots,i_m')$. The last set is of power $<\aleph_\tau$ by the induction hypothesis $(\mathbf{B}_m)$ since $i_1',\ldots,i_m'$ is a permutation of Λ'. Hence $\overline{\overline{Q}}<\aleph_\tau$.

Sufficiency. We shall prove the following two statements:

$(\mathbf{C})$. In the case $m=1$, there is no decomposition of X^{1+k} $(\overline{\overline{X}}=\aleph_{\tau+1})$ into $\binom{k+1}{1}=k+1$ sets $E_{(i)}$ $(i \in I_{k+1})$ such that

$$\overline{\overline{P\cdot E_{(i)}}}<\aleph_\tau \quad \text{for every } (i)\text{-set } P\subset X^{1+k}.$$

$(\mathbf{D})$. If there is a decomposition of X^{m+1+k} $(\overline{\overline{X}}=\aleph_{\tau+m+1})$ into $\binom{m+1+k}{m+1}$ sets E_Λ $(\Lambda \in \mathfrak{I}_{m+1,m+1+k})$ such that

$$\overline{\overline{P\cdot E_\Lambda}}<\aleph_\tau \quad \text{for every } \Lambda\text{-set } P\subset X^{m+1+k},\ \Lambda \in \mathfrak{I}_{m+1,m+1+k},$$

then there is a decomposition of X_0^{m+k} $(\overline{\overline{X}}_0=\aleph_{\tau+m})$ into $\binom{m+k}{m}$ sets $E_{\Lambda'}$ $(\Lambda' \in \mathfrak{I}_{m,m+k})$ such that

$$\text{(iv)}\qquad \overline{\overline{P'\cdot E_{\Lambda'}}}<\aleph_\tau \text{ for every } \Lambda'\text{-set } P'\subset X_0^{m+k},\ \Lambda' \in \mathfrak{I}_{m,m+k}.$$

The assertions $(\mathbf{C})$ and $(\mathbf{D})$ imply that, for every positive integer m, the decomposition (iv) of X_0^{m+k} is impossible whenever $\overline{\overline{X}}_0=\aleph_{\tau+m}$. Consequently, the decomposition (iv) is also impossible whenever $\overline{\overline{X}}_0\geqslant\aleph_{\tau+m}$.

Proof of $(\mathbf{C})$. Let X_0 be a subset of X with $\overline{\overline{X}}_0=\aleph_\tau$, and let $b_0 \in X_0$. For every $b \in X_0$, let P_b be the $(k+1)$-set [6] of all $(x_1,\ldots,x_k,b) \in X^{k+1}$. We have $\overline{\overline{P_b\cdot E_{(k+1)}}}<\aleph_\tau$. Therefore the projection Q of all sets $P_b\cdot E_{(k+1)}$ $(b \in X_0)$ on the „hyperplane" P_{b_0} has the power $\leqslant\aleph_\tau$. Consequently, there is a point $(a_1^0,\ldots,a_k^0,b_0) \in P_{b_0}-Q$. The set L (the „straight line") of all points $(a_1^0,\ldots,a_k^0,x)$ $(x \in X)$ has the properties:

(v) the set $M=L\cdot\sum_{b\in X_0}P_b$ has the power $\aleph_\tau$;

(vi) $M\cdot E_{(k+1)}=0$.

By (vi), $M\subset E_{(1)}+\ldots+E_{(k)}$. By (v), there is an integer $i_0 \in I_k$ such that $\overline{\overline{M\cdot E_{(i_0)}}}=\aleph_\tau$. Let P_0 be the (i_0)-set of all points $(x_1,\ldots,x_{k+1}) \in X^{k+1}$ such that $x_{i_0}=a_{i_0}$. We have $ME_{(i_0)}\subset P_0E_{(i_0)}$ which contradicts the assumption that $\overline{\overline{P_0E_{(i_0)}}}<\aleph_\tau$.

Proof of $(\mathbf{D})$. Let $X_0\subset X$, $\overline{\overline{X}}_0=\aleph_{\tau+m}$, and let $\mathfrak{P}$ be the class of all Λ-sets $P=Y_1\times\ldots\times Y_{m+1+k}\subset X^{m+1+k}$, where $Y_i=(a_i)\subset X_0$ for $i \in \Lambda \in \mathfrak{I}_{m+1,m+k}$ (i. e. $m+1+k$ non $\in\Lambda$).

Clearly $\overline{\overline{\mathfrak{P}}}=\aleph_{\tau+m}$. Since $\overline{\overline{PE_\Lambda}}<\aleph_\tau$ for every Λ-set P, the union S of all sets PE_Λ, where $\Lambda \in \mathfrak{I}_{m+1,m+k}$ and $P \in \mathfrak{P}$, has the power $\leqslant\aleph_{\tau+m}$. The projection of S on the $(m+1+k)$-th axis of coordinates also has the power $\leqslant\aleph_{\tau+m}$; therefore there is an element $a \in X$ such that

$$HPE_\Lambda=0 \quad\text{if}\quad P \in \mathfrak{P},\ \Lambda \in \mathfrak{I}_{m+1,m+k},\ P \text{ is a } \Lambda\text{-set},$$

where H is the set of all points [7] (x,a), $x \in X_0^{m+k}$.

Since H is the sum of all sets HP, where P is a Λ-set $\in\mathfrak{P}$ (Λ fixed), we obtain $HE_\Lambda=0$ for every $\Lambda \in \mathfrak{I}_{m+1,m+k}$. Hence

(vii) H is contained in the union of all E_Λ such that $m+1+k \in \Lambda \in \mathfrak{I}_{m+1,m+1+k}$.

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[6]) That is, P_b is a Λ-set, where $\Lambda=(k+1)$.

[7]) If $x=(x_1,\ldots,x_{m+k}) \in X_0^{m+k}$, then (x,a) denotes the point

$$(x_1,\ldots,x_{m+k},a) \in X^{m+k+1}.$$

For every $\Lambda' \in \mathfrak{I}_{m,m+k}$ let $E_{\Lambda'}$ be the set of all $x \in X_0^{m+k}$ such that $(x,a) \in E_\Lambda$, $\Lambda = \Lambda' + (m+1+k)$. By (vii), X_0^{m+k} is the union of all sets $E_{\Lambda'}$, $\Lambda' \in \mathfrak{I}_{m,m+k}$. We shall prove the property (iv).

Let $\Lambda' \in \mathfrak{I}_{m,m+k}$, $\Lambda = \Lambda' + (m+1+k) \in \mathfrak{I}_{m+1,m+1+k}$. Let $P' = Y_1' \times \ldots \times Y_{m+k}'$, where $Y_j' = (a_j)$ for $j \in \Lambda'$ be any Λ'-subset of X_0^{m+k}, and let P be the Λ-set of all points $(x_1, \ldots, x_{m+k}, a) \in X^{m+1+k}$ where $x_j = a_j$ for $j \in \Lambda$. We have $\overline{\overline{PE_\Lambda}} < \aleph_\tau$. Since $P'E_{\Lambda'}$ is the set of all $x \in X_0^{m+k}$ such that $(x,a) \in PE_\Lambda$, we infer that $\overline{\overline{P'E_{\Lambda'}}} < \aleph_\tau$.

Corollary 1. *Let k be any positive integer. The continuum hypothesis is equivalent to the assertion that the $(k+2)$-dimensional Euclidean space is the sum of $\binom{k+2}{2}$ sets $E_{(i,j)}$*[8] *such that the set $PE_{(i,j)}$ is finite for every k-dimensional hyperplane P perpendicular to the i-th and j-th axes of coordinates.*

Corollary 2. *Let k be any positive integer. The continuum hypothesis is equivalent to the assertion that the $(k+1)$-dimensional Euclidean space is the sum of $k+1$ sets E_i $(i=1,\ldots,k+1)$ such that, for every k-dimensional hyperplane P perpendicular to the i-th axis of coordinates, the set PE_i is at most denumerable.*

In order to prove the above corollaries it is sufficient to put in the Theorem

$$\tau = 0 \quad \text{and} \quad m = 2,$$

or:

$$\tau = 1 \quad \text{and} \quad m = 1.$$

[8]) Here $\Lambda = (i,j) \in \mathfrak{I}_{2,k+2}$, i. e. (i,j) is a two-element subset of I_{k+2}.

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The Space of Measures on a Given Set[1]).

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This paper is an attempt at a systematic discussion of the concept of weak convergence of measures. We shall introduce a neighborhood topology in the set M_R of all measures on a given set (or space) R, and discuss the relations between the properties of R and the topology of M_R. This topology specializes to weak convergence under certain conditions.

The space of measures. Let R be an abstract set with a class of subsets called „open", satisfying, for the present, only

Axiom I: *R is an open set.*

A *measure* is a set function defined for all sets, satisfying:

(1) $\varphi(A) \geqslant 0, \quad \varphi(0) = 0, \quad \Phi(R)$ *finite.*

(2) $A \subset B \implies \varphi(A) \leqslant \varphi(B)$

(3) $$\varphi\Big(\sum_{i=1}^{\infty} A_i\Big) \leqslant \sum_{i=1}^{\infty} \varphi(A_i)$$

(4) $\varphi(A) = LB\, \varphi(O)$ *for all open sets $O \supset A$ (Regularity).*

(5) *Open sets are (Carathéodory) measurable.*

Definition: A unitary neighborhood $\mathcal{O}(\varphi_0, O, a)$ of a measure φ_0 is the set of all measures φ for which $\varphi_0(O) < \varphi(O) + a$ and $|\varphi(R) - \varphi_0(R)| < a$, where O is open and $a > 0$.

Any finite product of unitary neighborhoods of φ_0 is called a neighborhood of φ_0.

The measures on R thus constitute a topological space M_R. Neighborhoods are open sets, but we shall not prove this.

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