plus all limit points of such elements. However

$$\Big\|\sum_{i\in\pi}\alpha_i v_i-\sum_{j\in\pi'}\beta_j v_j\Big\|=\Big\|\sum_{i\in\pi}\alpha_i v_i+\sum_{j\in\pi'}\beta_j v_j\Big\|=1$$

and the same holds for the limit points. Hence each $S_{\pi,\pi'}$ lies on the surface of the unit sphere. If $\pi\cup\pi'$ is all the positive integers, then $S_{\pi,\pi'}$ is a face of the unit sphere. Also every point of unit norm is in such a set, for otherwise it would be in a set $S_{\pi,\pi'}$ with $\pi\cap\pi'$ non empty and could be approximated by a point of the form

$$Z=\sum_{i\in\pi}\alpha_i v_i-\sum_{j\in\pi'}\beta_j v_j$$
$$=\sum_{i\in\pi-(\pi\cap\pi')}\alpha_i v_i+\sum_{k\in\pi\cap\pi'}(\alpha_k-\beta_k)v_k-\sum_{j\in\pi'-(\pi\cap\pi')}\beta_j v_j$$

with $\{\alpha_i\}$, $\{\beta_j\}$, finite sets of non negative numbers such that

$$\sum_i\alpha_i+\sum_j\beta_j=1.$$

By properties K 8 and K 9 it can be seen that $\|z\|<1$. Hence the $S_{\pi,\pi'}$ with $\pi\cap\pi'$ empty and $\pi\cup\pi'=1$, consist of all the faces of the unit sphere.

The faces S_k determined by v_k and $\{-v_j\}$, $j\neq k$, represent subsets σ_k of minimal positive measure, since the only face for which $F\lesssim S_k$ is F_0. If S_π is any face, then $S_k\lesssim S_\pi$ for all k for which $k\in\pi$.

Thus every measurable subset contains an atomic subset, and since the space Ω is of finite measure, the space of integrable functions represented is the space l.

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Undecidability of Some Topological Theories.

By

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The purpose of the present paper is to prove the essential undecidability of some elementary theories of closure algebra[1]), of Brouwerian algebra, of the algebra of bodies, of the algebra of convexity and of the semi-projective algebra. This is attained by means of the method of interpretation based on the general theorems of Tarski[2]) and on the theorems of Mostowski, Tarski and Robinson[3]) concerning the essential undecidability of a finitely axiomatizable arithmetic. The main idea of the proof is that this arithmetic can be interpreted as an arithmetic of finite sets. Hence each theory in which we can define the class of finite sets and some operations on the finite sets is undecidable.

The elementary theories $\mathcal{T}_1,\mathcal{T}_2,\ldots$ discussed herein are assumed to be formalized in the lower functional calculus, they all have the same logical constants (connectives, quantifiers, identity symbol), logical axiom schemes and rules of inference. Each theory $\mathcal{T}_n\langle\mathfrak{X},\mathfrak{D}_1,\mathfrak{D}_2,\mathfrak{D}_3\rangle$ is determined by its non-logical (primitive) constants: $\mathfrak{X},\mathfrak{D}_1,\mathfrak{D}_2,\mathfrak{D}_3$ and non-logical axioms. The terms: $\mathfrak{X},\mathfrak{D}_1,\mathfrak{D}_2,\mathfrak{D}_3$ denote „univers du discours", and certain relations. The non-elementary theories $\mathcal{S}_1,\mathcal{S}_2,\ldots$ of order ω contain the simple theory of (finite) types. (I. e. each type of the order $n+1$ is the class of all subclasses of the types of order n). The non-logical (primitive) constants: $\mathfrak{X}',\mathfrak{D}_1',\mathfrak{D}_2',\mathfrak{D}_3'$ of a non-elementary theory $\mathcal{S}_n\langle\mathfrak{X}',\mathfrak{D}_1',\mathfrak{D}_2',\mathfrak{D}_3'\rangle$ denote only the relations and operations defined over the individuals of the lowest type. The variable letters $\mathcal{A},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ will be used to designate theories which can be $\mathcal{T}_n$ or $\mathcal{S}_n$. $\mathcal{A}'$ will be

[1]) The undecidability of the closure algebra has been proved in another way by Stanisław Jaśkowski in 1939. See [2].

[2]) See Tarski [11] and Robinson [10].

[3]) See Mostowski [9].

the theory obtained from the theory $\mathcal{A}$ by adding to the theory $\mathcal{A}$ the rule of introducing of new terms defined by means of primitive constants of $\mathcal{A}$. In all cases which we shall encounter in the sequel the theory $\mathcal{A}_n^{\blacktriangledown}$ will contain all definitions established in the section concerning the theory $\mathcal{A}_n$. We say that a *relation* $\mathfrak{D}_n$ *is definable in an elementar way in* $\mathcal{A}$ if there exists in $\mathcal{A}^{\blacktriangledown}$ a definition D of the relation $\mathfrak{D}_n$ and D contains only the variables of the lowest type. A *theory* $\mathcal{A}$ *is contained in the theory* $\mathcal{A}_1$ if each theorem of $\mathcal{A}$ is a theorem of $\mathcal{A}_1$. $\mathcal{A}$ *is consistent with* $\mathcal{A}_1$ if there exists a consistent theory $\mathcal{A}_2$ which contains both $\mathcal{A}$ and $\mathcal{A}_1$. A space satisfying all the theorems of the theory $\mathcal{A}$ will be called a *space* $\mathcal{A}$.

§ 1. The arithmetic of finite sets.

Let $\mathcal{T}_1\langle\mathfrak{L}, \dagger, \times, \mathfrak{S}, \mathfrak{D}, \approx\rangle$ be the arithmetic of finite sets. Before writing the axioms defining the theory $\mathcal{T}_1$ as a formal system, we shall give an interpretation of its symbols. The primitive constants of $\mathcal{T}_1$ may have the following meaning:

$\mathfrak{L}(A)\cdot=\cdot A$ *is a finite set*, $\quad\mathfrak{S}(AB)\cdot=\cdot\bar{\bar{B}}=\bar{\bar{A}}+1$,

$\dagger(ABC)\cdot=\cdot\bar{\bar{A}}+\bar{\bar{B}}=\bar{\bar{C}}$, $\quad A\approx B\cdot=\cdot\bar{\bar{A}}=\bar{\bar{B}}$,

$\times(ABC)\cdot=\cdot\bar{\bar{A}}\cdot\bar{\bar{B}}=\bar{\bar{C}}$, $\quad\mathfrak{D}$ *is the empty set*,

where $\bar{\bar{A}}$ = the cardinal number of the set A and $+$, . are the familiar operations on natural numbers.

$\mathcal{T}_1$ has the following axioms:

1. $\mathfrak{L}(A)\to A\approx A$ $\qquad$ 2. $\mathfrak{L}(A)\cdot A\approx B\cdot\to\cdot\mathfrak{L}(B)$
3. $\mathfrak{L}(\mathfrak{D})$ $\qquad$ 4. $A\approx B\cdot B\approx C\cdot\to\cdot C\approx A$
5. $A\approx\mathfrak{D}\cdot\to\cdot\sim(\mathfrak{S}(BA))$
6. $\mathfrak{L}(A)\cdot\sim(A\approx\mathfrak{D})\cdot\to\cdot(\exists B)\cdot\mathfrak{L}(B)\cdot\mathfrak{S}(BA)$
7. $\mathfrak{L}(A)\cdot\to\cdot(\exists B)\cdot\mathfrak{L}(B)\cdot\mathfrak{S}(AB)$
8. $\mathfrak{S}(AB)\cdot\mathfrak{S}(CD)\cdot\to: A\approx C\cdot\equiv\cdot B\approx D$
9. $\mathfrak{L}(A)\cdot\mathfrak{L}(B)\cdot\to\cdot(\exists CD)\cdot\mathfrak{L}(C)\cdot\mathfrak{L}(D)\cdot\dagger(ABC)\cdot\times(ABD)$
10. $\mathfrak{L}(A)\to\dagger(A\mathfrak{D}A)$ $\qquad$ 11. $\mathfrak{L}(A)\to\times(A\mathfrak{D}\mathfrak{D})$
12. $\dagger(ABC)\cdot\dagger(DEF)\cdot A\approx D\cdot B\approx E\cdot\to\cdot C\approx F$
13. $\dagger(ABC)\cdot\dagger(AEF)\cdot\mathfrak{S}(FK)\cdot\mathfrak{S}(EB)\cdot\to\cdot C\approx K$
14. $\times(ABC)\cdot\times(DEF)\cdot A\approx D\cdot B\approx E\cdot\to\cdot C\approx F$
15. $\times(ABC)\cdot\mathfrak{S}(DB)\cdot\times(ADF)\cdot\dagger(FAE)\cdot\to\cdot C\approx E$

Theorem 1. *The theory* $\mathcal{T}_1$ *is essentially undecidable and finitely axiomatizable.*

Proof. $\mathcal{T}_1$ can be interpreted as the arithmetic of natural numbers: $\dagger, \times, \mathfrak{S}$ are interpreted as the operations of addition, multiplication and succession, $\mathfrak{L}$ = the class of natural numbers, $\mathfrak{D}$ = the number zero, $\approx$ is interpreted as the relation of identity. It is the result (yet not published) of Mostowski, Tarski and Robinson that this arithmetic is essentially undecidable.

§ 2. Undecidability of a closure algebra.

A theory $\mathcal{A}\langle\mathfrak{B}, +, \cdot, -, \mathbf{0}, \mathbf{1}, \mathbf{C}\rangle$[4]) will be called a *closure algebra* if $\mathcal{A}$ contains the axioms of a finitely additive Boolean algebra [5]) (A0) and the following specific axiom of the closure algebra [6]):

A 1 $\quad\mathfrak{B}(A)\cdot\mathfrak{B}(B)\cdot\to\cdot A+\mathbf{C}A+\mathbf{CC}B=\mathbf{C}(A+B)\cdot-\mathbf{C}(\mathbf{0})$.

The primitive symbols of a closure algebra denote respectively: the class of elements of the algebra, the Boolean sum, the Boolean product, the Boolean complementation, the zero-element, the unity-element, the closure operation. These operations will be interpreted in the sequel as set-theoretical and topological ones. The elements of the algebra will be called *sets of points*. The element 1 will be called the *space* of the algebra $\mathcal{A}$.

Let $\mathcal{S}_1\langle\mathfrak{B}, +, \cdot, -, \mathbf{0}, \mathbf{1}, \mathbf{C}\rangle$ be a non-elementary closure algebra. The remaining axioms of $\mathcal{S}_1$ state that:

A 2 *Each element* $A\neq\mathbf{0}$ *contains an atom. Each atom is closed and there exist at least two atoms.*

A 3 *is the axiom of normality.*

A 4 *is the second axiom of countability (of the basis)* [7]).

A 5 *The space* 1 *is connected.*

A 6 *If* A *and* B *are two closed, isolated disjoint sets such that* $A+B\subset E$, *and* E *is a connected open set, then there are two connected open disjoint sets* C *and* D, *such that* $A\subset C$, $B\subset D$ *and* $C+D\subset E$.

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[4]) The symbol „·“ will be used in continuation as logical sign (of conjunction). The Boolean product $A\cdot B$ will be denoted by AB.

[5]) Based e. g. on Huntington's axioms. See [1] and Tarski [13].

[6]) The axiom (A 1) is equivalent to four axioms of the closure algebra established first by K. Kuratowski in [4]. See also Mc Kinsey and Tarski [7] Definition 1.1 and Kiyosi Iseki [3].

[7]) Axiom A 1—A 4 are equivalent to the fundamental axiomatisation of topology of metrizable spaces given by Kuratowski (axioms I—V) plus the assumption that there exist two points. See [5]. A 4 is the single non-elementary axiom of $\mathcal{S}_1$.

Theorem 2. *The relations $\mathfrak{L}$, $\dagger$, $\times$, $\mathfrak{S}$, $\mathfrak{D}$, $\approx$, of the arithmetic of finite sets are definable in an elementary way in the theory $\mathcal{S}_1$.*

Proof. In $\mathcal{S}_1$ we can establish the elementary definitions: of the class of closed elements ($\mathfrak{Cl}$), of the class of open elements ($\mathfrak{Op}$), of the relation of inclusion ($\subset$), of the class of atoms ($\mathfrak{At}$), of the class of connected elements ($\mathfrak{Cn}$), of the relation of component ($A\,\mathfrak{Cmp}\,B$) and of the class of isolated sets ($\mathfrak{Is}$):

$$\mathfrak{Cl}(A)\cdot\equiv\cdot\mathfrak{B}(A)\cdot A=\boldsymbol{C}A$$
$$\mathfrak{Op}(A)\cdot\equiv\cdot\mathfrak{Cl}(-A)$$
$$A\subset B\cdot\equiv\cdot\mathfrak{B}(A)\cdot\mathfrak{B}(B)\cdot AB=A$$
$$\mathfrak{At}(A)\cdot\equiv:\mathfrak{B}(A)\cdot A\neq\boldsymbol{0}:(B):B\subset A\cdot B\neq\boldsymbol{0}\cdot\rightarrow\cdot B=A$$
$$\mathfrak{Cn}(A)\cdot\equiv:\mathfrak{B}(A):(BC):A=B+C\cdot B\neq\boldsymbol{0}\neq C\cdot\boldsymbol{C}BC=\boldsymbol{0}\cdot\rightarrow\cdot B\boldsymbol{C}C\neq\boldsymbol{0}$$
$$A\,\mathfrak{Cmp}\,B\cdot\equiv:\mathfrak{Cn}(A)\cdot A\subset B:(D):\mathfrak{Cn}(D)\cdot A\subset D\subset B\cdot\rightarrow\cdot A=D$$
$$\mathfrak{Is}(A)\cdot\equiv:\mathfrak{B}(A):(B):\mathfrak{At}(B)\cdot B\subset A\cdot\rightarrow\cdot(\exists C)\cdot\mathfrak{Op}(C)\cdot B=A\,C.$$

The primitive relations of the arithmetic of finite sets are obviously definable in $\mathcal{S}_1$, because $\mathcal{S}_1$ contains the simple theory of types. These relations may be defined in an elementary way in $\mathcal{S}_1$ in the following manner:

$$A\approx_G B\cdot\equiv:\mathfrak{Cl}(A)\cdot\mathfrak{Cl}(B)\cdot\mathfrak{Is}(A)\cdot\mathfrak{Is}(B)\cdot AB=\boldsymbol{0}\cdot A+B\subset G:$$
$$(D):D\,\mathfrak{Cmp}\,G\cdot\rightarrow\cdot\mathfrak{At}(DA)\cdot\mathfrak{At}(DB).$$

It is evident that if $A\approx_G B$ then there exists a one-one correspondence $\mathfrak{R}$:

$$C\mathfrak{R}D\cdot\equiv\cdot\mathfrak{At}(C)\cdot\mathfrak{At}(D)\cdot C\subset A\cdot D\subset B\cdot(\exists E)\cdot E\,\mathfrak{Cmp}\,G\cdot C=AE\cdot D=BE$$

between the atoms contained in the element A and the atoms contained in the element B. Conversely if there exists a one-one mapping ($\mathfrak{R}$) of A into B, and A and B are two closed isolated disjoint sets then for some G, $A\approx_G B$. Namely it follows from A0-A4 that A and B are finite or at most countable sets. Also we can assume that $A=A_1+A_2+\ldots$, $B=B_1+B_2+\ldots$ and $A_n\mathfrak{R}B_n$, where $A_1,A_2,\ldots,B_1,B_2,\ldots$, are atoms. From A5 and A6 it follows that there exist two open connected disjoint elements G_1 and D_1 such that $A_1+B_1\subset G_1$ and $(A+B)\cdot-(A_1+B_1)\subset D_1$. Similarly there exist two open connected disjoint sets G_2 and D_2 such that $A_2+B_2\subset G_2$, $G_2+D_2\subset D_1$, and $(A+B)\cdot-(A_1+A_2+B_1+B_2)\subset D_2$ etc. Also by induction we can show that for each pair A_n, B_n there exists an open connected element G_n such that $A_n=G_n\cdot A$ and $B_n=G_n\cdot B$ and $G_n\cdot G_m=0$ for $n\neq m$. Putting $G=\sum_n G_n$, we obtain that $A\approx_G B$.

$$A\approx B\cdot\equiv\cdot(\exists G)\cdot A-B\approx_G B-A$$
$$\mathfrak{L}(A)\cdot\equiv:\mathfrak{Is}(A)\cdot\mathfrak{Cl}(A):(BCD):\mathfrak{At}(B)\cdot B\subset C\cdot A=C+D\cdot CD=0.$$
$$C\approx D\cdot\rightarrow\cdot\sim(C-B\approx D)$$
$$\mathfrak{S}(AB)\cdot\equiv\cdot\mathfrak{L}(B)\cdot(\exists C)\cdot\mathfrak{At}(C)\cdot C\subset B\cdot A\approx B-C$$
$$\dagger(ABC)\cdot\equiv\cdot\mathfrak{L}(C)\cdot(\exists DE)\cdot C=D+E\cdot DE=0\cdot A\approx D\cdot B\approx E$$
$$\times(ABC)\cdot\equiv:\mathfrak{L}(C):(\exists DF):F\approx B\cdot F\subset D\cdot C\subset D:$$
$$(E):E\,\mathfrak{Cmp}\,D\cdot\rightarrow\cdot\mathfrak{At}(EF)\cdot EC\approx A$$
$$\mathfrak{D}=\boldsymbol{0}.$$

It is necessary to make remarks concerning only the necessity of the „definiens" of the last definition. If $\times(ABC)$ and $\overline{\overline{A}}=n$, $\overline{\overline{B}}=m$ and $\overline{\overline{C}}=n\cdot m$, then there exist sets: $B_1,B_2,\ldots,B_m$, such that $B_i\approx A$ for $1\leqslant i\leqslant m$, and $C=B_1+B_2+\ldots+B_m$, and $B_i\cdot B_j=0$ for $i\neq j$. From A5 and A6 it follows that there exist two open connected disjoint sets G_1 and D_1 such that $B_1\subset G_1$ and $B_2+\ldots+B_m\subset D_1$. By further similar applications of the axiom A6 we can prove that there exist m open connected sets $G_1,\ldots,G_m$ such that $B_i=G_i\cdot C$ and $G_i\cdot G_j=0$ for $i\neq j$. Let F be an element containing only one atom from each B_i, and put $D=G_1+\ldots+G_m$. It is easy to show that A, B, C satisfy the conditions of the „definiens" of the last definition.

Theorem 3. *The arithmetic $\mathcal{T}_1$ is contained in $\mathcal{S}_1^{\blacktriangledown}$.*

Proof. We obtain in $\mathcal{S}_1^{\blacktriangledown}$ all the theorems of the arithmetic of finite sets in a well know set-theoretical manner, since $\mathcal{S}_1^{\blacktriangledown}$ contains the whole simple theory of types and from A2, A3, A5 it follows that there exists an infinite number of atoms.

Theorem 4. a) *There exists an elementary theory $\mathcal{T}_2$ of the closure algebra which is* 1. *essentially undecidable,* 2. *finitely axiomatizable and* 3. *contained in $\mathcal{S}_1$.*

b) *Each theory $\mathcal{T}$ of closure algebra consistent with $\mathcal{T}_2$ (or with $\mathcal{S}_1$) is undecidable.*

Proof. a) Let $\mathcal{T}_2$ be a closure algebra containing all axioms of the arithmetic $\mathcal{T}_1$ written down by means of the primitive terms of the closure algebra. From Theorem 1 we can prove that $\mathcal{T}_2$ is essentially undecidable and finitely axiomatizable. From Theorem 3 it follows that $\mathcal{T}_2$ is contained in $\mathcal{S}_1$.

b) If $\mathfrak{T}$ is consistent with $\mathfrak{T}_2$, or with $\mathcal{S}_1$ then $\mathfrak{T}_2$ is consistently interpretable in $\mathfrak{T}$. By an application of Tarski's theorem [8]) it results that $\mathfrak{T}$ is undecidable.

A theory $\mathfrak{T}\langle \mathfrak{B}, +, \cdot, -, \mathbf{0}, \mathbf{1}, \varphi \rangle$ will be called an *algebra of topology based on the topological notion* φ if the terms $\mathfrak{B}, +, \cdot, -, \mathbf{0}, \mathbf{1}$ denote the class of the sets of points, the Boolean operations of sum, product, complementation, zero-element, unit-element, and φ denotes a topological notion, and all theorems of the theory $\mathfrak{T}$ are true in a certain topological space in a wide sense [9]). From Theorem 4 it follows that each algebra of topology $\mathfrak{T}$ is undecidable, if $\mathfrak{T}$ is consistent with $\mathcal{S}_1^{\blacktriangledown}$ and $\mathfrak{T}$ is based on one of the following 9 notions: 1. the operation of closure, 2. the operation of interior (or exterior), 3. the operation of derived set, 4. the operation of frontier (or boundary), 5. the relation of neighbourhood, 6. the relation of mutual separation, 7. the class of closed sets, 8. the class of open sets, 9. the class of isolated sets. It is well known that the closure operation is definable in an elementary way by means of each of these notion.

Let $\mathcal{S}_2$ be a theory of closure algebra containing all axioms of the theory $\mathcal{S}_1$ and the following axiom:

A 7. *If $A \subset B$, A is an atom and B is an open set, then there exists an open set C such that $A \subset C \subset B$ and $-C$ is connected.*

From Theorem 4 it follows that each algebra of topology based on the notion of the class of connected sets ($\mathfrak{Cn}$), and consistent with $\mathcal{S}_2^{\blacktriangledown}$ is undecidable. The closure operation is definable in each algebra of topology containing A 0, A 1, A 2, A 7, and based on the notion of connectivity, in the following manner:

$$A = \boldsymbol{C}B \cdot \equiv : \cdot (D) : \cdot \mathfrak{At}(D) \cdot D \subset A \cdot \equiv : \\ \mathfrak{At}(D) : (E) : \mathfrak{Cn}(E) \cdot B \subset E \cdot \rightarrow \cdot \mathfrak{Cn}(E+D).$$

The questions as to whether the closure operation can be defined by means of the notion of the class of boundary sets, or by means of the class of dense sets (or: non dense, or dense in themselves) remain open.

We can also draw from Theorem 4 some conclusions concerning the axiomatizability of topological theories:

[8]) Theorem IV from Tarski [11].
[9]) See Mc Kinsey and Tarski [7], Definition 1.1.

Corollary 1. *There exists no recursively countable set of axioms of the system of all sentences of the closure algebra true in an Euclidean space $\boldsymbol{E}_n$ for $n \geqslant 2$.*

Proof. Such a system is complete and undecidable. From the theory of recursive functions it follows that no complete and undecidable theory can be based on a recursively countable set of axioms.

From this it follows in particular that a well known non-elementary and categorical system of topology of the sphere [10]) $\boldsymbol{S}_2$ contains undecidable sentences of elementary closure algebra.

Corollary 2. *There exists no recursively countable set of axioms of the system of all sentences of the closure algebra true in all Euclidean spaces.*

Proof. If this system had a recursively countable set A of axioms, we could obtain the set of axioms of the complete closure algebra of the plane by adding to the set A an axiom stating that the space is strictly two dimensional.

It is evident that for the algebras considered in the next sections analogous corollaries can be proved.

The problem whether the algebras of topology of the straight line $\boldsymbol{E}_1$ are decidable remains unsettled.

§ 3. Undecidability of Brouwerian algebra.

A new decision problem arises, if we consider the partial algebras of topology, i. e. theories in which we consider not the class of all sets of points, but a narrower class of sets (e. g. the class of closed sets) and the operations restricted to the elements of this class. In this section the algebra of closed sets ($\mathfrak{Cl}$) will be examined.

Let $\dotplus, \odot, \dot{-}$, denote the operations of sum, product and difference, defined over the class of closed sets in the following way:

$$A \dotplus B = C \cdot \equiv \cdot \mathfrak{Cl}(A) \cdot \mathfrak{Cl}(B) \cdot C = A + B$$
$$(\text{where } \mathfrak{Cl}(A) \cdot \equiv \cdot \mathfrak{B}(A) \cdot A = \boldsymbol{C}(A))$$
$$A \odot B = C \cdot \equiv \cdot \mathfrak{Cl}(A) \cdot \mathfrak{Cl}(B) \cdot C = AB$$
$$A \dot{-} B = C \cdot \equiv \cdot \mathfrak{Cl}(A) \cdot \mathfrak{Cl}(B) \cdot C = \boldsymbol{C}(A \cdot -B).$$

A theory $\mathcal{A}\langle \mathfrak{Cl}, \dotplus, \odot, \dot{-}, \boldsymbol{0} \rangle$ will be called an *algebra of closed sets* if there exists a closure algebra $\mathcal{A}_1$ such that $\mathcal{A}$ is

[10]) See Kuratowski [6], p. 374 (with bibliography).

contained in $\mathcal{A}_1^{\blacktriangledown}$ (provided that $\mathcal{A}_1^{\blacktriangledown}$ contains the definitions stated above). In conformity with the theorem of Mc Kinsey and Tarski[11]) each algebra of closed sets contains the axioms of the Brouwerian algebras.

Let $\mathcal{S}_3\langle\mathfrak{B}, +, \cdot, -, \boldsymbol{0}, \boldsymbol{1}, C\rangle$ be a non-elementary closure algebra containing all the theorems of the theory $\mathcal{S}_1$ and the following axioms A 6′ and A 8:

A 6′. *If A and B are two closed isolated disjoint sets and $A+B\subset E$, and E is a connected open set, then there exist two connected open sets C and D such that $CC\cdot CD=\boldsymbol{0}$, $A\subset C$, $B\subset D$, and $C+D\subset E$.*

A 8. *If A and B are two closed isolated disjoint sets, and there exists a one-one mapping of A into B, then there exists a closed set C such that $A+B\subset C$, and every component D of the set C contains exactly one point of the set A and one point of the set B.*

(From the Axiom A 6′ it follows that there exists such a set C, but not always closed).

Let $\mathcal{T}_3\langle\mathfrak{Cl}, \dot{+}, \odot, \dot{-}, \boldsymbol{0}\rangle$ be the elementary algebra of closed sets containing all theorems provable in $\mathcal{S}_3^{\blacktriangledown}$ and meaningful in $\mathcal{T}_3$.

Theorem 5. *The arithmetic $\mathcal{T}_1\langle\mathfrak{L}, \dagger, \times, \mathfrak{S}, \mathfrak{D}, \approx\rangle$ is contained in $\mathcal{T}_3^{\blacktriangledown}$.*

Proof. In $\mathcal{T}_3^{\blacktriangledown}$ we can define notions similar to those considered in the preceding section.

$$A\subseteq B\cdot\equiv\cdot A\dot{+}B=B\cdot\mathfrak{Cl}(A)\cdot\mathfrak{Cl}(B)$$
$$A=\boldsymbol{1}\cdot\equiv:\mathfrak{Cl}(A):(B):\mathfrak{Cl}(B)\cdot\rightarrow\cdot B\subseteq A$$
$$\mathfrak{At}(A)\cdot\equiv:\mathfrak{Cl}(A)\cdot A\neq\boldsymbol{0}:(B):B\subseteq A\cdot B\neq A\cdot\rightarrow\cdot B=\boldsymbol{0}$$
$$\mathfrak{Cn}(A)\cdot\equiv.\mathfrak{Cl}(A):(DE):\mathfrak{Cl}(D)\cdot\mathfrak{Cl}(E)\cdot A= D\dot{+}E\cdot D\neq\boldsymbol{0}\neq E\cdot\rightarrow\cdot D\odot E\neq\boldsymbol{0}$$
$$A\,\mathfrak{Cmp}\,B\cdot\equiv:\mathfrak{Cn}(A)\cdot A\subseteq B:(D):A\subseteq D\subseteq B\cdot\mathfrak{Cn}(D)\cdot\rightarrow\cdot A=D$$
$$\mathfrak{Dz}(A)\cdot\equiv\cdot\mathfrak{Cl}(A)\cdot A=\boldsymbol{1}\dot{-}(\boldsymbol{1}\dot{-}A)$$
($\mathfrak{Dz}=$ the class of closed domains)
$$\mathfrak{Is}(A)\cdot\equiv:\mathfrak{Cl}(A):(C):\mathfrak{At}(C)\cdot C\subseteq A\cdot\rightarrow\cdot(\exists D)\cdot\mathfrak{Dz}(D)\cdot C= A\odot D\cdot C\odot(\boldsymbol{1}\dot{-}D)=\boldsymbol{0}$$
$$A\approx_G B\cdot\equiv:\mathfrak{Is}(A)\cdot\mathfrak{Is}(B)\cdot A\odot B=\boldsymbol{0}\cdot\mathfrak{Cl}(G)\cdot A\dot{+}B\subseteq G: (C):C\,\mathfrak{Cmp}\,G\cdot\rightarrow\cdot\mathfrak{At}(A\odot C)\cdot\mathfrak{At}(B\odot C$$
$$A\approx B\cdot\equiv\cdot\mathfrak{Is}(A)\cdot\mathfrak{Is}(B)\cdot(\exists G)\cdot A\dot{-}B\approx_G B\dot{-}A$$

[11]) Theorem 1.14 in Mc Kinsey and Tarski [8], p. 150.

$$\mathfrak{L}(A)\cdot\equiv:\mathfrak{Is}(A):(BCD):\mathfrak{At}(B)\cdot B\subseteq C\cdot A=C\dot{+}D\cdot C\odot D=\boldsymbol{0}\cdot C\approx D\cdot\rightarrow\cdot\sim(C\dot{-}B\approx D)$$
$$\mathfrak{S}(AB)\cdot\equiv:\mathfrak{L}(B)\cdot(\exists C)\cdot\mathfrak{At}(C)\cdot C\subseteq B\cdot A\approx B\dot{-}C$$
$$\dagger(ABC)\cdot\equiv\cdot\mathfrak{L}(C)\cdot(\exists DE)\cdot C=D\dot{+}E\cdot D\odot E=\boldsymbol{0}\cdot A\approx D\cdot B\approx E$$
$$\times(ABC)\cdot\equiv:\mathfrak{L}(C):(\exists DF):\mathfrak{Cl}(D)\cdot F\approx B\cdot F\subseteq D\cdot C\subseteq D: (E):E\,\mathfrak{Cmp}\,D\cdot\rightarrow\cdot\mathfrak{At}(E\odot F)\cdot E\odot C\approx A$$
$$\mathfrak{D}=\boldsymbol{0}.$$

The arithmetical relations defined above are the same as those of the preceding section. It is evident that they satisfy the axioms of $\mathcal{T}_1$.

Theorem 6. a) *There exists an elementary theory $\mathcal{T}_4$ of the algebra of closed sets such that $\mathcal{T}_4$ is 1. essentially undecidable, 2. finitely axiomatizable, and 3. contained in $\mathcal{T}_3$.*

b) *Every Brouwerian algebra $\mathcal{T}\langle\mathfrak{Cl}, \dot{+}, \odot, \dot{-}, \boldsymbol{0}\rangle$ consistent with $\mathcal{T}_4$ (or with $\mathcal{S}_3^{\blacktriangledown}$) is undecidable.*

Proof. From Theorem 5 in a manner similar to the proof of Theorem 4.

From Theorem 6 it follows in particular that the abstract algebra of projective geometry and the general lattice-theory are undecidable[12]). It is well known that the Brouwerian algebra as well as the abstract algebra of projective geometry and the general lattice theory can be based on two primitive terms: $\mathfrak{Cl}$, $\subseteq$, denoting the class of the elements of the algebra and the relation of inclusion defined in the familiar way. On the other hand every theory of the Brouwerian algebra contains both the axioms of the general lattice theory and the axioms of the abstract algebra of projective geometry. Hence each of these theories is consistent with $\mathcal{T}_4$ and undecidable according to Theorem 6.

§ 4. Undecidability of the algebra of bodies.

Every theory $\mathcal{A}\langle\mathfrak{Do}, \cup, \cap, -\!\!\!|, \Delta\rangle$ is said to be the *algebra of bodies*, provided that: 1. $\mathcal{A}$ contains all axioms of Boolean algebra for the terms: $\mathfrak{Do}$, $\cup$, $\cap$, $-\!\!\!|$, denoting respectively the class of elements of the algebra, and the Boolean operations of sum, product, and complementation, and 2. $\mathcal{A}$ contains the following axioms determining the meaning of the relation Δ (of mutually tangent sets):

[12]) The undecidability of the abstract algebra of projective geometry and of the general lattice theory was proved in another way by A. Tarski in [12].

1. $A\Delta B\cdot\rightarrow\cdot A\cap B=\mathbf{0}$
2. $A\cap B=\mathbf{0}\cdot C\subset A\cdot D\subset B\cdot C\Delta D\cdot\rightarrow\cdot B\Delta A$
3. $(C\cup D)\Delta B\cdot\sim(C\Delta B)\cdot D\rightarrow\cdot\Delta B$.

Let the terms $\mathfrak{Do}$, $\cup$, $\cap$, $\dashv$, Δ, be defined in closure algebra in the following manner:

$\mathfrak{Do}(A)\cdot\equiv\cdot\mathfrak{B}(A)\cdot A=-\mathbf{C}-\mathbf{C}A$ (the class of open domains)
$A\cup B=-\mathbf{C}-\mathbf{C}(A+B)$
$A\cap B=A\cdot B$ (where A and B are open domains)
$\dashv(A)=-\mathbf{C}(A)$
$A\Delta B\cdot\equiv\cdot\mathfrak{Do}(A)\cdot\mathfrak{Do}(B)\cdot AB=\mathbf{0}\cdot\mathbf{C}A\mathbf{C}B\neq\mathbf{0}$.

It is easy to verify that, if $\mathcal{A}\langle\mathfrak{B},+,\cdot,-,\mathbf{0},\mathbf{1},\mathbf{C}\rangle$ is a closure algebra, and $\mathcal{A}_1\langle\mathfrak{Do},\cup,\cap,\dashv,\Delta\rangle$ is contained in $\mathcal{A}^{\blacktriangledown}$ (provided that $\mathcal{A}^{\blacktriangledown}$ contains the definitions stated above) and $\mathcal{A}_1$ contains all theorems provable in $\mathcal{A}^{\blacktriangledown}$ and meaningful in $\mathcal{A}_1$, then $\mathcal{A}_1$ is an algebra of bodies. (The second dual topological interpretation of the algebra of bodies is the algebra of closed domains:

$\mathfrak{Do}(A)\cdot\equiv\cdot\mathfrak{B}(A)\cdot A=\mathbf{C}-\mathbf{C}-A$ (the class of closed domains)
$A\cup B=A+B$ (where A and B are closed domains)
$A\cap B=\mathbf{C}-\mathbf{C}-(A\cdot B)$
$\dashv(A)=\mathbf{C}-A$
$A\Delta B\equiv\mathfrak{Do}(A)\cdot\mathfrak{Do}(B)\cdot AB\neq\mathbf{0}\cdot-\mathbf{C}-A\cdot-\mathbf{C}-B=\mathbf{0}$).

Let $\mathcal{S}_4\langle B,+,\cdot,-,\mathbf{0},\mathbf{1},\mathbf{C}\rangle$ be a non-elementary closure algebra containing the axioms of $\mathcal{S}_1$, the axiom A 6′ and the following axioms A 9-A 11:

A 9. *If A is an isolated closed set, then there exist two open domains B and C such that $B\cdot C=\mathbf{0}$ and $\mathbf{C}B\cdot\mathbf{C}C=A$.*

A 10. *If A and B are open domains, D is an atom and $A\cdot B=\mathbf{0}$, $D\subset\mathbf{C}A\cdot\mathbf{C}B$, then there exist two open domains E and F such that $E\subset A$, $F\subset B$ and $D=\mathbf{C}E\cdot\mathbf{C}F$.*

A 11. *If A and B are closed isolated sets, $A\cdot B=\mathbf{0}$, and there exists a one-one mapping of A into B, then there exists an open domain C such that $A+B\subset C$ and every component D of set C is an open domain, and D contains exactly one point of the set A and one point of the set B.*

Let $\mathcal{T}_5\langle\mathfrak{Do},\cup,\cap,\dashv,\Delta\rangle$ be the elementary theory containing all theorems provable in $\mathcal{S}_4$ and meaningful in $\mathcal{T}_5$.

Theorem 7. *The arithmetic $\mathcal{T}_1\langle\mathfrak{L},\dagger,\times,\mathfrak{S},\mathfrak{D},\approx\rangle$ is interpretable in $\mathcal{T}_5$.*

Proof. As in the preceding sections the proof will consist in a series of definitions. The finite sets will be interpreted as pairs of open domains A,B, such that the set $\mathbf{C}A\cdot\mathbf{C}B$ is finite.

$A\nabla B\cdot\equiv:\mathfrak{Do}(A)\cdot\mathfrak{Do}(B)\cdot A\Delta B:(CD):C\subset A\cdot C\Delta B\cdot D\subset B\cdot D\Delta A\cdot\rightarrow\cdot C\Delta D.$

The open domains A and B are mutually tangent in only one point

$\mathfrak{Nb}(ABC)\cdot\equiv.\because\mathfrak{Do}(A)\cdot B\nabla C.\because(D):D\subset B\cdot D\Delta C\cdot\rightarrow\cdot D\cap A\neq\mathbf{0}.\because$
$(E).\because\mathfrak{Do}(E):(D):D\subset B\cdot D\Delta C\cdot\rightarrow\cdot E\Delta D:\rightarrow\cdot E\cap A\neq\mathbf{0}.$

The open domain A is a neighbourhood of the contact point of the domains B and C

$AB=CD\cdot\equiv:A\nabla B\cdot C\nabla D:(E):\mathfrak{Nb}(EAB)\cdot\equiv\cdot\mathfrak{Nb}(ECD).$

The contact point of the domains A and B is the same as the contact point of the domains C and D

$\mathfrak{Cn}(A)\cdot\equiv:\mathfrak{Do}(A):(BC):A=B\cup C\cdot B\neq\mathbf{0}\neq C\cdot B\cap C=\mathbf{0}\cdot\rightarrow\cdot$
$(\exists EF)\cdot E\subset B\cdot F\subset C\cdot\mathfrak{Nb}(AEF).$

The open domain A is connected

$A\,\mathfrak{Cmp}\,B\cdot\equiv:\mathfrak{Cn}(A)\cdot A\subset B:(C):\mathfrak{Cn}(C)\cdot A\subset C\subset B\cdot\rightarrow\cdot A=C.$

A is a component domain of the domain B

$\mathfrak{Is}(AB)\cdot\equiv:\mathfrak{Do}(A)\cdot\mathfrak{Do}(B)\cdot A\cap B=\mathbf{0}:(EF):E\subset A\cdot F\subset B\cdot E\nabla F\cdot\rightarrow\cdot$
$(\exists D)\cdot\mathfrak{Nb}(DEF)\cdot A\cap D\nabla B\cap D.$

The set of contact points of the domains A and B is isolated and closed

$ABE)(CDF\cdot\equiv:\mathfrak{Is}(AB)\cdot\mathfrak{Is}(CD)\cdot\mathfrak{Do}(E)\cdot\mathfrak{Do}(F)\cdot E\cap F=\mathbf{0}:$
$(GHKL):G\subset A\cdot H\subset B\cdot G\nabla H\cdot K\subset C\cdot L\subset D\cdot K\nabla L\cdot$
$\rightarrow\cdot\mathfrak{Nb}(EGH)\cdot\mathfrak{Nb}(FKL)$

$AB\approx_G CD\cdot\equiv.\because(\exists EF).\because ABE)(CDF\cdot\mathfrak{Do}(G):$
$(H):H\,\mathfrak{Cmp}\,G\cdot\rightarrow\cdot H\cap E\cap A\nabla H\cap E\cap B\cdot$
$H\cap F\cap C\nabla H\cap F\cap D.\because(KL):K\subset A\cdot L\subset B\cdot K\nabla L\cdot$
$\rightarrow\cdot(\exists H)\cdot H\,\mathfrak{Cmp}\,G\cdot\mathfrak{Nb}(HKL).\because(KL):K\subset C\cdot$
$L\subset D\cdot K\nabla L\cdot\rightarrow\cdot(\exists H)\cdot H\,\mathfrak{Cmp}\,G\cdot\mathfrak{Nb}(HKL)$

$$\mathfrak{L}(AB)\cdot\equiv::\mathfrak{Is}(AB)::(FHG\,DE)::D\cap E=\boldsymbol{0}\therefore(KL)\therefore K\nabla L\cdot K\subset A\cdot$$
$$L\subset B\cdot\rightarrow:\mathfrak{Nb}(DKL)\cdot\vee\cdot\mathfrak{Nb}(EKL)\therefore\mathfrak{Nb}(F,F\cap A,F\cap B)\cdot$$
$$F\subset D\cdot E\cap A,\ E\cap B\approx_G D\cap A,\ D\cap B\therefore\rightarrow\cdot\sim$$
$$(E\cap A,\ E\cap B\approx_H A\cap D\cap\dashv F,\ B\cap D\cap\dashv F)$$

$$AB\approx CD\cdot\equiv::\mathfrak{L}(AB)\cdot\mathfrak{L}(CD)::(\exists EFGH)::A\cap E,B\cap E\approx_G C\cap F,D\cap F$$
$$\mathfrak{Do}(E)\cdot\mathfrak{Do}(F)\cdot\mathfrak{Do}(H)\cdot E\cap F=\boldsymbol{0}=E\cap H=F\cap H\therefore$$
$$(KL):\cdot K\nabla L\cdot K\subset A\cdot L\subset B\cdot\rightarrow:\mathfrak{Nb}(EKL)\cdot\vee\cdot\mathfrak{Nb}(HKL)\therefore$$
$$(KL):\cdot K\nabla L\cdot K\subset C\cdot L\subset D\cdot\rightarrow:\mathfrak{Nb}(FKL)\cdot\vee\cdot\mathfrak{Nb}(HKL)\therefore$$
$$(MN):\cdot\mathfrak{Nb}(HMN)\cdot(\exists KL)\cdot K\subset C\cdot L\subset D\cdot MN=KL:\equiv:$$
$$\mathfrak{Nb}(HMN)\cdot(\exists KL)\cdot K\subset A\cdot L\subset B\cdot MN=KL\therefore(MNKL):$$
$$M\subset A\cdot N\subset B\cdot K\subset C\cdot L\subset D\cdot MN=KL\cdot\rightarrow\cdot\mathfrak{Nb}(HKL)$$

$$\mathfrak{S}(AB,CD)\cdot\equiv\cdot\mathfrak{L}(AB)\cdot\mathfrak{L}(CD)\cdot(\exists E)\cdot\mathfrak{Nb}(E,E\cap C,E\cap D)\cdot$$
$$A,B\approx C\cap\dashv E,\ D\cap\dashv E$$

$$\dagger(AB,CD,EF)\cdot\equiv\therefore\mathfrak{L}(AB)\cdot\mathfrak{L}(CD)\cdot\mathfrak{L}(EF)\cdot(\exists GH)\cdot G\cap H=\boldsymbol{0}\cdot$$
$$AB\approx G\cap E,\ G\cap F\cdot CD\approx H\cap E,\ H\cap F\therefore(KL):\cdot$$
$$K\nabla L\cdot K\subset E\cdot L\subset F\cdot\rightarrow:\mathfrak{Nb}(HKL)\cdot\vee\cdot\mathfrak{Nb}(GKL)$$

$$\times(AB,CD,EF)\cdot\equiv\therefore\mathfrak{L}(AB)\cdot\mathfrak{L}(CD)\cdot\mathfrak{L}(EF)\therefore(\exists GHK)\therefore GH\approx CD\cdot\mathfrak{Do}(K):$$
$$(L):L\,\mathfrak{Cmp}\,K\cdot\rightarrow\cdot\mathfrak{Nb}(L,L\cap G,L\cap H)\cdot L\cap E,L\cap F\approx AB\therefore$$
$$(MN):M\subset G\cdot N\subset H\cdot M\nabla N\cdot\rightarrow\cdot(\exists L)\cdot L\,\mathfrak{Cmp}\,K\cdot\mathfrak{Nb}(LMN)\therefore$$
$$(MN):M\subset E\cdot N\subset F\cdot M\nabla N\cdot\rightarrow\cdot(\exists L)\cdot L\,\mathfrak{Cmp}\,K\cdot\mathfrak{Nb}(LMN)$$

$$\mathfrak{O}=\{\boldsymbol{0},\boldsymbol{0}\}.$$

It is evident that these relations satisfy the axioms of $\mathcal{T}_1$. From Theorems 1 and 7, by an application of a theorem of Tarski [8]) it results

Theorem 8. *Every algebra of bodies consistent with $\mathcal{T}_5$ is undecidable.*

§ 5. Undecidability of the algebra of convexity.

In the present and in the next sections we shall give some examples of the applications of the Theorems 4-8 to prove the undecidability of some geometrical but not strictly topological theories.

A theory $\mathcal{A}\langle\mathfrak{B},+,\cdot,-,\mathcal{O}\rangle$ will be called an *algebra of convexity* provided that 1. the primitive terms of $\mathcal{A}$ denote respectively: the class of all sets of points, the Boolean operations of sum, product, and complementation on the sets of points, and the convex-operation, and 2. all theorems of $\mathcal{A}$ are true in a certain Euclidean space $\boldsymbol{E}_n$. (The result of the convex operation $\mathcal{O}(A)$ is the smallest convex set containing A).

Theorem 9. *Every algebra of convexity true in an Euclidean space $\boldsymbol{E}_n$, for $n\geqslant 2$, is undecidable.*

Proof. For each Euclidean space $\boldsymbol{E}_n$ we can establish a definition of closure-operation built by means of the primitive terms of the algebra of convexity. For example it is easy to define the relation of neighbourhood. Let $n(A)$ mean that the set A contains strictly n atoms, and $\mathfrak{Nb}(AB)$ denote the relation of neighbourhood. The following equivalence is true in $\boldsymbol{E}_n$:

$$\mathfrak{Nb}(AB)\cdot\equiv:\mathfrak{At}(B)\mathfrak{B}(A):(\exists C):n+1(C)\cdot B\subset\mathcal{O}(C)\subset A\cdot(D):\mathfrak{At}(D)\cdot$$
$$B\neq D\cdot\rightarrow\cdot(\exists E)\cdot E\neq B\cdot\mathfrak{At}(E)\cdot E\subset\mathcal{O}(B+D)\cdot E\subset\mathcal{O}(C).$$

If A is a neighbourhood of the atom B then there exists a simplex $\mathcal{O}(C)$ which satisfies the conditions of the „definiens" of this definition. Conversely if C satisfies the „definiens", then B is contained in the interior of the simplex $\mathcal{O}(C)$.

Analogously we can prove the undecidability of some similar geometrical theories. For example, the algebra $\mathcal{T}\langle\mathfrak{B},+,\cdot,-,\mathfrak{X}\rangle$ is undecidable if $\mathcal{T}$ is true in an Euclidean space $\boldsymbol{E}_n$, for $n\geqslant 2$, provided that the symbols $\mathfrak{B},+,\cdot,-$ denote the class of sets of points and the familiar Boolean operations, and $\mathfrak{X}$ denotes the class of closed spheres. The definition of the relation of neighbourhood has the following form

$$\mathfrak{Nb}(AB)\cdot\equiv\cdot(\exists CDE)\cdot\mathfrak{X}(C)\cdot\mathfrak{X}(D)\cdot\mathfrak{X}(E)\cdot\sim(\mathfrak{At}(C))\cdot\sim(\mathfrak{At}(D))\cdot$$
$$C+D\subset E\subset A\cdot\mathfrak{At}(B)\cdot B=CD.$$

(If $\mathfrak{X}$ denotes the class of parallelepipeds (or the class of rectangular parallelepipeds) $\mathcal{T}\langle\mathfrak{B},+,\cdot,-,\mathfrak{X}\rangle$ is also undecidable).

Similarly the algebra $\mathcal{T}\langle\mathfrak{Do},\cup,\cap,\dashv,\mathfrak{X}\rangle$ is undecidable if $\mathcal{T}$ is true in $\boldsymbol{E}_n$, for $n\geqslant 2$, provided that the symbols $\mathfrak{Do},\cup,\cap,\dashv$ denote the class of closed domains and the Boolean operations defined over closed domains in the same way as in the preceding section and:

(α) $\mathfrak{X}$ denotes the class of convex closed domains,

(β) $\mathfrak{X}$ denotes the class of closed spheres [13]),

[13]) This theory is similar to Tarski's theory of spheres. See [14].

(γ) $\mathfrak{X}$ denotes the class of parallelepipeds (or rectangular parallelepipeds),

(δ) $\mathfrak{X}$ denotes the class of parallelepipeds with faces parallel to certain given directions.

In each of these theories we can construct:

(1) the elementary definition of the relation $C \nabla_0 D$ as having only one contact point for two closed domains C and D which are both: (α) convex, (β) spheres, (γ)-(δ) parallelepipeds;

(2) the elementary definition of the relation of neighbourhood $\mathfrak{Nb}_0(ECD)$ of the unique contact point of two domains C and D which are both: (α) convex, (β) spheres, (γ)-(δ) parallelepipeds;

(3) the elementary definition of the relation $A \Delta B$ of mutual tangency of two closed domains A and B:

$$A \Delta B \equiv : \mathfrak{Do}(A) \cdot \mathfrak{Do}(B) \cdot A \cap B = \boldsymbol{0} \cdot (\exists CD) \cdot \mathfrak{Do}(C) \cdot \mathfrak{Do}(D) \cdot C \nabla_0 D : (E) : \mathfrak{Nb}_0(ECD) \cdot \rightarrow \cdot E \cap A \neq \boldsymbol{0} \cdot E \cap B \neq \boldsymbol{0}.$$

The construction of such definitions presents no difficulties. Hence, by the application of Theorem 8, we obtain the theorems on the undecidability of these theories.

The decision problem remains unsettled for the algebras $\mathfrak{T}\langle \mathfrak{X}, \subset \rangle$, where $\mathfrak{X}$ denotes: (α) the class of closed convex domains, (β) the class of spheres, (γ) the class of parallelepipeds (or rectangular paralellepipeds); $\subset$ denotes the relation of inclusion defined over the elements of the class $\mathfrak{X}$.

§ 6. Undecidability of semi-projective algebras.

A theory $\mathcal{A}\langle \mathfrak{B}, +, \cdot, -, \boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_{n \ldots n<\alpha} \rangle$ will be called a *semi-projective algebra* provided that: 1. $\mathcal{A}$ contains as theorems all axioms of the Boolean algebra for the terms: $\mathfrak{B}$, $+$, $\cdot$, $-$, and 2. $\mathcal{A}$ contains as theorems all formulae of the forms:

$$\boldsymbol{C}_i \boldsymbol{0} = \boldsymbol{0}$$
$$A \subset \boldsymbol{C}_i(A)$$
$$\boldsymbol{C}_i \boldsymbol{C}_j A = \boldsymbol{C}_j \boldsymbol{C}_i A$$
$$\boldsymbol{C}_i A \cdot \boldsymbol{C}_i B = \boldsymbol{C}_i(A \cdot \boldsymbol{C}_i B) + \boldsymbol{C}_i(B \cdot \boldsymbol{C}_i A) \qquad \text{(for each } j, i < \alpha).$$

Many semi-projective algebras may be interpreted as theories of the geometrical operations of semi-projection in the Euclidean spaces.

We will now consider such an interpretation. Let $\boldsymbol{E}_n$ be the n-dimensional Euclidean space. Each point p of $\boldsymbol{E}_n$ is an n-tupel $p = \{\xi_1^p, \xi_2^p, \ldots, \xi_n^p\}$, where the real numbers: $\xi_1^p, \ldots, \xi_n^p$ are the co-ordinates of the point p. We define the result of the operation of the semi-projection as follows:

1. if p is a point:

$$\boldsymbol{C}_i p = \underset{q}{E} \{\xi_j^q = \xi_j^p, \quad \textit{for } j \neq i, \textit{ and } j \leqslant n, \textit{ and } \xi_i^q \leqslant \xi_i^p\},$$

where $i \leqslant n$;

2. if A is a set of points:

$$\boldsymbol{C}_i A = \sum_{p \in A} \boldsymbol{C}_i p.$$

If we assume this interpretation the following theorem can be proved:

Theorem 10. *Every semi-projective algebra $\mathfrak{T}\langle \mathfrak{B}, +, \cdot, -, \boldsymbol{C}_1, \ldots, \boldsymbol{C}_n \rangle$ true in the Euclidean space $\boldsymbol{E}_n$, for $n \geqslant 2$, is undecidable.*

Proof. In each semi-projective algebra $\mathfrak{T}$ with n semi-projective operators we can define in an elementary way the relation of neighbourhood in the Euclidean space $\boldsymbol{E}_n$. For simplicity we shall indicate such a definition in the case of the algebra with two semi-projective operation $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$:

$$\mathfrak{Nb}(AB) \cdot \equiv .\therefore \mathfrak{B}(A) \cdot \mathfrak{At}(B) \cdot (\exists CDEFGHIK) \cdot B, C, D, E, F, G \textit{ are different atoms} \cdot G \subset \boldsymbol{C}_1 D \subset \boldsymbol{C}_1 E \cdot F \subset \boldsymbol{C}_2 C \subset \boldsymbol{C}_2 E \cdot B = \boldsymbol{C}_1 D \, \boldsymbol{C}_2 C \cdot \mathfrak{B}(H) \cdot \mathfrak{B}(I) \cdot \mathfrak{B}(K) \cdot K = \boldsymbol{C}_1 H \, \boldsymbol{C}_2 I \cdot K \subset A : (M) : \mathfrak{At}(M) \cdot G \subset \boldsymbol{C}_1 M \subset \boldsymbol{C}_1 E \cdot \rightarrow \cdot M \subset H .\therefore (M) : \mathfrak{At}(M) \cdot F \subset \boldsymbol{C}_2 M \subset \boldsymbol{C}_2 E \cdot \rightarrow \cdot M \subset I.$$

It is evident that this equivalence is true in $\boldsymbol{E}_2$. Similar definitions for each $n < \aleph_0$ and $n \geqslant 2$ can easily be established.

Similarly by an application of Theorem 8 we can prove as in the preceding section, that every semi-projective algebra $\mathfrak{T}\langle \mathfrak{Do}, \cup, \cap, \dashv, \boldsymbol{C}_1, \ldots, \boldsymbol{C}_{n, \ldots n<\alpha} \rangle$ is undecidable if $\mathfrak{T}$ is true in $\boldsymbol{E}_n$ (for $n \geqslant 2$) provided that the symbols: $\mathfrak{Do}$, $\cup$, $\cap$, $\dashv$ denote the class of closed domains and the Boolean operations defined over the closed domains.

The decision problem for semi-projective algebras of convex sets of points remains unsettled [14]).

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[14]) I am indebted to Professor A. Mostowski for his remarks.

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Dimension Theory in Closure Algebras.

By

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This paper is the continuation of my paper *Closure Algebras*[1]) cited hereafter as CA.

The generalization of the concept of dimension to the case of closure algebras presents no difficulty. The definition assumed in this paper is inductive by means of separation of closed elements. For C-algebras[2]), this definition is equivalent (Theorem 3.5) to Lebesgue's definition which clearly can be formulated without difficulty for arbitrary closure algebras.

The generalization of fundamental theorems from Dimension Theory to the case of arbitrary C-algebras is easy. Some theorems can be proved analogously to the case of metric spaces (see § 1); their proofs are omitted. Other theorems follow from analogous statements for metric spaces (see e. g. 3.2).

The specification of all theorems which hold for C-algebras is not the purpose of this paper. As in my earlier paper CA, I shall only show the method of generalization. Roughly speaking, all theorems from Dimension Theory which hold for separable metric spaces are also true for arbitrary C-algebras.

It was stated in CA that every quotient algebra $\boldsymbol{A}/\boldsymbol{I}$, where $\boldsymbol{A}$ is a C-algebra and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{A}$, is also a C-algebra. This fact suggests the following general problem: Suppose the topological properties of $\boldsymbol{A}$ and $\boldsymbol{I}$ are known; what topological properties has the quotient algebra $\boldsymbol{A}/\boldsymbol{I}$?

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[1]) See *References* at the end of this paper.

The knowledge of Parts I and II of CA is assumed. Theorems from CA will be cited by their numbers together with the letters "CA".

[2]) See the definition on p. 154.