A system which can define its own truth.

By

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Tarski has shown [1]) that for a certain class of logical systems S the following holds:

It is impossible to define in S the class of Gödel-numbers of true statements of S.

The essence of his proof consists of the following version of the Epimenides. Let *Fmla* be the class of Gödel-numbers of meaningful statements of S; then we can define the class of false statements of S as follows

$$x \in Fals \equiv x \in Fmla \cdot \sim x \in True$$

where *True* is the class of Gödel-numbers of true statements of S.

Let „$Nom\,(y,x)$" say that y is the Gödel-number of the numeral designating x, and let „$Subst\,(z,y,x)$" say that z is the Gödel-number of the result of writing the expression whose Gödel-number is y for all free occurrences of „v" in the expression whose Gödel-number si x. Let n be the numeral designating the Gödel-number of

Ep. 1 $\quad (Ey)\,(Ez)\,(Nom\,(y,v)\cdot Subst\,(z,y,v)\cdot z \in Fals)$.

Then the formula

Ep. 2 $\quad (Ey)\,(Ez)\,(Nom\,(y,n)\cdot Subst\,(z,y,n)\cdot z \in Fals)$

says that the result of writing n for all free occurrences of „v" in Ep. 1 is false. But this result is Ep. 2 itself; i. e. Ep. 2 affirms its own falsehood, an evident contradiction [2]).

[1]) A. Tarski, *Pojęcie prawdy w językach nauk dedukcyjnych*, Warszawa, 1933.

[2]) We have $(Ey)\,(Ez)\,(Nom\,(y,n)\cdot Subst\,(z,y,n)\cdot z \in Fals) \equiv (Ep.\ 2\ is\ true)$; but by Tarski's schema for truth (see Tarski, op. cit.), also $(Ey)\,(Ez)\,(Nom\,(y,n)\cdot Subst\,(z,y,n)\cdot z \in Fals) \equiv (Ep.\ 2\ is\ true)$; the contradiction follows by the theory of deduction.

It is obvious that this proof of Tarski's depends upon S's containing a certain amount of conceptual apparatus; in particular it depends upon S's containing negation. The question has hitherto remained undecided, whether *any* system, even without negation, can define its own truth. The purpose of this paper is to answer this question in the affirmative.

Rósza Péter [3]) has constructed a number-theoretic function Φ such that for every primitive recursive function f of two arguments there is a number x such that

$$f(y,z) = \Phi(x,y,z)$$

for all y and z. Further, it is evident from the definition of this function that it is general recursive.

Let S_1 be a system consisting of the recursion equations for Φ and everything which can be deduced from them by the use of extensionality and substitution of constants for variables.

Let S_2 be the class of all formulae of S_1 which contain no free variables.

Let S_3 be a system consisting of all formulae of S_2 and everything which can be obtained from them by means of the rule:

From „$\ldots n$—", where „n" is a numeral, infer „$(Ex)\,(\ldots x$—$)$", where „x" is a variable not occurring in „$\ldots n$—".

We shall show that S_3 *can define its own truth.*

It is evident that S_3 is a system, i. e. that the class of Gödel-numbers of theorems of S_3 forms the range of values of a general recursive function, say a. Further S_2, and hence S_3, is clearly complete and consistent, in the sense that all true formulae expressible in the notation of S_2 and S_3, and no others, are provable in S_2 and S_3 respectively. Hence the class of true statements of S_3 coincides with the class of theorems of S_3.

Rosser [4]) has shown that the range of values of every general recursive function coincides with the range of values of some primitive

[3]) R. Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann., **111** (1935), pp. 42-60.

[4]) B. Rosser, *Extensions of Some Theorems of Gödel and Church*, Journal of Symbolic Logic, **1**, p. 88, Lemma I, Corollary 1.

recursive function. Hence we may suppose α primitive recursive. We have

$$x \text{ is the Gödel-number of a true statement of } S_3 \equiv (Ey)\,(x=\alpha(y))$$
$$\equiv (Ey)\,(x=\beta(y,0))$$
$$\equiv (Ey)\,(x=\Phi(m,y,0))$$

for some primitive recursive β and for some m and this is clearly expressible in S_3; hence S_3 can define its own truth.

Q. E. D.

A Proof of the Completeness Theorem of Gödel.

By

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In this paper we shall give a new proof of the following well-known theorem of Gödel[1]):

(*) *If a formula α of the functional calculus is valid in the domain of positive integers, then α is provable.*

Three ideas play an essential part in our proof: Mostowski's algebraic interpretation of a formula α as a functional the values of which belong to a Boolean algebra; Lindenbaum's construction of a Boolean algebra from formulas of the functional calculus; and a theorem on the existence of prime ideals in Boolean algebras, the proof of which is topological and uses the well-known category method.

1. *The functional calculus.* By the *functional calculus* (of first order) we understand the system which can be briefly described as follows:

The symbols of the system are: *individual variables* $x_1, x_2, \ldots$; *functional variables* $F_1^k, F_2^k, \ldots$ with k arguments ($k=1,2,\ldots$); and *constants*. The constants are: the negation sign $'$, the disjunction sign $+$, the existential quantifier $\sum\limits_{x_k}$, and the brackets.

$F_j^k(x_{i_1},\ldots,x_{i_k})$ is a (elementary) *formula* of this system; if α and β are formulae, then $\alpha+\beta$, α' and $\sum\limits_{x_k}\alpha$ are also formulae.

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[1]) K. Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik **37** (1930), pp. 349-360. See also D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Band II, Berlin 1939; and L. Henkin, *The completeness of the first-order functional calculus*, Journal of Symbolic Logic **14** (1949), pp. 159-166.